MSRI Discrete and Computational Geometry
Workshop on Combinatorial and Discrete Geometry
Berkeley, November 17, 2003

The Generalized Lower Bound Theorem for Polytopes and j-Facets of Finite Point Sets

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Lower Bound Theorem and GLBT

Entering and Leaving j-Facets

Applications to a Generalized UBT

Open Questions

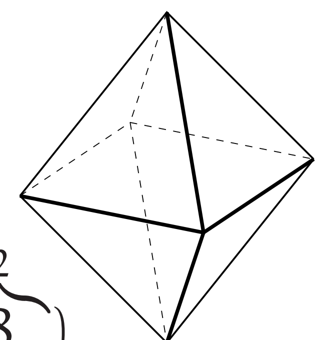
f-Vector of Polytopes

$\mathcal{P}$ simplicial d-polytope:

$f_i = f_i(\mathcal{P}) :=$ # of i-faces, $0 \le i \le d-1$.

$(f_i)_{i=-1}^{d-1}$, the f-*vector of* $\mathcal{P}$, $\quad f_{-1} := 1$.

$(\overbrace{1}^{-1}, \overbrace{6}^{0}, \overbrace{12}^{1}, \overbrace{8}^{2})$



$f_0 \ge d+1$ for $d \ge 1$

$f_1 \ge d\, f_0 - \binom{d+1}{2}$ for $d \ge 3$

Lower Bound Theorem [Barnette '70]

GLBT

Generalized Lower Bound Theorem

$$g_1 := f_0 - \binom{d+1}{1} \geq 0, \qquad d \geq 1$$

$$g_2 := f_1 - \binom{d}{1} f_0 + \binom{d+1}{2} \geq 0, \qquad d \geq 3$$

$$g_3 := f_2 - \binom{d-1}{1} f_1 + \binom{d}{2} f_0 - \binom{d+1}{3} \geq 0, \quad d \geq 5$$

$$\vdots$$

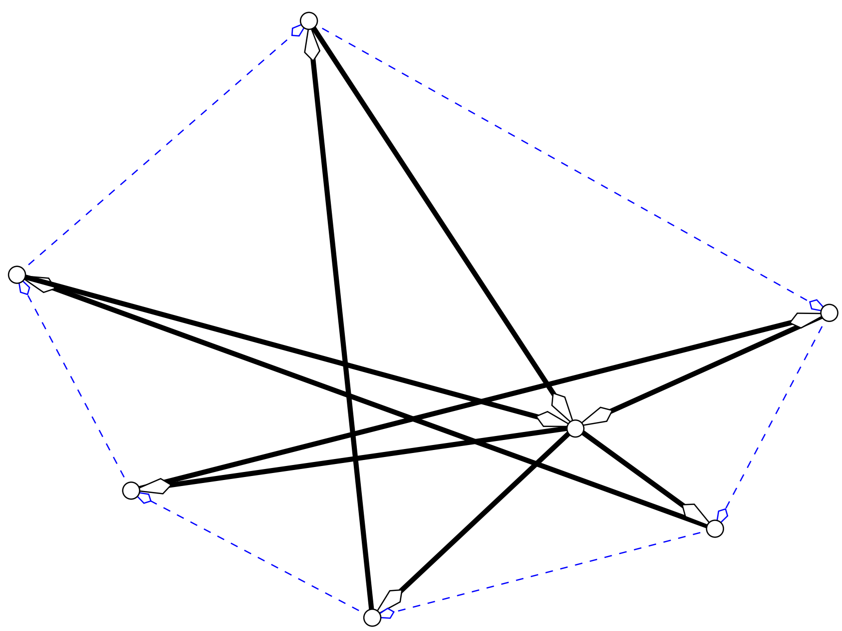
which is part of necessity part of g-Theorem, conjectured by McMullen,

proved by [Stanley '80].

j-Facets

S a set of points in $\mathbb{R}^d$, in general position.

An oriented simplex spanned by d points in S is called *j-facet*, if there are exactly j points from S on its positive side (i.e. on the positive side of the hyperplane it spans).



0-facets ≡ facets

0- and 2-facets
$e_0 = 6, e_2 = 9$

Number of j-Facets

For $j \in \mathbb{Z}$,

$$e_j = e_j(S) \quad := \quad \# \text{ of } j\text{-facets of } S,$$

$$E_j = E_j(S) \quad := \sum_{i \le j} e_i \qquad (\le j)\text{-}\textit{facets}.$$

Note for $n := |S|$

$$e_0(S) = E_0(S) = f_{d-1}(\operatorname{conv} S)$$

$$e_j = e_{n-d-j}$$

$$E_{n-d} = 2\binom{n}{d}$$

Bounds on e_j and E_j:

[Lovász'71], [Erdős,L.,Simmons,Straus'73],...

Entering j-Facets

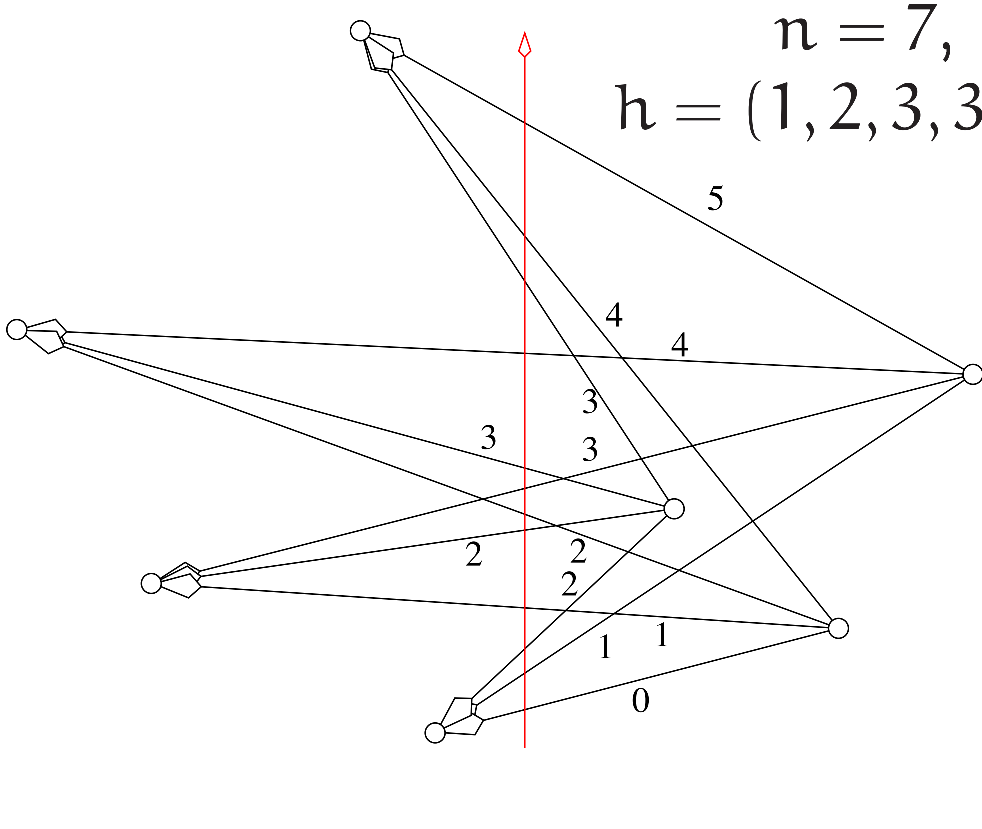
A directed line ℓ *enters* a j-facet, if it intersects the j-facet in its relative interior, directed from its *positive to* its *negative* side.

$$h_j = h_j(S, \ell) := \# \text{ of j-facets entered by } \ell.$$

$(h_j)_{j=0}^{n-d}$, the h-*vector of* S *and* ℓ.

$n = 7,\ d = 2$

$h = (1, 2, 3, 3, 2, 1)$



How many j-facets can be entered by a line?

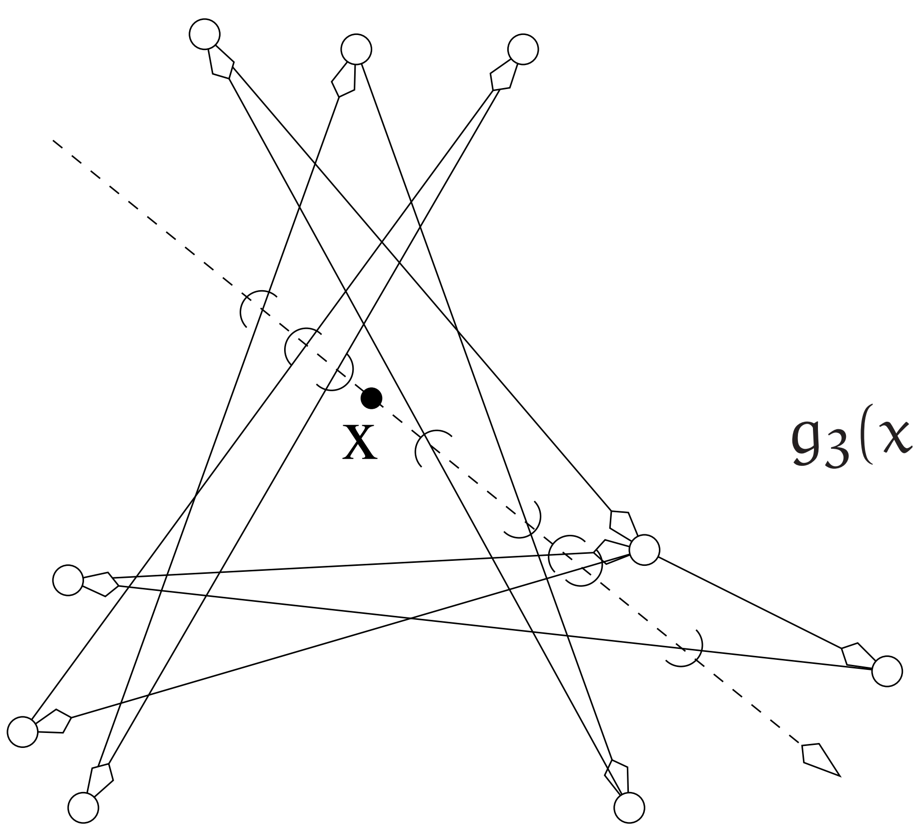
At most $\binom{j+d-1}{d-1}$.

Entering and Leaving

A directed line ℓ *leaves* a j-facet, if it intersects the j-facet in its relative interior, directed from its *negative to* its *positive* side. (... enters the 'opposite' $(n-j-d)$-facet ...)

For a point x on ℓ $(S \cup \{x\}$ in gen. pos.)

$$g_j(x) = g_j(x, S, \ell) := \# \text{ of j-facets entered by } \ell \text{ up to } x - \# \text{ of j-facets left by } \ell \text{ up to } x$$



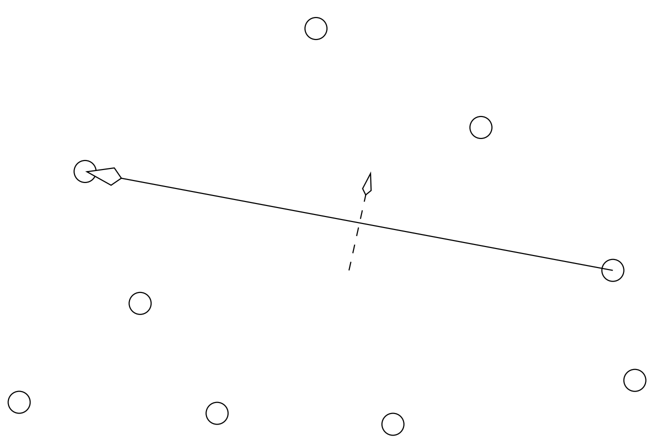
$g_3(x) = 2 - 1 = 1$

You May Come Either Way

Claim. $g_j(x) = g_j(x, S, \ell)$ does not depend on the line ℓ.

For a proof, let $s_k(x)$ be the number of $Q \subseteq S$ with $|Q| = (k + d + 1)$ and $x \in \operatorname{conv} Q$.

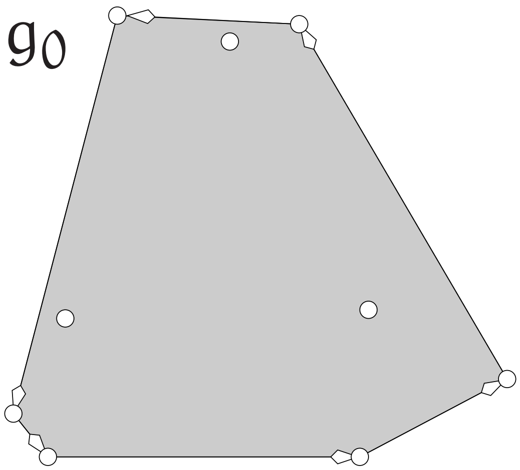
As x enters a j-facet, it loses $\binom{j}{k+1}$ such sets and gains $\binom{n-d-j}{k+1}$ such sets. Therefore, the vector $(s_k(x))_k$ is determined by the vector $(g_j(x))_j$ …and vice versa.



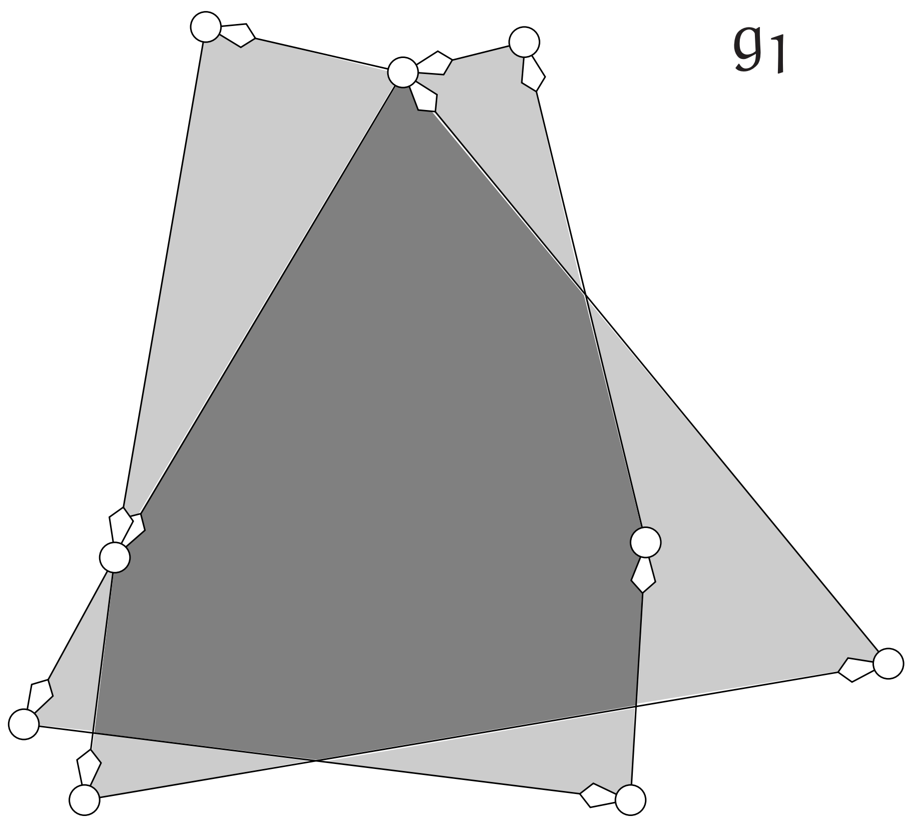
Entering a 5-facet: We lose $\binom{5}{2}$ sets of size 4, and we gain $\binom{2}{2}$ such sets.

$\Rightarrow$ every line enters the same number of j-facets as it leaves).

Areas of Equal g_j-Value



$$g_0(x) = 1 \text{ iff } x \in \operatorname{conv} S$$



g_j's measure interiorisity of a point relative to S?

First Enter, then Leave

$g_{n-j-d}(x) = -g_j(x)$, so values for $j \le \frac{n-d}{2}$ determine the whole vector.

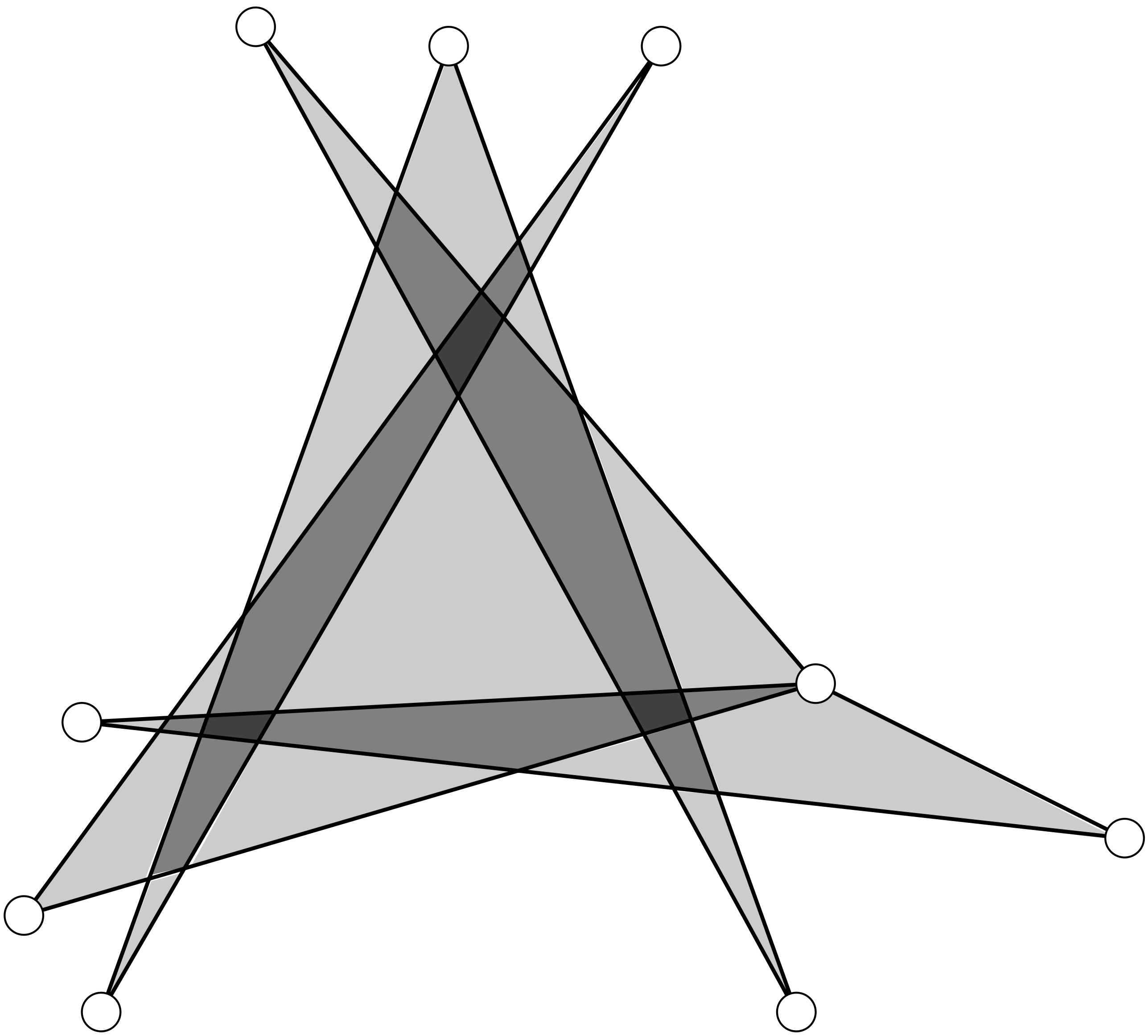
Theorem. $g_j(x) \ge 0$ for $j \le \frac{n-d}{2}$.

We can never leave more j-facets than we have entered.

The $\{\text{enter}, \text{leave}\}$-sequence is a well-formed string of parenthesis.

Equivalent to GLBT for polytopes via Gale Transform [Carl Lee '91, W. '01]

Example



Entered minus left 3-facets.
The darker, the larger $g_3(x)$ is.

Moving Towards the Center Point

Proof of $g_j(x) \geq 0$ for $j < \frac{n}{d+1} - 1$.

Choose a *center point* c, i.e. a point so that every hyperplane containing c has at most $\frac{dn}{d+1}$ points on either side. c is on the negative side of every j-facet with

$$n - j - 1 > \frac{dn}{d+1} \;\Leftrightarrow\; j < \frac{n}{d+1} - 1.$$

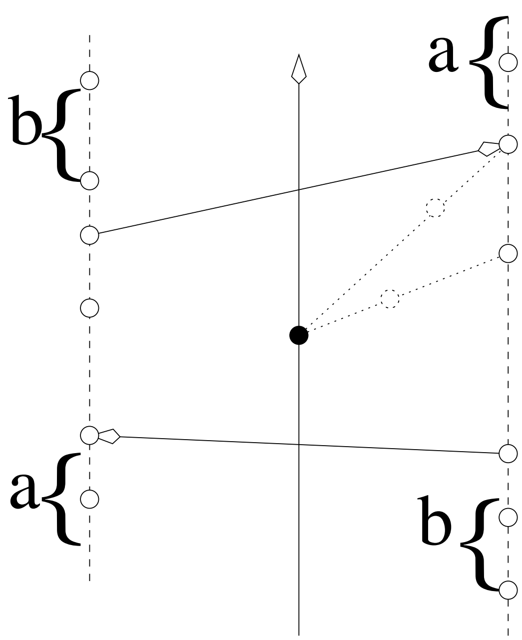
Now choose a line $\ell \supseteq \{c, x\}$ directed from x to c. Moving on ℓ towards c, we never leave a j-facet with $j < \frac{n}{d+1} - 1$, so $g_j(x, S, \ell)$ cannot be negative.

If c is on or on the positive side of a j-facet, then there is a hyperplane through c that has at least $(n-j-d)+(d-1) = n-j-1$ points on one side. Because of the center point property, $n-j-1 \leq \frac{dn}{d+1}$. So if $n - j - 1 > \frac{dn}{d+1}$, c has to be on the negative side of any j-facet.

In the Plane

Proof of $g_j(x) \geq 0$ for $j \leq \frac{n}{2} - 1$ in $\mathbb{R}^2$.

(Assume n even.) Choose a line $\ell \ni x$ that has the same number of points in S on either side. Project (from x) the points in S on two lines parallel to ℓ. In this way the $s_k(x)$'s don't change, and so the $g_j(x)$'s don't.



For a, b,
$a + b =: j \leq \frac{n}{2} - 1$,
match the j-facet entered with a points to the left and b points to the right, with the j-facet left, with b points to the left and a points to the right.

An Application

Upper Bound for E_j in $\mathbb{R}^3$. [W. '01]

For every set S of n points in $\mathbb{R}^3$ and all $0 \leq j \leq n-4$

$$E_j(S) + 2\sum_{p \in S} g_j(p, S \setminus \{p\}) = E_j(C_n^3)$$

C_n^3, n points on moment curve (conv. pos.).

Since $g_j \geq 0$ for $j \leq \frac{(n-1)-3}{2}$, it follows that $E_j \leq E_j(C_n)$ for all $j \leq \frac{n-4}{2}$.

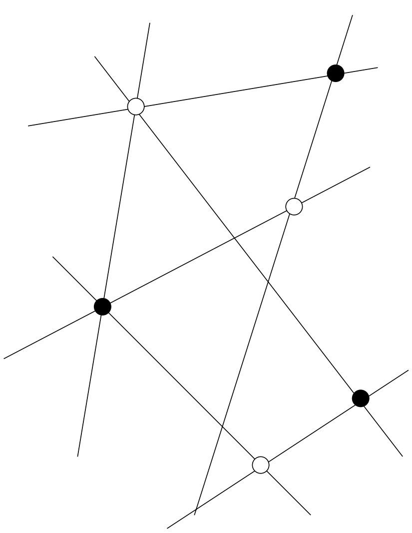
'$E_j \leq E_j(C_n)$' is equivalent to the GLBT for d-polytopes with at most $d+4$ vertices.

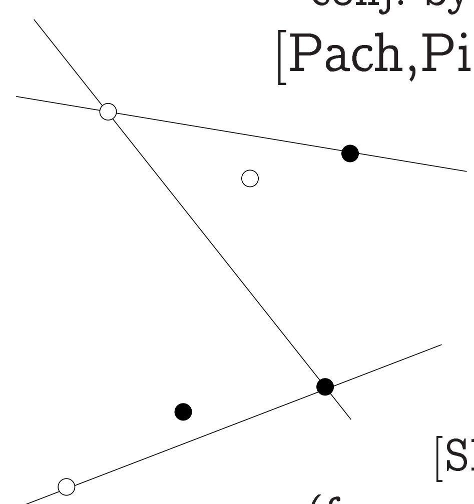
Further Implications

Every set of $2n+1$ points in $\mathbb{R}^3$ has at least n^2 *halving triangles* (which is tight).

n red and n blue points in the plane always allow at least n *balanced lines* (lines through two points that have on either side the same number of red and blue points.

conj. by G. Baloglou
[Pach,Pinchasi '01]





[Sharir,W. '03]
(for conn. to GLBT)

Questions

Simple(r) proof of GLBT,

i.e. $g_j(x) \geq 0$ for $d \geq 3$.

Generalized Upper Bound Theorem in R^d?

Is it true that $E_j \leq E_j(C_n^d)$ for all d and all $j \leq \frac{n-d-1}{2}$?

known for $d \leq 3$;

known to be true asymptotically [Clarkson,Shor'89]

More generally: Exact linear inequalities for the e-vector $(e_j)_{j=0}^{n-d}$.

Known: e_0 (UBT); bounds above; $E_j \geq 3\binom{j+2}{2}$ in $\mathbb{R}^2$ for $j \leq n/3 - 1$ with implications to K_n crossing number [Lovász, Vesztergombi, Wagner,W.'03]