The Gromov invariant of $\Sigma_m \times \Sigma_n$ and Minimal triangulations of $C_m \times C_n$

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Joint work with L. Bowen, J.A. De Loera, M. Develin

The Problem(s)

$\Sigma_m =$ the (orientable, compact) surface of genus m.

$C_m =$ the (convex) polygon with m vertices.

We want to compute:

- The Gromov invariant of $\Sigma_m \times \Sigma_n$.
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(1) The Gromov invariant of $\Sigma_n \times \Sigma_m$

The Gromov invariant

For any d-manifold M, let

$$||M|| := \inf\{\text{weight of a chain representing the fundamental class in } H_d(M,\mathbb{R})\}$$

$$= \inf\left\{\frac{\#K}{|\deg(f)|} : K \text{ is an oriented } d\text{-pseudo-manifold and } f : |K| \to M \text{ is a map}\right\}.$$

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Intuitively, it is the minimum amount of (singular, fractional) simplices needed to cover M “face to face”.

$||M||$ is called the **Gromov invariant**, **Gromov norm**, or **simplicial volume** of M (Gromov 1982, Thurston lecture notes 1978).

Example

Let $S^1 = \{e^{i\alpha} : \alpha \in \mathbb{R}\} \subset \mathbb{C}$ be the 1-sphere. Then:

- $\sigma_1 : [0,1] \to S_1$ defined by $\sigma_1(t) = e^{i\pi t}$ is a singular simplex giving the fundamental class $\Rightarrow ||S^1|| \leq 1$.

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$$\Rightarrow ||S^1|| \leq 1/k, \quad \forall k \qquad \Rightarrow ||S^1|| = 0.$$

More generally

Lemma (Gromov) *For any map $f : M \to N$ (M and N manifolds of the same dimension),*

$$||M|| \geq |\deg(f)|\ ||N||.$$

Proof: If c_M is a fundamental cycle of M, then $f_*(c_M)$ gives $|\deg(f)|$ times the fundamental cycle of N.

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Lemma (Gromov) *For any map* $f : M \to N$ *(*M *and* N *manifolds of the same dimension),*

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Corollary: If there is a map $f : M \to M$ with $|\deg(f)| > 1$ then $||M|| = 0$.

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. . . and, actually, $||M|| = 0$ and $||M \times N|| = 0$ if $||M||$ is an elliptic or Euclidean (and compact) manifold!

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with equality if f is a covering map.

Proof: In this case the fundamental cycle c_N of N can be pulled back to a fundamental cycle $c_M := f^*(c_N)$ of M such that $f_*(c_M) = |\deg(f)|c_N$.

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Summing up:

$$||\Sigma_m \times \Sigma_n|| = \begin{cases} 0, & \text{if } \min\{m,n\} \leq 1. \\ (n-1)(m-1)||\Sigma_2 \times \Sigma_2||, & \text{if } m,n \geq 2. \end{cases}$$

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Well, of course yes.

Hyperbolic manifolds

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Lemma 1: $v_d < \infty$. Actually, v_d=volume of regular ideal simplex (Haagerup-Munkholm, 1981) and $v_{d+1} < v_d/d$.

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Lemma 2: For a hyperbolic manifold M, the Gromov norm can be computed using geodesic ("straight") simplices.

Proof: If a singular class c represents a fundamental cycle, its "streightening"[1] does too.

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[1] replace every singular simplex by the unique straight simplex with the same vertex set (and homotopic to it)

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Corollary: For a hyperbolic d-manifold $||M||$, $||M|| \geq \frac{Vol(M)}{v_d}$.

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Example:

$$||\Sigma_n|| = \frac{(4n-2)\pi - 2\pi}{\pi} = 4(n-1).$$

Proof: Σ_n is represented as a $4n$-gon in $\mathbb{H}^2$ with total angle sum 2π. Its volume is hence $(4n-2)\pi - 2\pi$. The volume of the regular ideal triangle is π.

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Corollary: $||\Sigma_n \times \Sigma_m|| \leq 6||\Sigma_n||\ ||\Sigma_m|| = 96(n-1)(m-1).$

(2) The Gromov invariant and triangulations

Precedents

Hyperbolic geometry has been used in the past to study triangulations of polytopes:

[Sleanor-Tarjan-Thurston, 1988] (using Gromov norm) prove that the flip-graph of triangulations of an$n$-gon has diameter **equal** to $2n - 10$ for every sufficiently large n.

[W. Smith, 1998] (relating the volumes of the regular ideal simplex and regular ideal cube in $\mathbb{H}^n$) gives the best lower bound on the size of triangulations of the n-cube known so far.

Warm up: $||\Sigma_n|| = 4(n-1)$ revisited

Σ_n can be represented by a $4n$-gon. A triangulation of it (with $4n-2$ triangles) covers Σ_2 $n-1$ times. Hence:

$$||\Sigma_2|| = \frac{4n-2}{n-1} = 4 + \frac{2}{n-1} \quad \forall n \quad \Rightarrow \quad ||\Sigma_2|| \leq 4.$$

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Observe that **we haven't used hyperbolic volume computations at all**.

$||\Sigma_2 \times \Sigma_2||$ and triangulations of $C_n \times C_m$

Let $t_{m,n} :=$ minimal number of simplices in a triangulation of $C_n \times C_m$.

$\Sigma_n \times \Sigma_m$ can be represented by a $C_{4n} \times C_{4m}$ in $\mathbb{H}^2 \times \mathbb{H}^2$, and it covers $\Sigma_2 \times \Sigma_2$ $(n-1)(m-1)$ times. Hence:

Proposition:

$$||\Sigma_2 \times \Sigma_2|| \leq \frac{t_{4n,4m}}{(n-1)(m-1)}.$$

(Some technical details need to be filled-in: a triangulation T of $C_{4n} \times C_{4m}$ does not ion general represent a cycle in $\Sigma_n \times \Sigma_m$, because identified faces need to be triangulated equal. What we prove is that replicating T a finite number of times one does get a cycle).

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Remark: we can relax a lot the concept of "triangulation" in the definition of $t_{m,n}$; what we are interested in is basically the Gromov norm of $C_n \times C_m$ when singular cycles are required to have good restrictions on faces of $C_n \times C_m$. We call that the **polytopal Gromov norm of** $C_m \times C_n$.

$||\Sigma_2 \times \Sigma_2||$ and triangulations of $C_n \times C_m$

Equality does not follow from Gromov's theorem because:

- $\mathbb{H}^2 \times \mathbb{H}^2$ is not hyperbolic (Gromov's theorem does not apply), and
- As $n, m \to \infty$ the simplices used to triangulate $C_{4n} \times C_{4m}$ do not have maximal volume in $\mathbb{H}^2 \times \mathbb{H}^2$ (actually, they do not all have the same volume!).

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Still, we can prove that:

Theorem:

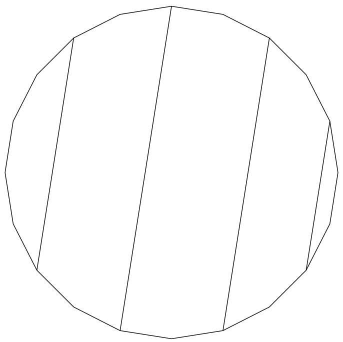
$$||\Sigma_2 \times \Sigma_2|| = \lim_{n,m\to\infty} \frac{16 t_{n,m}}{nm} = \inf_{n,m} \frac{16 t_{n,m}}{nm}.$$

$||\Sigma_2 \times \Sigma_2||$ and triangulations of $C_n \times C_m$

Sketch of proof: $\lim_{n,m\to\infty} \frac{16t_{n,m}}{nm} = \inf_{n,m} \frac{16t_{n,m}}{nm}$ follows from

Lemma: If $n-2$ and $m-2$ divide $n'-2$ and $m'-2$, then

$$t_{n',m'} \geq \frac{(n'-2)(m'-2)}{(n-2)(m-2)} t_{n,m}.$$



A 22-gon chopped into 5 hexagons.

$||\Sigma_2 \times \Sigma_2||$ and triangulations of $C_n \times C_m$

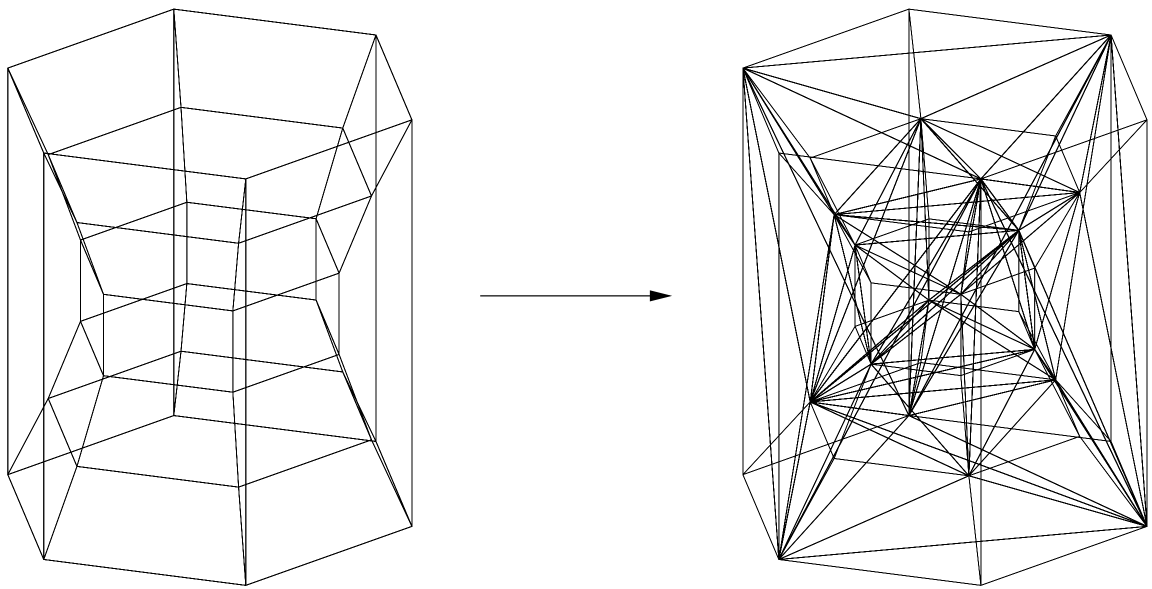
Sketch of proof: For $||\Sigma_2 \times \Sigma_2|| \geq \inf_{n,m} \frac{16 t_{n,m}}{nm}$:

1. One can consider only "straight" (in the Klein model of $\mathbb{H}^2 \times \mathbb{H}^2$) simplices to compute $||\Sigma_2 \times \Sigma_2||$ (easy),

2. For any straight cycle c representing $||\Sigma_2 \times \Sigma_2||$ and for any sufficiently large integer h, each simplex of c is embedded in $||\Sigma_h \times \Sigma_h||$ (by properties of discrete subgroups of isometries in $\mathbb{H}^2 \times \mathbb{H}^2$),

3. Consider the covering map $\pi : \Sigma_h \times \Sigma_h \to \Sigma_2 \times \Sigma_2$ and the pull-back cycle $c_h := \pi^*(c)$ (for which $\pi(c_h) = (h-1)^2 c$). Then, c_h can be homotoped to a "polytopal cycle" (the ones in the definition of $t_{4h,4h}$). Hence,
$$||\Sigma_2 \times \Sigma_2|| \geq \frac{t_{4h,4h}}{(h-1)^2}.$$

(3) Triangulations of $C_n \times C_m$



Our goal

Compute $t_{m,n}$, or, at least,

$$\lim_{n,m\to\infty} \frac{t_{n,m}}{nm} = \frac{1}{16}||\Sigma_2 \times \Sigma_2||.$$

Empirical data

Using the so-called **universal polytope** $U(P)$ one can compute the minimum size of a triangulation of a polytope P. By definition, each vertex of $U(P) \subset [0,1]^{\binom{n}{d+1}}$ is a 0/1 vector indicating which simplices appear in a specific triangulation. A facet definition of $U(P)$ is easy to derive from the oriented matroid of P [de Loera-Hoşten-S.-Sturmfels, 1996].

Remark: usually one needs to do integer programming on $U(P)$ to compute the minimum triangulation. But in our context fractional solutions are valid cycles in the definition of the polytopal Gromov norm. Hence, we can use linear programming on $U(P)$.

Empirical data

We calculated the following for the minimal size triangulations of $C_n \times C_m$:

$_n\backslash^m$	3	4	5	6	7	8	9			
3	6	10	15	19	24	28	33	$=$	$\frac{9}{2}m-8$	$+\frac{1}{2}[1-(-1)^m]$
4	10	16	26	32	42	≤ 48	≤ 58	$=?$	$8m-16$	$+2[1-(-1)^m]$
5	15	26	38	≤ 49	≤ 61	≤ 72		$=?$	$\frac{23}{2}m-20$	$+\frac{1}{2}[1-(-1)^m]$
6	19	32	≤ 49	≤ 60	≤ 77	≤ 90		$\neq$	$15m-28$	$+2[1-(-1)^m]$

Remark: The numbers on the right column all obey the rule:

$$\left(\frac{7}{2}mn - 6m - 6n + 8\right) \quad + \quad 2[2-(-1)^m-(-1)^n] - \frac{3}{2}[1-(-1)^{mn}]$$

Statement of results

1. For $n = 3$, $t_{3,m} = \lceil \frac{9m}{2} \rceil - 8$.

2. For $n = 4$, $\frac{7}{2}(m-2) \leq t_{4,m} \leq 8(m-2)$.

3. Lower bound:
$$t_{m,n} \geq 2mn - \frac{8}{3}m - \frac{8}{3}n + \Omega(1)$$

4. Upper bound:
$$t_{m,n} \leq \frac{7}{2}mn - 6m - 6n + 8$$

Warm-up: the minimum triangulation of a prism

Theorem (de Loera, S., Takeuchi, 2001): the minimum triangulation of $C_m \times I$ has $\left\lceil \frac{5}{2}(m-2) \right\rceil$ tetrahedra.

Proof: Lower bound: you need $m-2$ for the lower m-gon, $m-2$ for the upper m-gon and $(m-2)/2$ for the "middle" region.

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Upper bound: assume m even. Chop-off every other vertex (m tetrahedra used) and use a pulling triangulation of the remaining $\frac{m}{2}$-antiprism.You use $m-3+\frac{m}{2}-2$ tetrahedron there.

If m is odd then first cut a triangular prism and then proceed.

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Remark: Both proofs are purely topological. They apply to every combinatorial prism and they are valid for fractional and/or "singular" triangulations (as long as they restrict to triangulations of faces, as required in the definition of the polytopal Gromov norm).

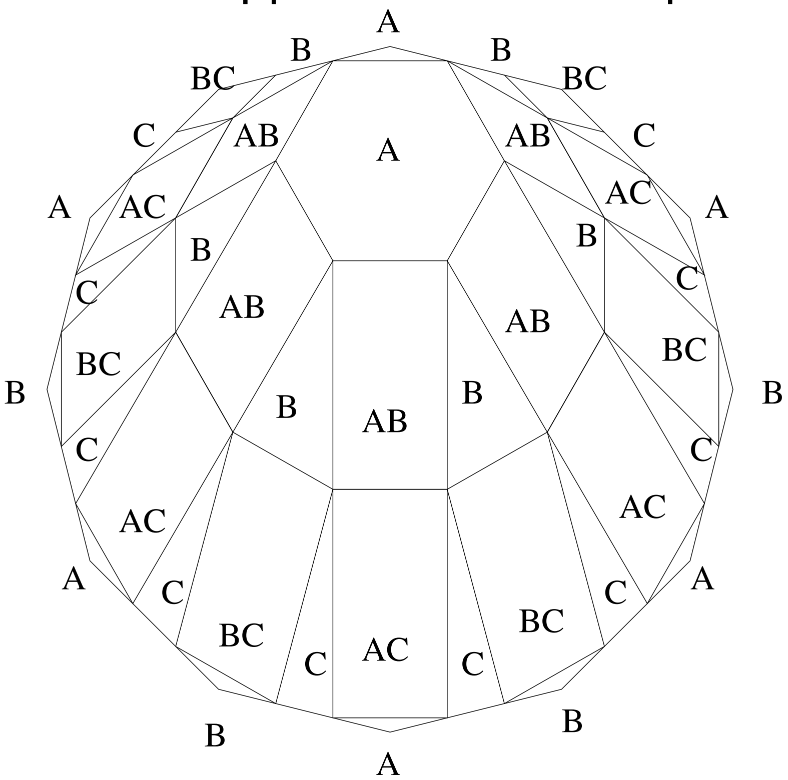
The minimum triangulation $C_3 \times C_m$

Theorem: the minimum triangulation of $C_3 \times C_m$ has $\lceil \frac{9}{2}m - 8 \rceil$ 4-simplices.

Proof: Lower bound: you need $m-2$ for each of the three copies of C_m and $\frac{m-2}{2}$ for the interior of each of the three $C_m \times I$. This gives $\frac{9}{2}m - 9$. But to triangulate with that number you would need to chop-off alternate vertices in the three prisms simultaneously, which is impossible.

The minimum triangulation $C_3 \times C_m$

Proof: Upper bound: an explicit construction that gives exactly that number:



A triangulation of $C_3 \times C_{12}$ with 46 4-simplices.

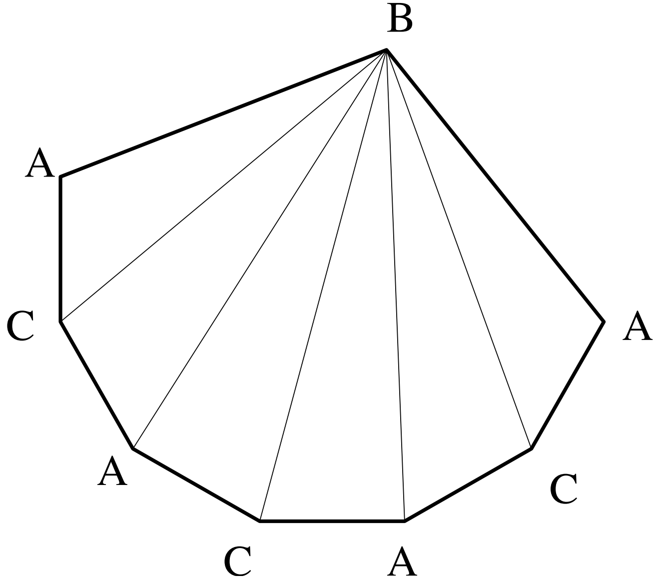
Triangulations of $C_m \times C_4$

Lower bound: look at $C_m \times C_4$ as a prism over $C_m \times I$: you need $\frac{5m}{2}$ to triangulate the bottom $C_m \times I$, $\frac{5m}{2}$ to triangulate the top $C_m \times I$, and $\frac{5m}{2}$ to triangulate the middle.

Upper bound: divide $C_m \times C_4$ into $(m-2)/2$ copies of $C_4 \times C_4$ (a 4-cube) and triangulate each with 16 simplices.

A triangulation of $C_n \times C_m$ of size $\frac{7}{2}mn - 6m - 6n + 8$:

Divide $C_n \times C_m$ into $n-2$ copies of $C_3 \times C_m$ and triangulate them as before.

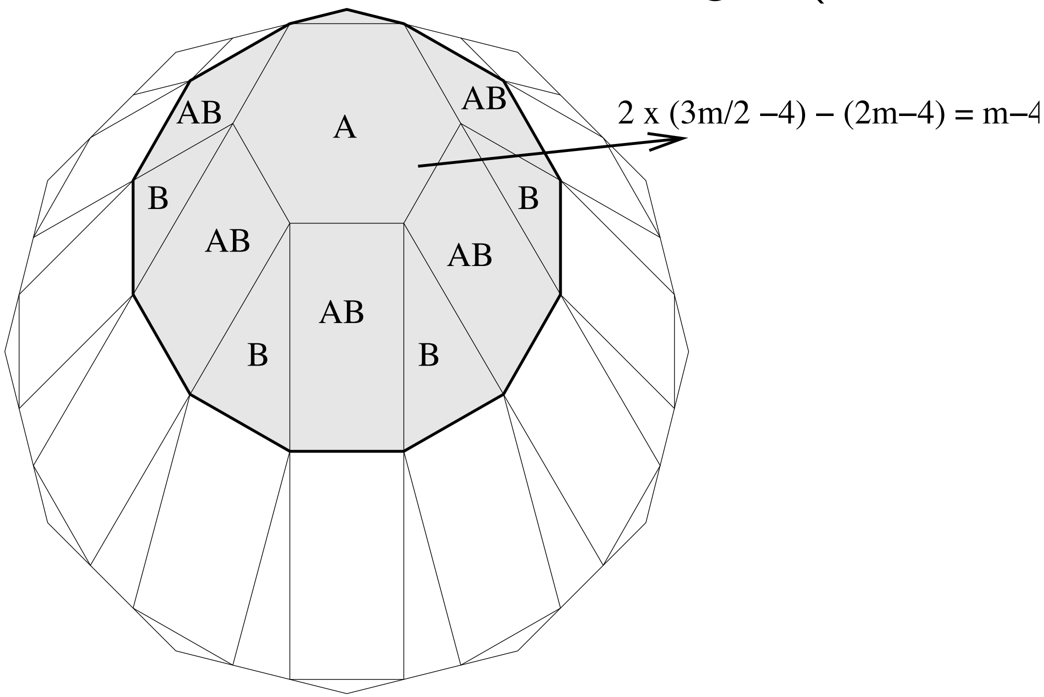


A triangulation of $C_3 \times C_{12}$ with 46 4-simplices.

This gives (about) $\frac{9mn}{2}$. But looking at how the different $C_3 \times C_m$ are glued, you can reduce locally the number of simplices, (magically) giving that number.

A triangulation of $C_n \times C_m$ of size $\frac{7}{2}mn - 6m - 6n + 8$:

For example, the following shaded region represents a $m/2$-antiprism that is coned to two "C" in different triangles (about $3m$ 4-simplices).

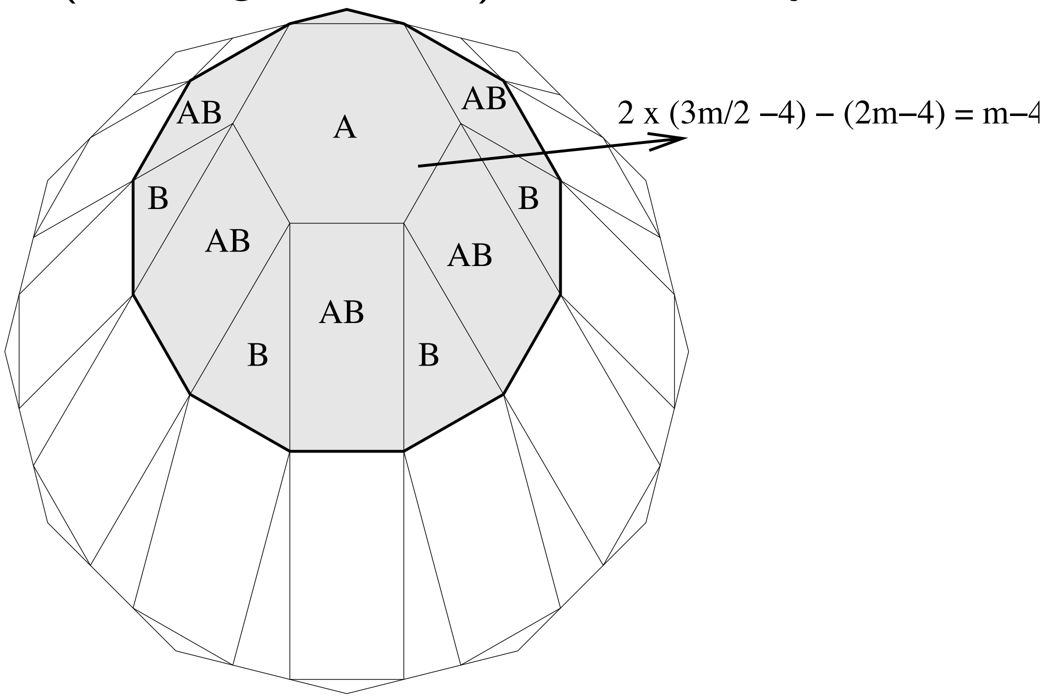


total savings: m–4

Saving about $mn/2$ simplices

A triangulation of $C_n \times C_m$ of size $\frac{7}{2}mn - 6m - 6n + 8$:

Certainly, it is more efficient to re-triangulate this "bi-pyramid" by coning the axis to (a triangulation of) the boundary of the antiprism (about $2m$ 4-simplices).



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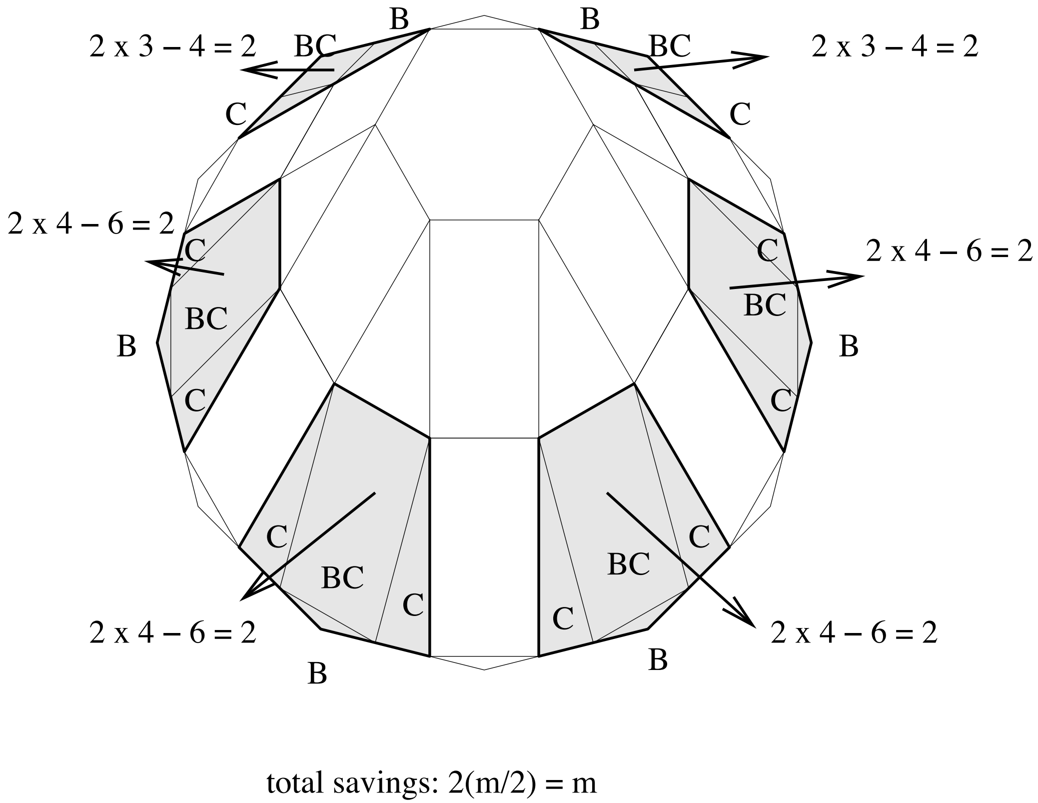
A similar trick can be played on the BC-prisms:



Saving another $mn/2$ simplices

A lower bound of $2mn - O(m+n)$

Lemma 1: *In every triangulation of an m-prism there are at least $3m - O(1)$ interior triangles.*

Corollary: *In every triangulation of $C_n \times C_m$ there are at least $6mn - O(m+n)$ "semi-interior" triangles (triangles interior to a boundary prism)*

Corollary: *In every triangulation of $C_n \times C_m$ there are at least $12mn - O(m+n)$ incidences* (4-simplex, facet tetrahedron, semi-interior triangle).

Lemma 2: *No 4-simplex in $C_n \times C_m$ has more than 6 incidences* (4-simplex, facet tetrahedron, semi-interior triangle).

Conclusions/conjectures

- (We think that) we can improve the upper and lower bounds to, respectively, $\frac{13}{4}mn + O(m+n)$ and $\frac{25}{12}mn - O(m+n)$.

- We beleive that our upper bound is very close to optimal, and that room for improvement is in the lower bound (because all the lower bounds that we have tried follow only from "boundary constraints").

- One "nice" possibility is that

$$t_{m,n} = 3mn \pm \Theta(m+n), \quad \Rightarrow \quad ||\Sigma_m \times \Sigma_n|| = 48(m-1)(n-1)$$