Coefficients and Roots of Ehrhart Polynomials

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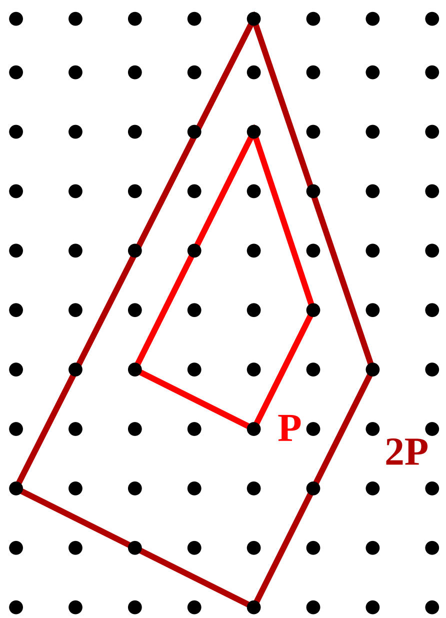
Measuring volume by counting

A convex d-dimensional polytope $P \subset \mathbb{R}^d$ is a **lattice polytope** if $\mathrm{vert}\, P \subset \mathbb{Z}^d$.

- Approximate $\mathrm{vol}\, P$ by counting lattice points



Theorem (Ehrhart 1967) The function

$$i_P : \mathbb{N} \to \mathbb{N}, \qquad i_P(n) \;=\; \#\{nP \cap \mathbb{Z}^d\}$$

- is a polynomial in n of degree d,
- with leading coefficient $\mathrm{vol}\, P$, and constant term 1.

Calculate $\mathrm{vol}\, P$ by counting lattice points in d dilated copies of P.

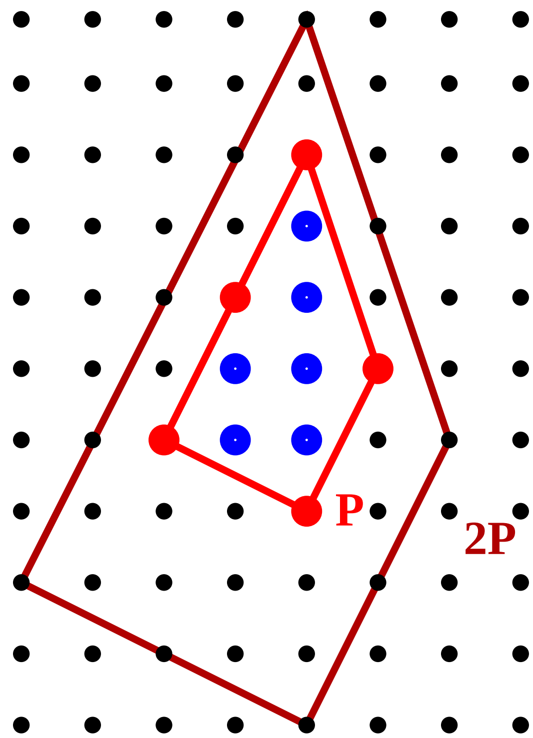
Ehrhart reciprocity

- $i_P(n)$ is a polynomial in n
- therefore, it is defined for $n \in \mathbb{Z}$ (even $n \in \mathbb{C}$)

$$i_P(n) \;=\; \frac{15}{2}n^2 + \frac{5}{2}n + 1$$

Ehrhart reciprocity: count the number of interior lattice points by evaluating i_P at negative integers n:

$$\#\{\,\mathrm{relint}(nP) \cap \mathbb{Z}^d\} \;=\; (-1)^d\, i_P(-n)\,.$$



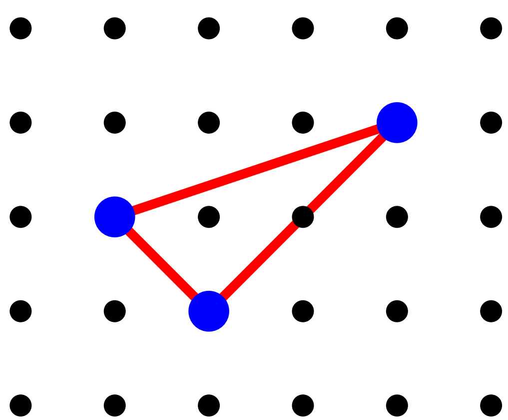
Bases for polynomials, I

Two bases for the vector space of polynomials $p \in \mathbb{R}[n]$ of degree d:

Power basis: $$p(n) = \sum_{i=0}^{d} c_i \, n^i$$

if $p = i_P$, highest coefficients express **geometry** of polytope P

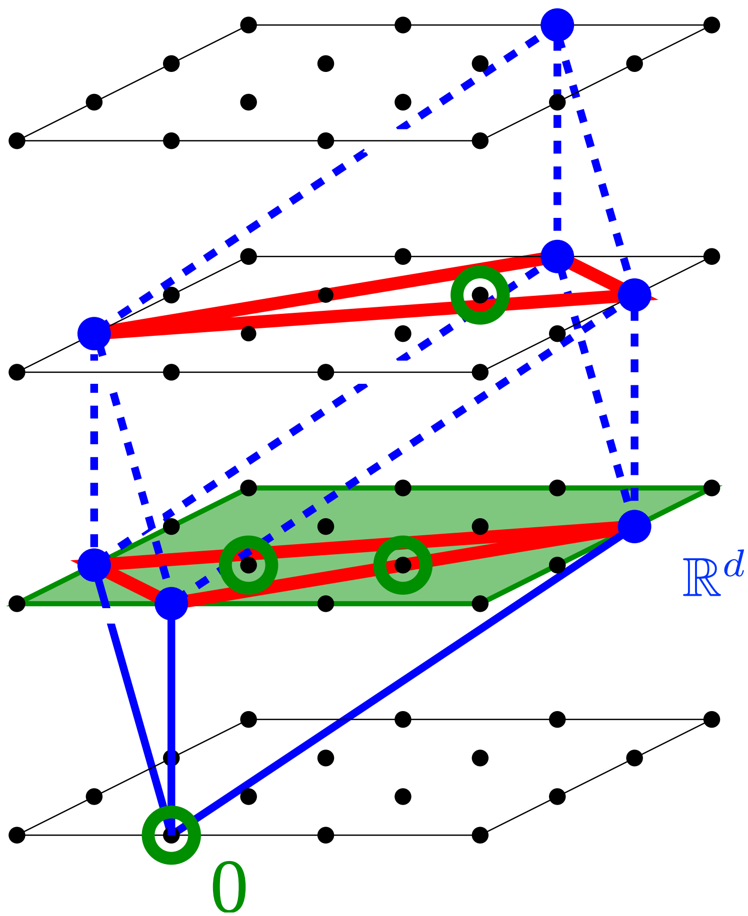
- $c_d = \mathrm{vol}_d \, P$, normalized w.r.t. $\mathbb{Z}^d$
- $c_{d-1} = \frac{1}{2} \sum_{F \text{ facet of } P} \mathrm{vol}_{d-1} \, F$, normalized w.r.t. $\mathbb{Z}^{d-1} \cong \mathbb{Z}^d \cap \mathrm{aff}\, F$
- $c_0 = 1$



$$i_P(n) = 2n^2 + \frac{4}{2} n + 1$$

Bases for polynomials, II

Binomial coefficient basis: $p(n) \;=\; \sum_{i=0}^{d} \boldsymbol{a_i} \binom{n+d-i}{d}$



e.g. $P = \Delta^d$ embedded in $x_{d+1} = 1$:

$\boldsymbol{a_i}$ counts the number of lattice points at height i in the **half-open parallelopiped**

$$\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \boldsymbol{x} = \sum_{i=0}^{d+1} \lambda_i \boldsymbol{v}_i,\; 0 \le \lambda < 1\}$$

$$\begin{aligned} i_{\Delta^2} &= \mathbf{1} \cdot \binom{n+2}{2} \\ &\quad + \mathbf{2} \cdot \binom{n+1}{2} + \mathbf{1} \cdot \binom{n}{2} \\ &= 2n^2 + 2n + 1 \end{aligned}$$

Linear inequalities, I

$$p(n) = \sum_{i=0}^{d} c_i\, n^i = \sum_{i=0}^{d} a_i \binom{n+d-i}{d}$$

Theorem [Stanley 1980]

$$a_i \geq 0, \qquad i = 0, 1, \ldots, d$$

Theorem [Betke & McMullen, 1984]

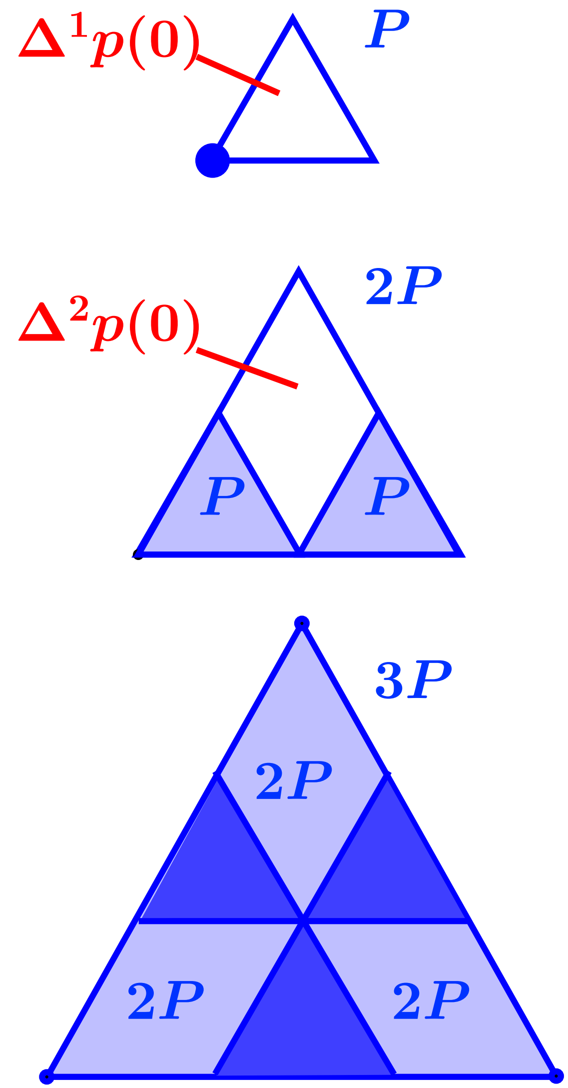
$$c_j \leq \left[{d \atop j}\right] c_d + \frac{1}{(d-1)!} \left[{d \atop j+1}\right] \qquad j = 0, 1, \ldots, d$$

Linear inequalities, II

k-th iterated difference:

$$\Delta p(n) = p(n+1) - p(n)$$

$$\Delta^k p(n) = \sum_{i=0}^{d} a_i \binom{d+n-i}{d-k} \qquad k \geq 0$$



Theorem If $a_i \geq 0$ for $0 \leq i \leq d$, then

$$\binom{d}{\ell} \Delta^k p(0) \leq \binom{d}{k} \Delta^\ell p(0), \quad \text{for } 0 \leq k < \ell \leq d\,.$$

In particular,

$$\binom{d}{k} \leq \Delta^k p(0) \leq \binom{d}{k} d!\, c_d \qquad \text{for } 0 \leq k \leq d\,.$$

Roots of Ehrhart polynomials

- P is lattice polytope $\Longrightarrow$ no $n \in \mathbb{N}$ is root of i_P
- Ehrhart reciprocity $\Longrightarrow$ if $(nP)^\circ \cap \mathbb{Z}^d = \emptyset$, then $i_P(-n) = 0$ for $n \in \mathbb{N}$

Standard simplex: $\Delta^d \; = \; \operatorname{conv}\left\{0, e_1, \ldots, e_d\right\}$

$$i_{\Delta^d}(n) \; = \; \binom{n+d}{d} \quad \Longrightarrow \quad \text{roots are} \quad -d, -d+1, \ldots, -1$$

Standard cross-polytope: $\Diamond^d \; = \; \left\{\boldsymbol{x} \in \mathbb{R}^d : \; |x_1| + \cdots + |x_d| \leq 1\right\}$

Theorem [Bump et al. 1999, Rodriguez 2000]

$$i_{\Diamond^d}(z) = 0, \quad z \in \mathbb{C} \quad \Longrightarrow \quad \operatorname{Re}(z) = \frac{1}{2}$$

Real roots of (Ehrhart) polynomials, I

Proposition. Let $a_i \geq 0$ for $i = 0, 1, \ldots, d$, and

$$p(n) = \sum_{i=0}^{d} a_i \binom{n+d-i}{d}.$$

(a) For $d \geq 1$, all real roots of p lie in the interval $[-d, d-1)$.

(b) These bounds are tight. ($-d$ is obvious.)

Proof.

- If $n > d-1$, then $\binom{n+d-i}{d} > 0$.
- If $n < -d$, then $(-1)^d \binom{n+d-i}{d} > 0$.
- (b) easy by adjusting a_i's.

Real roots of Ehrhart polynomials, II

Theorem. Let $a_i \geq 0$ for $i = 0, 1, \ldots, d$ and $\boldsymbol{c_{d-1} \geq 0}$.
Then all roots of

$$p = \sum_{i=0}^{d} a_i \binom{n+d-i}{d} = \sum_{i=0}^{d} c_i \, n^i$$

are contained in $[\boldsymbol{-d, \lfloor d/2 \rfloor})$.

Proof. Use $c_{d-1} = \frac{1}{(d-1)!} \sum_{i=0}^{d} a_i \, (d - 2i + 1)$ and the following lemma:

Lemma. *(Newton Bound)*

Let $f \in \mathbb{R}[n]$ be a polynomial of degree d and $B \in \mathbb{R}$ be such that

$$f^{(\ell)}(B) > 0 \quad \text{for} \quad \ell = 0, 1, \ldots, d \, .$$

Then all real roots of f are contained in $(-\infty, B)$. $\square$

Real roots of Ehrhart polynomials, IIa

Proof, contd. Put $B = \lfloor d/2 \rfloor$. Now $\sum_{i=0}^{d} a_i \, (d - 2i + 1) \; > \; 0$ and

$$i_P^{(\ell)}(B) \; = \; \frac{\ell!}{d!} \sum_{i=0}^{d} a_i \, g_i(B, \ell) \, .$$

Claim. For each $0 \le \ell \le d$, there exists a $\lambda(\ell) > 0$ with

$$(*) \qquad g_i(B, \ell) \; > \; \lambda(\ell) \, (d - 2i + 1) \qquad \text{for all } \; 0 \le i \le d.$$

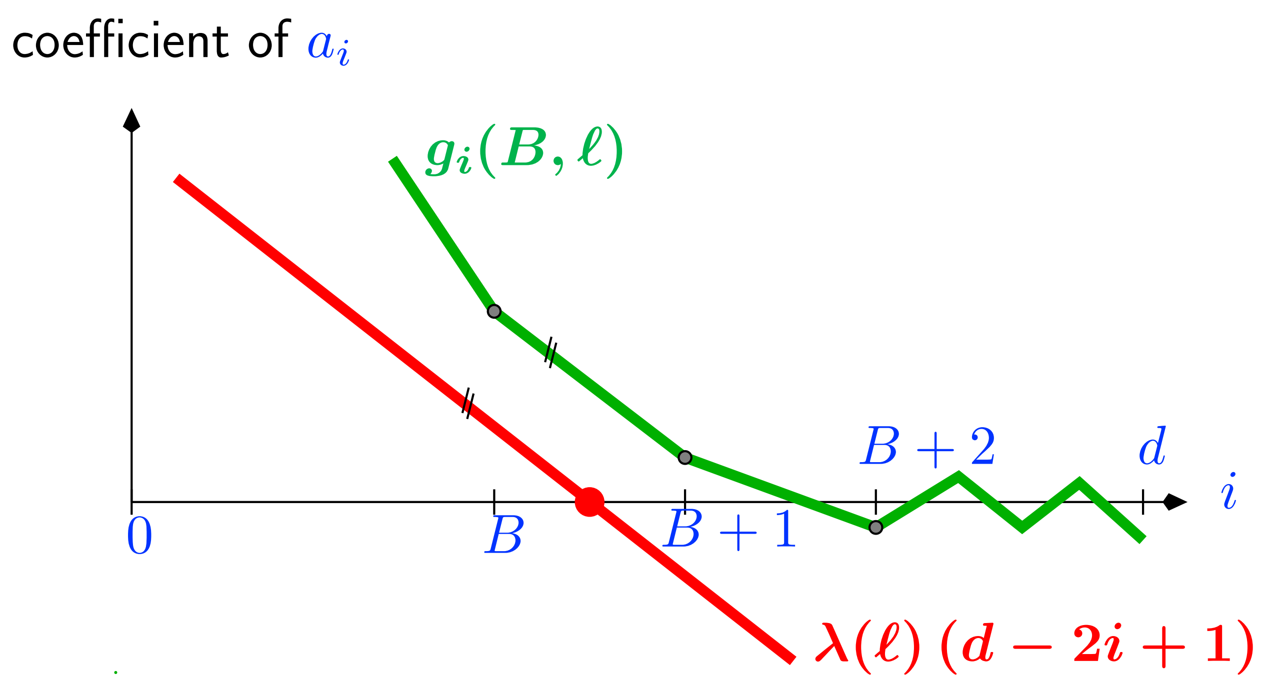
The theorem now follows from

$$\begin{aligned} 0 \quad &< \quad \sum_{i=0}^{d} \big(g_i(B, \ell) - \lambda(\ell) \, s(i)\big) a_i \\ &< \quad \sum_{i=0}^{d} \big(g_i(B, \ell) - \lambda(\ell) \, s(i)\big) a_i + \lambda(\ell) \sum_{i=0}^{d} a_i \, s(i) \; = \; \frac{d!}{\ell!} \, i_P^{(\ell)}(B) \, . \end{aligned}$$

Real roots of Ehrhart polynomials, IIb

Claim. For each $0 \le \ell \le d$, there exists a $\lambda(\ell) > 0$ with

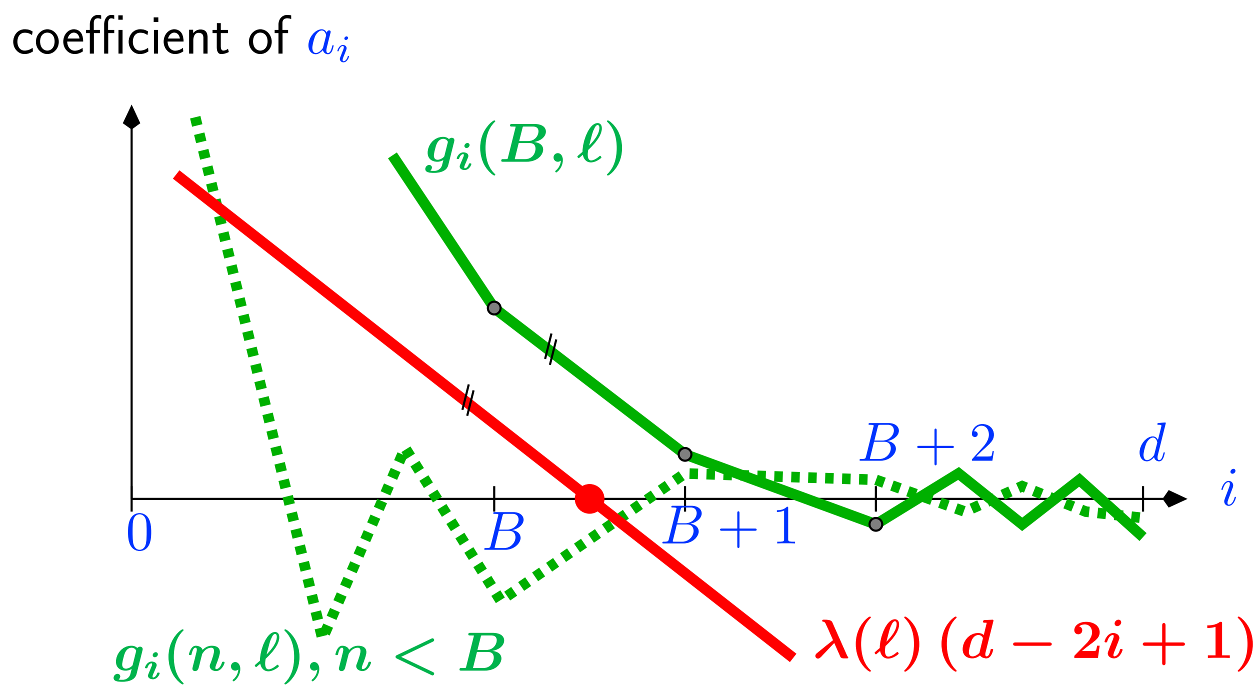
$(*)$ $\quad \boldsymbol{g_i(B,\ell) \; > \; \lambda(\ell)\,(d - 2i + 1)}$ $\quad$ for all $0 \le i \le d$.



Real roots of Ehrhart polynomials, IIb

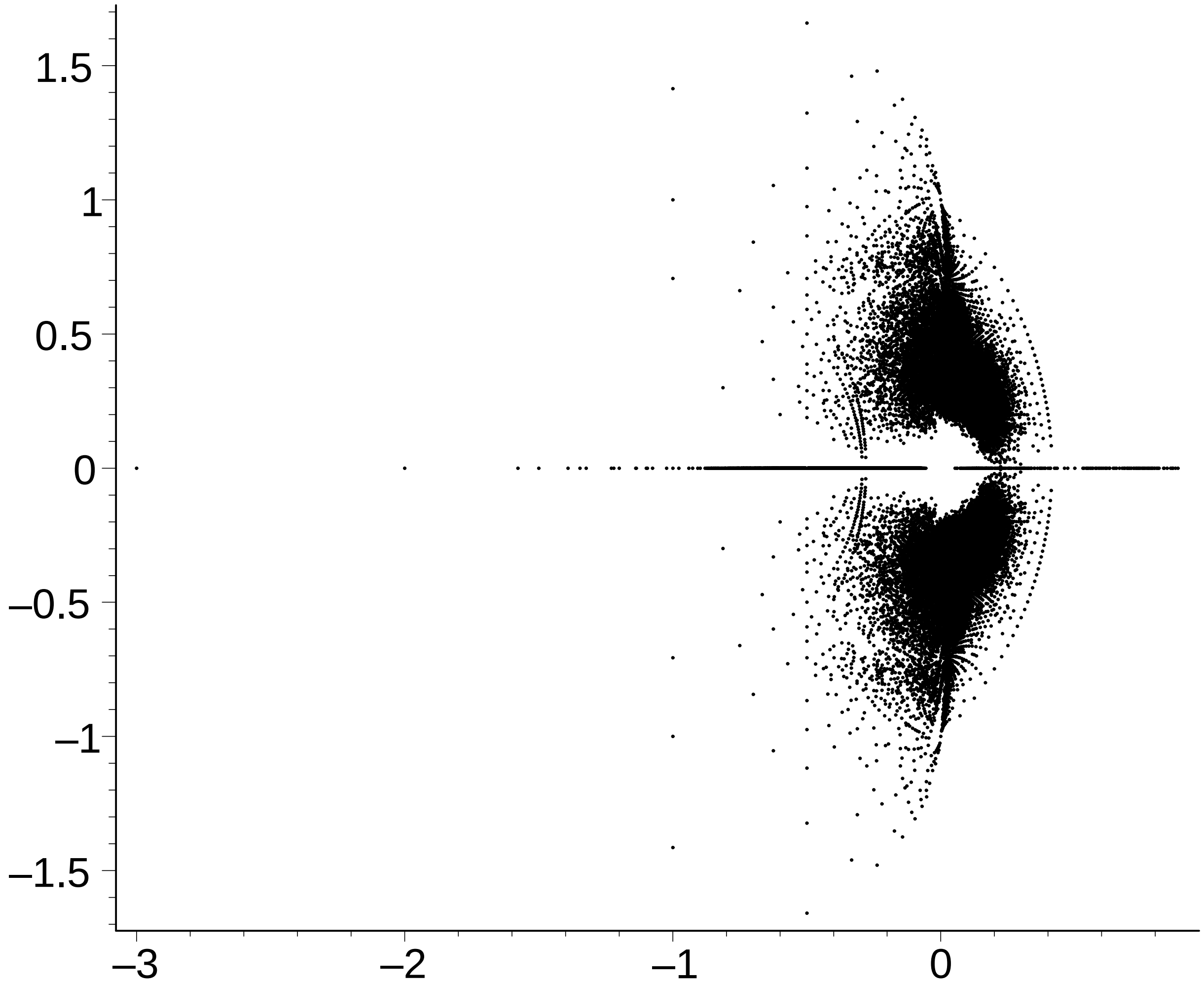
Claim. For each $0 \le \ell \le d$, there exists a $\lambda(\ell) > 0$ with

$$(*) \qquad g_i(B, \ell) > \lambda(\ell)\,(d - 2i + 1) \qquad \text{for all } 0 \le i \le d.$$

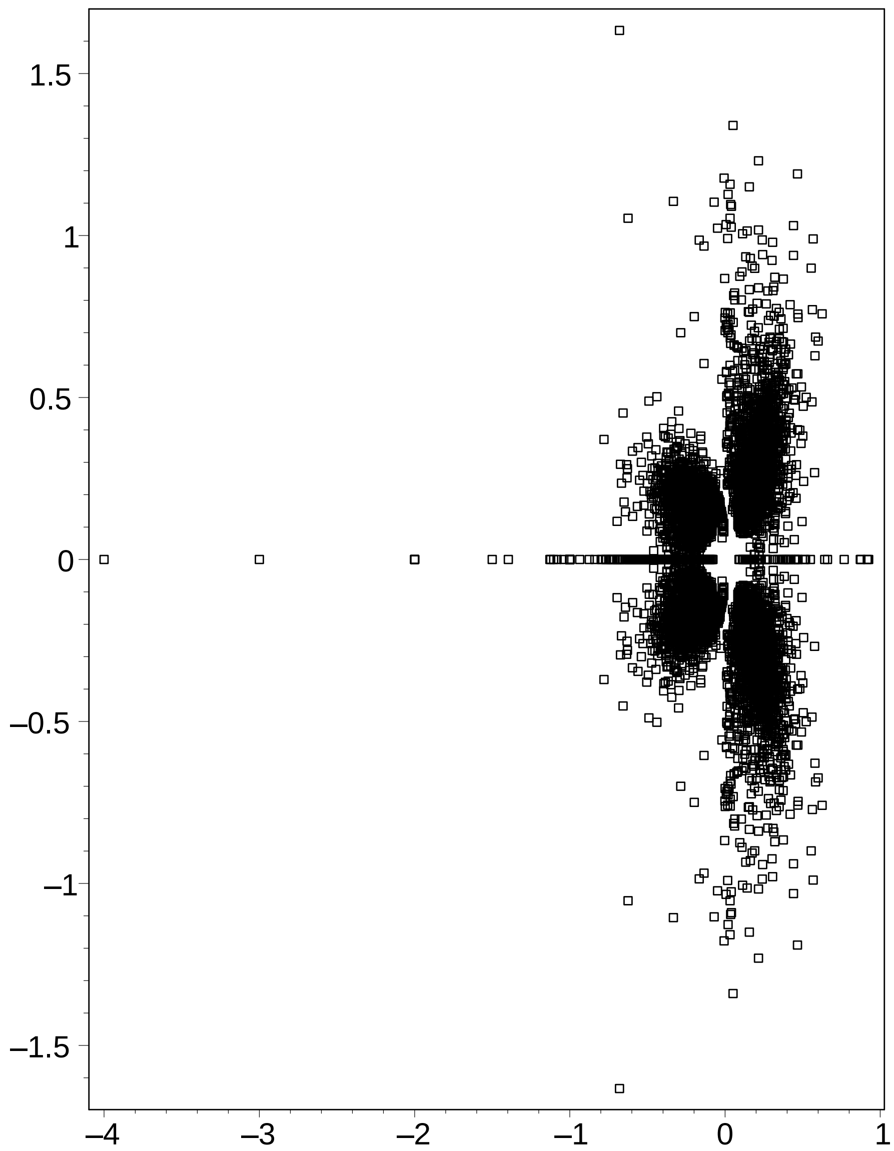


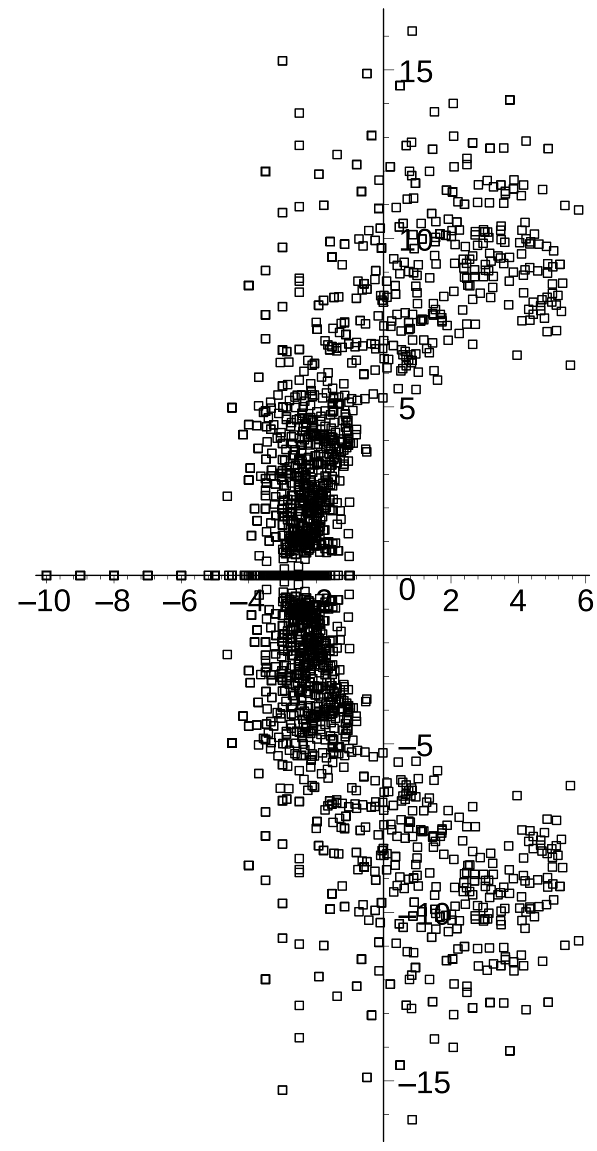
- $c_{d-1} \ge 0$ is the *only* known inequality with negative a-coefficients
- $g_i(n, \ell)$ can be negative for $n < B \implies$ cannot apply Newton Bound

Complex roots of Ehrhart polynomials



Complex roots of Ehrhart polynomials, II





Some conjectures

Conjecture 1. All real roots α of Ehrhart polynomials of lattice d-polytopes satisfy $-d \leq \alpha < 1$. (True for $d = 4$; the upper bound 1 is tight)

Conjecture 2. Set $T = \{t_1, t_2, \ldots, t_m\} \in \mathbb{Z}$, and let

$$C(T, d) \ = \ \operatorname{conv}\left\{(t_i, t_i^2, \ldots, t_i^d) : t_i \in T\right\}$$

be an integral cyclic polytope. Then

$$i_{C(T,d)}(n) \ \stackrel{?}{=} \ \sum_{k=0}^{d} \operatorname{vol}_k \left(C(T,k)\right) n^k.$$

Conjecture 3. The Ehrhart polynomial of any $0/1$-polytope has only non-negative coefficients.