THE KISSING NUMBER IN FOUR DIMENSIONS

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Abstract. The kissing number τ_n is the maximal number of equal size nonoverlapping spheres in n dimensions that can touch another sphere of the same size. The number τ_3 was the subject of a famous discussion between Isaac Newton and David Gregory in 1694. The Delsarte method gives an estimate $\tau_4 \leq 25$. In this paper we present an extension of the Delsarte method and use it to prove that $\tau_4 = 24$. We also present a new proof that $\tau_3 = 12$.

Keywords: Kissing numbers, contact numbers, spherical codes, thirteen spheres problem, Gegenbauer (ultraspherical) polynomials, Delsarte method

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1. Introduction

The *kissing number* (contact number, coordination number, ligancy, or Newton number) τ_n is the highest number of equal nonoverlapping spheres in $\mathbf{R}^n$ that can touch another sphere of the same size. Here the verb *kiss* refers to the game of billiards, where it signifies two balls that just touch each other. In three dimensions the kissing number problem is asking how many white billiard balls can kiss a black ball.

It is easy to see $\tau_1 = 2$ and $\tau_2 = 6$. The kissing number in three dimensions was the subject of a famous discussion between Isaac Newton and David Gregory in 1694 (May 4, 1694, see details in [G. Szpiro, Newton and the kissing problem, http://plus.maths.org/issue23/features/kissing/). Newton believed the answer was 12, while Gregory thought that 13 might be possible. The correct answer is $\tau_3 = 12$.

If the 12 spheres are placed at positions corresponding to the vertices of a regular icosahedron concentric with the central sphere, these 12 spheres do not touch each other and may all be moved freely.

This problem is often called the *thirteen spheres problem*. Several German papers in 1874/75 described approaches to the problem, but "certain ideas emerged in... (these papers) ...only to be ignored ... so that they waited until 1950 to be rediscovered and expanded in the joint works of W. Habicht, K. Schütte and B.L. van der Waerden " [Danzer]. Schütte and van der Waerden gave a detailed proof in 1953. A subsequent proof by Leech [**?**] in 1956 "... although elementary and straightforward, it cannot be called trivial"[Conway - Sloane].

Coxeter proposed upper bounds on τ_n in 1963; for $n = 4, 5, 6, 7$, and 8 these bounds were 26, 48, 85, 146, and 244, respectively. Coxeter's bounds are based on the conjecture that equal size spherical caps on a sphere $\mathbf{S}^k$ can be packed no denser than $k+1$ spherical caps on $\mathbf{S}^k$ that simultaneously touch one another. Böröczky proved this conjecture in 1978.

The main progress in the kissing number problem was in the end of 1970's. Levenshtein, Odlyzko and Sloane independently proved that $\tau_8 = 240,\ \tau_{24} = 196\,560$ in 1979. In Odlyzko - Sloane's paper the *Delsarte method* was applied in dimensions up to 24. For comparison with the values of Coxeter's bounds on τ_n for $n = 4, 5, 6, 7$, and 8 this method gives 25, 46, 82, 140, and 240, respectively. (For $n = 3$ Coxeter's and Delsarte's methods only gave $\tau_3 \leq 13$.) Kabatiansky and Levenshtein have found an asymptotic upper bound $2^{0.401n(1+o(1))}$ for τ_n in 1978. The lower bound $2^{0.2075n(1+o(1))}$ was found by Wyner in 1965.

Note that $\tau_4 \geq 24$. Indeed, the unit sphere in $\mathbf{R}^4$ centered at $(0,0,0,0)$ has 24 unit spheres around it, centered at the points $(\pm\sqrt{2}, \pm\sqrt{2}, 0, 0)$, with any choice of signs and any ordering of the coordinates.

Arestov and Babenko in 1997 proved that the bound $\tau_4 \leq 25$ cannot be improved using Delsarte's method.

The kissing number problem can be stated in other way: How many points can be placed on the surface of an unit sphere $\mathbf{S}^{n-1}$ in Euclidean space $\mathbf{R}^n$ so that the angular separation between any two points is at least 60°? This leads to an important generalization: a finite subset X of $\mathbf{S}^{n-1}$ is called a *spherical z-code* if for every pair (x, y) of X the scalar product $x \cdot y \leq z$. Spherical codes have many applications. The main application outside mathematics is in the design of signals for data transmission and storage. There are interesting applications to the numerical evaluation of n-dimensional integrals.

For any $X \subset \mathbf{S}^{k-1}$ denote by $\varphi(X)$ the minimum of the angular separation between any two points of X: $\varphi(X) = \min\{\phi_{ij}, i \neq j\}$. Let

$$\varphi_k(M) = \max\{\varphi(X), \quad |X| = M, \quad X \subset \mathbf{S}^{k-1}\}.$$

It is clear that $\varphi_2(M) = 360^\circ/M$. In three dimensions $\varphi_3(M)$ is the largest angular separation that can be attained in a spherical code on $\mathbf{S}^2$ containing M points. This is sometimes called *Tammes' problem*, after the Dutch botanist who was led to this question by studying the distribution of pores on pollen grains. Equivalently we can ask: where should M inimical dictators build their palaces on a planet so as to be as far away from each other as possible?

The best codes and the values $\varphi_3(M)$ presently known for
$M \leq 12$ and $M = 24$:
M=3,4,6,12 - L. Fejes-Tóth, 1943;
M=5,7,8,9 - K. Schütte and B.L. van der Waerden, 1951;
M=10,11 - L. Danzer, 1963;
M=24 - R.M. Robinson, 1961.

For instance, $\varphi_3(5) = \varphi_3(6) = 90^\circ, \quad \varphi_3(7) = 77.86954...^\circ$
$(\cos\varphi_3(7) = \cot 40^\circ \cot 80^\circ)$.

2. Gegenbauer polynomials and Schoenberg's theorem.

Let $X = \{x_1, x_2, \ldots, x_M\}$ be any finite subset of the unit sphere $\mathbf{S}^{n-1}$.

By ϕ_{ij} we denote the spherical (angular) distance between x_i, x_j. It is clear that for any real numbers $u_1, u_2, \ldots, u_M$ the relation

$$|| \sum u_i x_i ||^2 = \sum_{i,j} \cos\phi_{ij} u_i u_j \geq 0$$

holds, or equivalently the Gram matrix $T(X)$ is positive definite, where $T(X) = (t_{ij})$, $t_{ij} = \cos\phi_{ij} = x_i \cdot x_j$.

Example. Let $X \subset S^1 \subset \mathbf{C}$, $x_k = \exp(i\phi_k)$. Suppose

$$X^{(m)} = \{x_1^m, \ldots, x_M^m\} = \{\exp(i\, m\, \phi_1), \ldots, \exp(i\, m\, \phi_M)\},$$

$$T(X^{(m)}) = (\cos(m\, \phi_{ij})), \quad \phi_{ij} = \phi_i - \phi_j.$$

Therefore, the matrix $(f(t_{ij}))$ is positive definite, where

$$f(t) = \cos m\phi, \quad t = \cos\phi, \quad m = 1, 2, 3, ...$$

Schoenberg extended this property for all dimensions n. He considered functions $f(\cos\phi)$ that give positive definite matrix $(f(t_{ij}))$ for arbitrary subset X of $\mathbf{S}^{n-1}$. Denote by $G_k^{(n)}(t)$ Gegenbauer (ultraspherical) polynomials.

Theorem (Schoenberg, 1942) *If $g_{ij} = G_k^{(n)}(t_{ij})$, then the matrix (g_{ij}) is positive definite.*

The converse holds also: if $f(t)$ is a real polynomial and for any finite $X \subset \mathbf{S}^{n-1}$ the matrix $(f(t_{ij}))$ is positive definite, then f is a sum of $G_k^{(n)}$ with nonnegative coefficients.

Let us recall the definition of Gegenbauer polynomials. Suppose $C_k^{(n)}(t)$ be the polynomials defined by the expansion

$$(1 - 2rt + r^2)^{1-n/2} = \sum_{k=0}^{\infty} r^k C_k^{(n)}(t).$$

Then the polynomials $G_k^{(n)}(t) = C_k^{(n)}(t)/C_k^{(n)}(1)$ are called *Gegenbauer* or *ultraspherical* polynomials. (So the normalization of $G_k^{(n)}$ is determined by the condition $G_k^{(n)}(1) = 1$.)

The Gegenbauer polynomials $G_k^{(n)}$ can be defined another way:

$$G_0^{(n)} = 1, \;\; G_1^{(n)} = t, \;\; \ldots, \;\; G_k^{(n)} = \frac{(2k+n-4)\, t\, G_{k-1}^{(n)} - (k-1)\, G_{k-2}^{(n)}}{k+n-3}$$

They are orthogonal on the interval $[-1, 1]$ with respect to the weight function $\rho(t) = (1-t^2)^{(n-3)/2}$. In the case $n = 3$, $G_k^{(n)}$ are Legendre polynomials P_k, and $G_k^{(4)}$ are Chebyshev polynomials of the second kind (with normalization $U_k(1) = 1$),

$$G_k^{(4)}(t) = U_k(t) = \frac{\sin((k+1)\phi)}{(k+1)\sin\phi}, \quad t = \cos\phi, \quad k = 0, 1, 2, \ldots$$

For instance, $\quad U_0 = 1, \quad U_1 = t, \quad U_2 = (4t^2 - 1)/3,$

$$U_3 = 2t^3 - t, \quad U_4 = (16t^4 - 12t^2 + 1)/5,$$

$$U_9 = (256t^9 - 512t^7 + 336t^5 - 80t^3 + 5t)/5.$$

3. Delsarte's method

The Delsarte method (also known in coding theory as Delsarte's linear programming method, Delsarte's scheme, polynomial method) is described in [Conway-Sloane, Levenshtein etc]. Let $f(t)$ be a real polynomial such that $f(t) \leq 0$ for $t \in [-1, z]$, the coefficients c_k's in the expansion of $f(t)$ in terms of Gegenbauer polynomials $G_k^{(n)}$ are nonnegative, and $c_0 = 1$. Then the maximal number of points in a spherical z-code in $\mathbf{S}^{n-1}$ is bounded by $f(1)$. Suitable coefficients c_k's can be found by the linear programming method.

Let us now prove the bound of Delsarte's method. If a matrix (g_{ij}) is positive definite, then for any real u_i the inequality $\sum g_{ij} u_i u_j \geq 0$ holds, and then for $u_i = 1$, we have $\sum\limits_{i,j} g_{ij} \geq 0$. Therefore, for $g_{ij} = G_k^{(n)}(t_{ij})$, we obtain

$$\sum_{i=1}^{M} \sum_{j=1}^{M} G_k^{(n)}(t_{ij}) \geq 0 \tag{3.1}$$

Suppose

$$f(t) = c_0 G_0^{(n)}(t) + \ldots + c_d G_d^{(n)}(t), \quad \text{where } \ c_0 \geq 0, \ldots, \, c_d \geq 0. \tag{3.2}$$

Let $F(X) = \sum\limits_i \sum\limits_j f(t_{ij})$. Using (3.1), we get

$$F(X) = \sum_{k=0}^{d} \sum_{i=1}^{M} \sum_{j=1}^{M} c_k G_k^{(n)}(t_{ij}) \geq \sum_{i=1}^{M} \sum_{j=1}^{M} c_0 G_0^{(n)}(t_{ij}) = c_0 M^2. \tag{3.3}$$

Theorem (Delsarte, 1972) *Let $X = \{x_0, \ldots, x_M\} \subset \mathbf{S}^{n-1}$ be a spherical z-code. Suppose $f(t)$ satisfies* (3.2) *and $f(t) \leq 0$ for $t \in [-1, z]$. If $c_0 > 0$, then*

$$M \leq \frac{f(1)}{c_0}$$

Proof. Note that $t_{ii} = 1$. Then

$$F(X) = Mf(1) + 2f(t_{12}) + \ldots + 2f(t_{M-1\,M}) \leq Mf(1).$$

If we combine this with (3.2), then for $c_0 > 0$ we get $M \leq f(1)/c_0$. $\square$

This inequality play a crucial role in the Delsarte method. If $z = 1/2$ and $c_0 = 1$, then it implies $\tau_n \leq f(1)$. Levenshtein, Odlyzko and Sloane have found the polynomials $f(t)$ such that $f(1) = 240$, when $n = 8$; and $f(1) = 196\,560$, when $n = 24$. Thus $\tau_8 \leq 240, \tau_{24} \leq 196\,560$. //When $n = 8, 24$, there exist sphere packings (E_8 and Leech lattices) with these kissing numbers. Thus $\tau_8 = 240$ and $\tau_{24} = 196\,560$.

When $n = 4$, a polynomial f of degree 9 with $f(1) = 25.5585...$ was found by Odlyzko and Sloane. This implies $24 \leq \tau_4 \leq 25$.

Let us prove that $\tau_8 \leq 240$.

$$f(t) = \frac{320}{3}(t+1)\,(t+1/2)^2\,t^2\,(t-1/2)$$

$$= G_0^{(8)} + \frac{16}{7}G_1^{(8)} + \frac{200}{63}G_2^{(8)} + \frac{832}{231}G_3^{(8)} + \frac{1216}{429}G_4^{(8)} + \frac{5120}{3003}G_5^{(8)} + \frac{2560}{4641}G_6^{(8)}.$$

We have $c_0 = 1,\ f(1) = 240$; then $M \leq 240$.

4. An extension of Delsarte's method.

If $A=[-1,z]\cup\{1\}$, then $t_{ij}\in A$ for all i,j.
Let $A_+=\{t:\ t\in A \text{ and } f(t)>0\}$ and

$$F_i(X)=\sum_{j:t_{ij}\in A_+} f(t_{ij}),$$

then

$$F(X)\le\sum_{i=1}^{M} F_i(X). \tag{4.1}$$

Definition. Suppose m and $Y=\{y_0,y_1,\ldots,y_m\}\subset \mathbf{S}^{n-1}$ satisfy

$$y_i\cdot y_j\le z \ \text{ for all } \ i\ne j, \quad f(y_0\cdot y_i)\ge 0 \ \text{ for } \ 1\le i\le m. \tag{4.2}$$

Denote by μ the highest value of m such that the constraints in (4.2) define a non-empty set of solutions $(y_0,\ldots,y_m)$.

Suppose $0\le m\le\mu$. Let

$$h(Y)=h(y_0,y_1,\ldots,y_m):=f(1)+f(y_0\cdot y_1)+\ldots+f(y_0\cdot y_m),$$

$$h_m:=\max_Y h(Y), \quad h_{max}:=\max\{h_0,h_1,\ldots,h_\mu\}.$$

It is clear that $F_i(X)\le h_{max}$. Since (4.1), we have $F(X)\le M\,h_{max}$. Combining this with (3.3), we obtain

Proposition. Suppose $X\subset\mathbf{S}^{n-1}$ is a spherical z-code, $|X|=M$, and f satisfies (3.2). If $c_0=1$, then $M\le h_{max}$.

Note that $h_0=f(1)=\sum c_k>0$, i.e. $\{1\}\in A_+$. In the Delsarte method $A_+=\{1\}$, $\mu=0$, $h_{max}=h_0=f(1)$.

The problem of evaluating of h_{max} in general case looks even more complicated than the upper bound problem for spherical z-codes. Here we consider this problem only for a very restrictive class of functions $f(t)$:

$$f(t) \text{ is a monotone decreasing function on the interval } [-1, t_0],$$
$$f(t) \leq 0 \text{ for } t \in [t_0, z], \quad t_0 < -z \leq 0 \tag{4.3}$$

.

Denote by ϕ_k for $k > 0$ the distance between y_k and y_0^*, where $y_0^* = -y_0$ is the antipodal point to y_0. Then $y_0 \cdot y_k = -\cos\phi_k$, and $h(Y)$ is represented in the form:

$$h(Y) = f(1) + f(-\cos\phi_1) + \ldots + f(-\cos\phi_m). \tag{4.4}$$

A subset C of $\mathbf{S}^{n-1}$ is called (spherical) *convex* if it contains, with every two nonantipodal points, the small arc of the great circle containing them. If, in addition, C does not contain antipodal points, then C is called strongly convex. The closure of a convex set is convex and is the intersection of closed hemispheres. If a subset Z of $\mathbf{S}^{n-1}$ lies in a hemisphere, then the convex hull of Z is well defined, and is the intersection of all convex sets containing Z.

Suppose $f(t)$ satisfies (4.3), then from (4.2) it follows that $Q_m = \{y_1, \ldots, y_m\}$ lies in the hemisphere of center y_0^*. Denote by Δ_m the convex hull of Q_m in $\mathbf{S}^{n-1}$, $\Delta_m = \operatorname{conv} Q_m$.

Lemma 1. *Suppose f satisfies (4.3) and $Y = \{y_0, \ldots, y_m\} \subset \mathbf{S}^{n-1}$ is optimal, i.e. $h(Y) = h_m$ and Y has the maximal number of $\phi_{ij} = \psi$ $(y_i \cdot y_j = z)$. Then*
(i) $y_0^* \in \Delta_m$ and any $y_k \in Q_m$ is a vertex of Δ_m, i.e. $\Delta_m^0 = Q_m$;
(ii) if $m \leq n$, then Δ_m is a regular spherical simplex with edge length ψ;
(iii) if $m > n$, then for any $y_k \in Q_m$ there are at least $n-1$ distinct points in Q_m at the distance of ψ from y_k.

Proof. Let $\phi_0 = \cos^{-1}(-t_0)$, then from the assumptions follow $\phi_k \le \phi_0 < \psi$. The function $f(t)$ is monotone decreasing on $[-1, t_0]$. By (4.4) it follows that the function $h(Y)$ increases whenever ϕ_k decreases. This means that for an optimal Y no $y_k \in Q_m$ can be shifted towards y_0^*.
(i) If $y_0^* \notin \Delta_m$, then whole Q_m can be shifted to y_0^*. If $\phi_k = 0$, then $y_k = y_0^*$ and $m = 1$ because in the converse case $\phi_{kj} = \phi_j < \psi$. If $\phi_k > 0$, then consider the great $(n-2)$-sphere S_k such that $y_k \in S_k$, and S_k is orthogonal to the arc $y_0^* y_k$. Suppose H_0 is the hemisphere in $\mathbf{S}^{n-1}$ such that its boundary is S_k and H_0 contains y_0^*. Let us prove that Q_m belongs to H_0. Note that this implies (i).

Consider the triangle $y_0^* y_k y_j$ and denote by γ_{kj} the angle $\angle y_0^* y_k y_j$ in this triangle. From the law of cosines for spherical triangles follows

$$\cos\phi_j = \cos\phi_k \cos\phi_{kj} + \sin\phi_k \sin\phi_{kj} \cos\gamma_{kj}$$

If y_j does not belong to H_0, then $\gamma_{kj} > 90^0$, and $\cos\gamma_{kj} < 0$ (Fig. 1). Therefore,
$\cos\phi_j < \cos\phi_k \cos\phi_{kj} < \cos\phi_{kj} \le \cos\psi$, then $\phi_j > \psi -$ a contradiction.

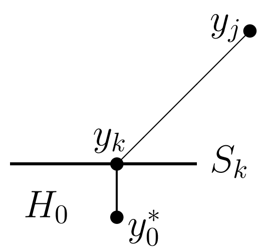

Fig. 1

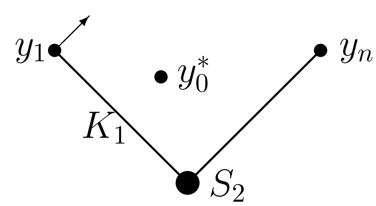

Fig. 2

$(ii), (iii)$. Note that if $m \le n$, then $Q_m \subset \mathbf{S}^{m-1}$ and (i) implies $Y \subset \mathbf{S}^{m-1}$. Hence, if $\ell = \min\{m, n\}$, then $Y \subset \mathbf{S}^{\ell-1}$

Let d_k = number of points in Q_m at the distance of ψ from y_k. Suppose $d_1 < l - 1$ and let after suitable permutations of labels we have

$$\phi_{12} = \ldots = \phi_{1d_1} = \phi_{1d_1+1} = \psi, \quad \phi_{1i} > \psi \ \text{ for } \ i = d_1 + 2, \ldots, m.$$

Consider $K_1 = \operatorname{conv}\{y_1, y_2, \ldots, y_{\ell-1}\}$ and $K_2 = \operatorname{conv}\{y_2, \ldots, y_{\ell-1}\}$. Denote by $S_i, i = 1, 2$, the great $(\ell - i - 1)$-sphere in $\mathbf{S}^{\ell-1}$ such that S_i contains K_i. Now we prove that $y_0^* \in S_2$.

Assume the converse. If $y_0^* \notin K_1$, then y_1 can be shifted towards y_0^* (rotation of y_1 about S_2 by a small angle, see Fig. 2) decreasing ϕ_1- a contradiction. Thus, $y_0^* \in K_1$. If K_1 lies in the boundary $\partial\Delta_m$ of Δ_m, then from convexity of Δ_m follows that K_1 (with y_0^*) can be rotated about S_2 by a small angle towards other y_k- a contradiction. When K_1 does not belong to $\partial\Delta_m$ (i.e. some of the edges of K_1 are internal in Δ_m), then S_1 separates $y_k,\ k \geq \ell$, into two subsets in accordance to which of the hemispheres (bounded by S_1) they belong. Take one of them and shift it towards K_1. This shift decreases ϕ_k- a contradiction.

We have $y_0^* \in S_2$. In fact, we proved that $y_0^* \in \widetilde{S}$, where $\widetilde{S}$ is the great $(d_1 - 1)$-sphere that contains $\{y_2, \ldots, y_{d_1+1}\}$. Moreover, when $k \geq d_1 + 2$, then $d_k = \ell - 1$ or $\phi_{ik} = \psi$ for $2 \leq i \leq d_1 + 1$. Thus any rotation about $\widetilde{S}$ does not change ϕ_k and ϕ_{1i} for $i = 2, \ldots, d_1 + 1$. Let us show that y_1 can be rotated so as to bring one of $\phi_{1k} = \psi,\ k \geq d_1 + 2$, that increases d_1 and contradicts to optimality of Y.

(ii) Since $\ell = m$ it follows that in any case $\phi_{ik} = \psi$ for $2 \leq i \leq d_1 + 1 < k$. It is clear that y_1 can be rotated about $\widetilde{S}$ so as to bring $\phi_{1k} = \psi$ for $k \geq d_1 + 2$ (see Fig. 3).

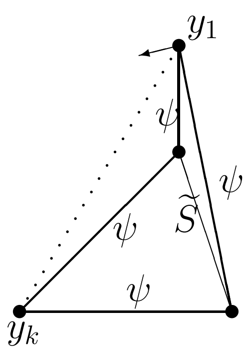

Fig. 3

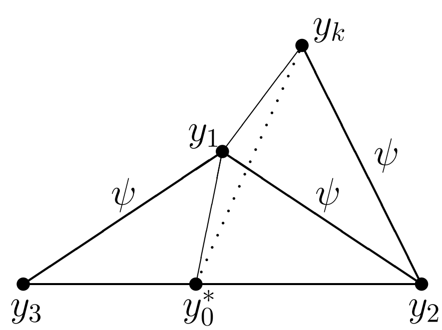

Fig. 4

(iii) For simplicity we consider here only the case $n \leq 4$.

When $n = 3$, S_2 consist of the one point y_2, i.e. $y_0^* = y_2$. Then $\phi_k \geq \psi > \phi_0 -$ a contradiction. When $n = 4$, K_2 is the spherical segment $y_2 y_3$ and $y_0^* \in [y_2, y_3]$. Since the sum of the angles $y_1 y_0^* y_2$ and $y_1 y_0^* y_3$ equals $180°$, then one of them (suppose the first one) is not exceed $90°$. Note that for y_2 there is y_k, $k \neq 1, 3$, at the distance of ψ from y_2. Then y_1 can be rotated about S_2 towards y_k so as to bring $\phi_{1k} = \psi$. Indeed, consider an arrangement of $\{y_1, y_2, y_3, y_k\}$ in $\mathbf{S}^3$ such that it gives the minimal distance between y_1 and y_k. Then y_1 lies in the great 2-sphere defined by $\{y_2, y_3, y_k\}$ (see Fig. 4). It is easy to see that $\operatorname{dist}(y_1, y_k) < \operatorname{dist}(y_0^*, y_k)$. Since $\operatorname{dist}(y_0^*, y_k) = \phi_k \leq \phi_0 < \psi$, we obtain $\operatorname{dist}(y_1, y_k) < \psi$. $\square$

Suppose f satisfies (4.3). Then the function

$$h(y_0, y_1) = f(1) + f(-y_0 \cdot y_1)$$

attains its maximum at $y_1 = y_0^*$. Therefore,

$$h_1 = f(1) + f(-1).$$

Denote by

$$\Lambda_m = \{y : y \in \Delta_m, \quad y \cdot y_k \geq -t_0, \ 1 \leq k \leq m\}.$$

Note that Λ_m is a convex set in $\mathbf{S}^{n-1}$. Let

$$H_m(y) = f(1) + f(-y \cdot y_1) + \ldots + f(-y \cdot y_m).$$

Then h_m is the maximum of $H_m(y)$ on Λ_m. Now we have

$$h_0 = f(1), \quad h_1 = f(1) + f(-1),$$

$$h_m = \max_{y \in \Lambda_m} H_m(y), \quad \Lambda_m \subset \Delta_m \subset \mathbf{S}^{n-1}, \quad 2 \leq m \leq \mu. \tag{4.5}$$

Define the function $\rho(z,\phi_0)$ in z,ϕ_0 by the equation:

$$\cos\rho(z,\phi_0) = \frac{z-\cos^2\phi_0}{\sin^2\phi_0}.$$

Lemma 2. *Suppose* $Y=\{y_0,y_1,\ldots,y_m\}\subset \mathbf{S}^{n-1}$, *where* $\phi_{ij}\ge\psi$ *for* $i\neq j$; $\phi_i=\cos^{-1}(y_0^*\cdot y_i)\le\phi_0$ *for* $1\le i\le m$; *and* $\phi_0\le\psi$, $\cos\psi=z\ge 0$. *If* $\rho(z,\phi_0)>\varphi_{n-1}(M)$, *then* $m<M$.

Proof. Let $q(\alpha)=(z-\cos\alpha\cos\beta)/\sin\alpha$, then $q'(\alpha)=(\cos\beta-z\cos\alpha)/\sin^2\alpha$. From this follows, if $0<\alpha,\beta\le\phi_0$, then $\cos\beta\ge z$; so then $q'(\alpha)\ge 0$, and $q(\alpha)\le q(\phi_0)$ $(*)$.

Let Π be the projection of $\{y_1,\ldots,y_m\}$ onto equator $\mathbf{S}^{n-2}$ from pole y_0^*. Then the distances γ_{ij} between points of Π in $\mathbf{S}^{n-2}$ can not be less than $\rho(z,\phi_0)$. Indeed, combining $(*)$ and the inequality $\cos\phi_{ij}\le z$, we get

$$\cos\gamma_{ij}=\frac{\cos\phi_{ij}-\cos\phi_i\cos\phi_j}{\sin\phi_i\sin\phi_j}\le\frac{z-\cos^2\phi_0}{\sin^2\phi_0}=\cos\rho(z,\phi_0).$$

Thus $\gamma_{ij}\ge\rho(z,\phi_0)$. From other side, $\Pi\subset\mathbf{S}^{n-2}$, then $\min\limits_{i\neq j}\gamma_{ij}\le\varphi_{n-1}(m)$, so then $\rho(z,\phi_0)\le\varphi_{n-1}(m)$. □

Corollary 1. *Suppose* $f(t)$ *satisfies* (4.3). *If* $n=4$, $z=1/2$, *and* $t_0\le -0.6058$, *then* $\mu\le 6$.

Proof. Since $\cos\phi_0=-t_0\ge 0.6058$, then $\rho(1/2,\phi_0)\ge 77.8707...^\circ>\varphi_3(7)$. Lemma 2 implies $m<7$, i.e. $\mu=\max\{m\}\le 6$. □

5. $\tau_4 = 24$

For $n = 4$, $z = \cos 60^\circ = 1/2$ we apply this extension of Delsarte's method with

$$f(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.384t^4 - 9.832t^3 - 4.128t^2 - 0.434t - 0.016$$

The expansion of f in terms of $U_k = G_k^{(4)}$ is

$$f = U_0 + 2U_1 + 6.12U_2 + 3.484U_3 + 5.12U_4 + 1.05U_9$$

This polynomial f has two roots: $t_0 = -0.60794...$ and $t = 1/2$ on $[-1, 1]$, $f(t) \leq 0$ for $t \in [t_0, 1/2]$, and f is a monotone decreasing function on the interval $[-1, t_0]$. The last property holds because there are no zeros of the derivative $f'(t)$ on $[-1, t_0]$. Therefore, f satisfies (4.3) for $z = 1/2$.

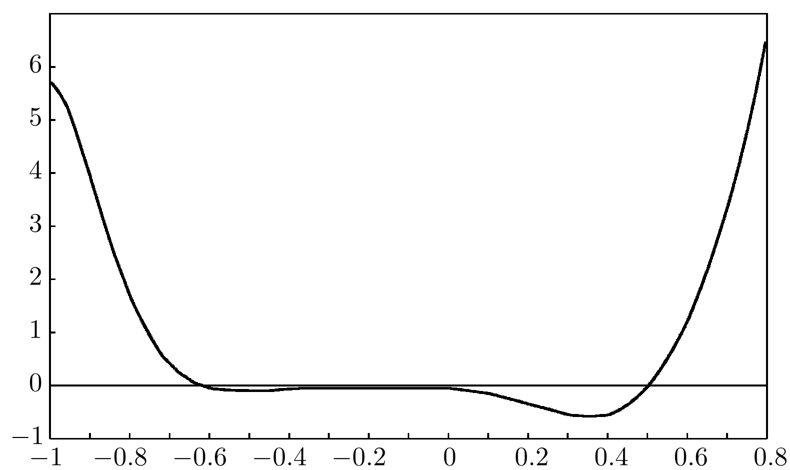

Fig. 5. The graph of the function $f(t)$

Remark. The polynomial f was found by using the algorithm in Appendix I. This algorithm for $n = 4$, $z = 1/2$, $d = 9$, $N = 2000$, $t_0 = -0.6058$ gives $E = 24.7895$. For the polynomial f the coefficients c_k were changed to "better looking" ones with $E = 24.8644$.
(*Here and below numbers are shown to* 4 *decimal places.*)

We have $t_0 < -0.6058$. Then Corollary 1 gives $\mu \leq 6$. Consider all $m \leq 6$.

$$h_0 = f(1) = 18.774, \quad h_1 = f(1) + f(-1) = 24.48.$$

When $m = 2$, Lemma 1 implies that Δ_2 is an arc (spherical segment) y_1y_2 with length $\psi = 60°$. We obviously have $\phi_1 + \phi_2 = 60°$. Then (4.4) implies

$$h(Y) = \widetilde{H}(\phi_1) := f(1) + f(-\cos\phi_1) + f(-\cos(60° - \phi_1)).$$

From (4.5) it follows that h_2 is the maximum of $\widetilde{H}(\phi_1)$ on the interval $\Lambda_2 = [\psi_0, \phi_0]$, where

$$\phi_0 = \cos^{-1}(-t_0) = 52.5588°, \quad \psi_0 = 60° - \phi_0.$$

The graph of the function $\widetilde{H}(\phi_1)$ (see Fig. 6) shows that this function achieves its maximum at $\phi_1 = 30°$. It can be proven by the following method.

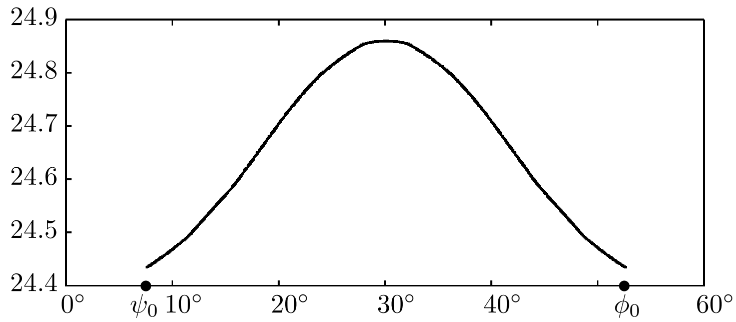

Fig. 6. The graph of the function $\widetilde{H}(\phi_1)$

Let $\alpha = \phi_1 - 30°$, $s = \cos\alpha$, and $\Phi(s) := \widetilde{H}(\phi_1)$. It is easy to see that $\Phi(s)$ is a polynomial of degree 9 in the variable s. The inequality $\phi_1 \leq \phi_0$ implies $s \geq s_0 = \cos(\phi_0 - 30°)$. Therefore, h_2 is the maximum of $\Phi(s)$ on $[s_0, 1]$.

The calculations show that there are no critical points of the function $\Phi(s)$ on $(s_0, 1)$. In other words, there are no roots of the polynomial $\Phi'(s)$ on $(s_0, 1)$, then $\Phi(s)$ achieves its maximum at $s = s_0$ or at $s = 1$. Since $\Phi(1) > \Phi(s_0)$, then

$$h_2 = \Phi(1) = \widetilde{H}(30°) = f(1) + 2f(-\cos 30°) = 24.8644.$$

The cases $m = 3, 4, 5, 6$ are considered below. Corollaries 2, 3 and Lemmas 6, 7 give that

$$h_2 > h_3 = 24.8345 > h_4 = 24.8180 > h_5 = 24.6856, \quad h_6 < h_2.$$

Thus $h_{max} = h_2$.

Theorem 1. $\tau_4 = 24$

Proof. Let X be a spherical 1/2-code in $\mathbf{S}^3$ with $M = \tau_4$ points. The polynomial f is such that $h_{max} < 25$, then combining this and Proposition, we get
$\tau_4 \leq h_{max} < 25$. Recall that $\tau_4 \geq 24$. Consequently, $\tau_4 = 24$. □

The thirteen spheres problem: a new proof

Theorem 2. $\tau_3 = 12$

Proof. Let

$$f(t) = \frac{2431}{80}t^9 - \frac{1287}{20}t^7 + \frac{18333}{400}t^5 + \frac{343}{40}t^4 - \frac{83}{10}t^3 - \frac{213}{100}t^2 + \frac{t}{10} - \frac{1}{200}.$$

The expansion of f in terms of $P_k = G_k^{(3)}$ is

$$f = P_0 + 1.6P_1 + 3.48P_2 + 1.65P_3 + 1.96P_4 + 0.1P_5 + 0.32P_9.$$

The function $f(t)$ is a monotone decreasing function on the interval $[-1, t_0]$, $f(t_0) = 0$, $\quad f(t) < 0 \quad$ for $\quad t_0 < t \leq 1/2$, $\quad$ and

$$t_0 = -0.5907, \quad \phi_0 = \cos^{-1}(-t_0) = 53.7940°.$$

Since $\rho(1/2, \phi_0) = 76.5821°$ and $\varphi_2(5) = 72°$, we have $m < 5$ (Lemma 2).

$$h_0 = f(1) = 10.11, \quad h_1 = f(1) + f(-1) = 12.88.$$

$\Phi(s)$ achieves its maximum on the interval $[s_0, 1]$ at $s = 1$. Thus

$$h_2 = f(1) + 2f(-\cos 30°) = 12.8749 < h_1.$$

Corollary 2 and Lemma 5 give that

$$h_3 = 12.8721 < h_1, \quad h_4 = 12.4849 < h_1.$$

Therefore, all $h_m \leq h_1$. Thus $12 \leq \tau_3 \leq h_{max} = h_1 = 12.88 < 13$. $\square$

6. On calculations of h_m for $m \le n$

From here on $f(t) = f_0 + f_1\,t + \ldots + f_d\,t^d$ be a real polynomial of degree d that satisfies (4.3).

When $m \le n$, Lemma 1 (ii) implies that Δ_m is a regular $(m-1)$-dimensional spherical simplex. Let the vertices of the simplex $\Delta_m \subset \mathbf{S}^{m-1}$ have coordinates

$$y_1 = (a+b, a, \ldots, a),\ y_2 = (a, a+b, \ldots, a), \ldots,\ y_m = (a, a, \ldots, a+b);$$

where $a = (\sqrt{1+(m-1)z} - \sqrt{1-z}\,)/m$, $\ b = \sqrt{1-z}$. Suppose $y \in \mathbf{R}^m$ has coordinates $(u_1, \ldots, u_m)$, then (4.5) implies

$$h_m = \max_{u_1,\ldots,u_{m-1}} \widetilde{H}_m(u_1, \ldots, u_{m-1}), \quad \text{where}$$

$$\widetilde{H}_m(u_1, \ldots, u_{m-1}) = H_m(u_1, \ldots, u_{m-1}, u_m), \quad u_m = \sqrt{1 - u_1^2 - \ldots - u_{m-1}^2}\,,$$

$$\text{subject to} \ \ t_k = y \cdot y_k = a(u_1 + \ldots + u_m) + b u_k \ge -t_0 \ \ \text{for} \ \ 1 \le k \le m. \quad (6.1)$$

For the proofs of Theorems 1 and 2 we need to consider the cases $n=3$, $m=3$, and $n=4$, $m=3,4$. For $m=3$ and 4 Δ_m are an a regular triangle and a regular tetrahedron, respectively, so h_m can be found by (6.1).

The equality (6.1) show that h_m is the maximum of the function $\widetilde{H}_m$. We have a classical computational problem: to find the maximum of a function in $m-1$ variables. Numerical Analysis methods can be used for calculation of this maximum. In the first version of this paper (see short communication [**?**]) the Nelder-Mead simplex method (see [**?**, **?**]) was applied. For the polynomial $f(t)$ from Section 4 the calculations give that $h_3 = 24.8345$ and $h_4 = 24.8180$, i.e. $h_4 < h_3 < h_2$. For f from Section 5 this method gives $h_3 = 12.8721$.

For $m \le n$ the values h_m can be calculated another way. Let us show that the problem of calculations of h_m for $m \le 4$ can be reduced to calculations of zeros of some polynomials in one variable. (It is important that these calculations can be independently verified. If you have approximate values for all (real and complex) roots of a polynomial , then you can check the existence of these roots by simple computations.)

Let us consider $H_m(y)$ as the symmetric polynomial $F_m(t_1,\ldots,t_m)$ in the variables $t_1,\ldots,t_m$: $F_m(t_1,\ldots,t_m) = f(1)+f(-t_1)+\ldots+f(-t_m)$. Denote by $s_k = s_k(t_1,\ldots,t_m)$ the power sum $t_1^k+\ldots+t_m^k$. Then

$$F_m(t_1,\ldots,t_m) = \Psi_m(s_1,\ldots,s_d) = f(1)+mf_0 - f_1\,s_1+\ldots+(-1)^d f_d\,s_d.$$

The equality $u_1^2+\ldots+u_m^2 = 1$ in (6.1) holds if and only if

$$s_2 = \sigma(s_1) := \frac{z}{(m-1)z+1}s_1^2+1-z. \tag{6.2}$$

Any symmetric polynomial in m variables can be expressed as a polynomial of $s_1,\ldots,s_m$. Therefore, in the case $k>m$ the power sum s_k is $R_k(s_1,\ldots,s_m)$. Combining this with (6.2), we get

$$\Psi_m(s_1,\sigma(s_1),s_3,\ldots,s_d) = \Phi_m(s_1,s_3,\ldots,s_m).$$

Therefore, we have

$$h_m = \max\Phi_m(s_1,s_3,\ldots,s_m), \quad (s_1,s_3,\ldots,s_m)\in D_m\subset\mathbf{R}^{m-1},$$

where D_m is the domain in $\mathbf{R}^{m-1}$ defined by the constraints $t_i\geq -t_0$ and (6.2).

Let us show now how to determine D_m for $m>2$. The equation (6.2) defines the ellipsoid $E: s_2=\sigma(s_1)$ in space $\{t_1,\ldots,t_m\}$. Then $s_1 = t_1+\ldots+t_m$ attains its maximum on E at the point with $t_1=t_2=\ldots=t_m$, and s_1 achieves its minimum on $E\bigcap\{t_i\geq -t_0\}$ at the point with $t_2=\ldots=t_m=-t_0$. From this follows $w_1\leq s_1\leq w_2$, where

$$w_1 = \frac{\sqrt{(p-t_0^2)\,(p-z^2)}-z\,t_0}{p}-(m-1)\,t_0, \quad p=\frac{1+(m-2)\,z}{m-1},$$

$$w_2 = \sqrt{m\,(m-1)\,z+m}.$$

The equation $s_1 = \omega$ gives the hyperplane, and the equation $s_2 = \sigma(\omega)$ gives the $(m-1)$-sphere in space: $\{(t_1, \ldots, t_m)\}$. Denote by $S(\omega)$ the $(m-2)$-sphere that is the intersection of these hyperplane and sphere. Let $l_k(\omega)$ be the minimum of s_k on $S(\omega) \bigcap \{t_i \geq -t_0\}$, and $v_k(\omega)$ is its maximum. Now we have

$$h_m = \max_{s_1} \max_{s_3} \ldots \max_{s_m} \Phi_m(s_1, s_3, \ldots, s_m), \quad \text{where}$$

$$w_1 \leq s_1 \leq w_2, \quad l_k(s_1) \leq s_k \leq v_k(s_1), \quad k = 3, \ldots, m.$$

For the polynomials f from Sections 4 and 5 we can give more details about calculations of h_m for $m = 3, 4$.

Let us consider the case $m = 3$ with $d = 9$. In this case $F_\omega(s_3) = \Phi_3(\omega, s_3)$ is a polynomial of degree 3 in the variable s_3.

Lemma 3. *Let f be a 9th degree polynomial $f(t) = f_0 + f_1 t + \ldots + f_9 t^9$ such that $f_9 > 0$, $f_6 = f_8 = 0$, and $f_7 > -15 f_9/7$. If $F'_\omega(s) \leq 0$ at $s = l_3(\omega)$, then the function $F_\omega(s)$ achieves its maximum on the interval $[l_3(\omega), v_3(\omega)]$ at $s = l_3(\omega)$.*

Proof. The expansion of s_9 in terms of $s_1^i s_2^j s_3^k$, $i + 2j + 3k = 9$, is

$$s_9 = \frac{1}{9} s_3^3 + s_3^2 (\frac{2}{3} s_1^3 + s_2 s_1) + s_3 (\frac{3}{8} s_2^3 - \frac{3}{8} s_2^2 s_1^2 - \frac{7}{8} s_2 s_1^4 + \frac{5}{24} s_1^6) + R(s_1, s_2).$$

The coefficient of $s_3^2 s_1$ in s_7 equals $7/9$. Thus

$$F_\omega(s) = -s^3\, f_9/9 - s^2\, (f_9\, \omega\, \sigma(\omega) + 2 f_9\, \omega^3/3 - 7 f_7\, \omega/9) + s R_1(\omega) + R_0(\omega).$$

$F_\omega(s)$ is a cubic polynomial with negative coefficient of s^3. Then $F_\omega(s)$ is a concave function for $s > r$, where $r : F''_\omega(r) = 0$. Therefore, if $r < l_3(\omega)$, then $F_\omega(s)$ is a concave function on the interval $[l_3(\omega), v_3(\omega)]$. $r < l_3(\omega)$ iff

$$B(\omega) := 3 l_3(\omega) + 6\omega^3 + 9\omega\, \sigma(\omega) > -7\omega f_7/f_9.$$

This inequality holds for $t_0 < -z \leq 0$. Indeed,

$$\omega \geq w_1 \geq 1 + 2z, \quad \sigma\,(\omega) \geq 1, \quad l_3(\omega) > 0;$$

so then

$$B(\omega) > 15\omega > -7\omega f_7/f_9.$$

The inequality $F'_\omega(l_3(\omega)) \leq 0$ implies that $F_\omega(s)$ is a decreasing function on the interval $[l_3(\omega), v_3(\omega)]$. $\square$

The polynomials f from Sections 4 and 5 satisfy the assumptions in Lemma 3. Then $\Phi_3(\omega, s)$ attains its maximum at the point $s = l_3(\omega)$, i.e. at the point with $t_1 = t_2 \geq t_3$, or with $t_1 \geq t_2 \geq t_3 = -t_0$. If $t_1 = t_2 \geq t_3$, then $p(\omega) = \Phi_3(\omega, l_3(\omega))$ is a polynomial in ω. This polynomial is a decreasing function in the variable ω on the interval $t_3 \geq -t_0$. Therefore, $p(\omega)$ achieves its maximum on this interval at the point with $t_3 = -t_0$. The calculations show that for f from Section 4 $h_3 = \max p(\omega) = 24.8345$, when $\phi_3 = \phi_0$, $\phi_1 = \phi_2 = 30.0715°$, and for f from Section 5 $h_3 = 12.8721$, when $\phi_3 = \phi_0$, $\phi_1 = \phi_2 = 30.0134°$.

Corollary 2. *Let f be the polynomial from Section 4 (Section 5), then* $h_3 = 24.8345$ ($h_3 = 12.8721$).

Consider the function $F_\omega(s_3, s_4) = \Phi_4(\omega, s_3, s_4)$ on $S(\omega)$. Let $q_i \in S(\omega)$ and $q_1 : t_1 = t_2 > t_3 = t_4$, $q_2 : t_1 = t_2 = t_3 > t_4$, and $q_3 : t_1 > t_2 = t_3 = t_4$.

Lemma 4. *Let f be a 9th degree polynomial $f(t) = \sum f_i t^i$. If $f_9 > 0$ and $f_6 = f_8 = 0$, then the function $F_\omega(s_3, s_4)$ achieves its maximum on $S(\omega)$ with $\omega > 1$ at one of the points $(s_3(q_i), s_4(q_i))$, $i = 1, 2, 3$.*

Proof. The expansion of s_9 in terms of $s_1^i s_2^j s_3^k s_4^l$ is

$$s_9 = \frac{9}{16}s_4^2 s_1 + \frac{1}{9}s_3^3 - \frac{1}{3}s_3^2 s_1^3 + \frac{3}{4}s_4 s_3 s_1 + \frac{3}{8}s_4 s_2 s_1^3 - \frac{3}{8}s_3 s_2^2 s_1^2 - \frac{1}{24}s_3 s_1^6 + R(s_1, s_2).$$

The coefficient of $s_3^2 s_1$ in s_7 equals 0. We have $f_6 = f_8 = 0$, then $F_\omega(s_3, s_4) = -f_9\, s_9 + \ldots = -f_9(s_3^3/9 - s_3^2\,\omega^3/3) + \ldots$ Therefore,

$$F_{33} = \frac{\partial^2 F_\omega(s_3, s_4)}{\partial^2 s_3} = -f_9(\frac{2}{3}s_3 - \frac{2}{3}\omega^3) = \frac{2f_9}{3}(\omega^3 - s_3).$$

If $F_\omega(s_3, s_4)$ has its maximum on $S(\omega)$ at the point x, and x is not a critical point of s_3 on $S(\omega)$, then $F_{33} \leq 0$. From other side, for all $t_i \in [0,1]$ and $s_1 = \omega > 1$ we have $s_3 \leq \omega < \omega^3$, so then $F_{33} > 0$. The function s_3 on $S(\omega)$ (up to permutation of labels) has critical points at $q_i,\ i = 1, 2, 3$. $\square$

Corollary 3. *Let f be the polynomial from Section* 4*. Then* $h_4 = 24.8180$.

Proof. By direct calculations it can be shown that
$F_\omega(s_3(q_1), s_4(q_1)) > F_\omega(s_3(q_i), s_4(q_i))$ for $i = 2, 3$. Then Lemma 4 implies $h_4 = \max p(\omega)$, where $p(\omega) = F_\omega(s_3(q_1), s_4(q_1)) = \Phi_4(\omega, s_3(q_1), s_4(q_1))$.

The polynomial $p(\omega)$ attains its maximum $h_4 = 24.8180$ at the point with $\phi_1 = \phi_2 = 30.2310°$, $\phi_3 = \phi_4 = 51.6765°$. $\square$

7. On calculations of h_m for $m > n$

When $m > n$, Q_m is not uniquely (up to isometry) defined by Lemma 1 *(iii)*. For the proofs of Theorem 1 and Theorem 2 we need to consider the cases $n = 4,\ m = 5, 6$ and $n = 3,\ m = 4$. In these cases Δ_m are rather simple.

Let Γ_m denotes the graph of the edges of Δ_m with length ψ. From Lemma 1 it follows that the degree of any vertex of Γ_m is equal to or greater than $n - 1$.

For instance, let $n = 3$ and $m = 4$. Then Lemma 1 implies that $\Delta_4 \subset \mathbf{S}^2$ is a spherical equilateral quadrangle (rhomb) with edge length ψ. If its smallest diagonal has length α, then $\psi \le \alpha \le \gamma$, where γ is the length of the diagonal of the regular quadrangle with edge length ψ, $\cos\gamma = 2\cos\psi - 1$ (Fig. 7). Denote this rhomb by $\Delta_4(\alpha)$ and its vertices by $y_k(\alpha),\ k = 1, 2, 3, 4$.

Consider an 1-parametric family of $\Delta_4(\alpha),\ \ \psi \le \alpha \le \gamma,$ on $\mathbf{S}^2$. Let $H_4(y, \alpha) = f(1) + f(-y \cdot y_1(\alpha)) + \ldots + f(-y \cdot y_4(\alpha))$. Then from Definition and Lemma 1 follow

$$h_4 = \max_{y,\,\alpha} H_4(y, \alpha), \quad y \in \mathbf{S}^2, \quad y \cdot y_k(\alpha) \ge -t_0, \quad \psi \le \alpha \le \gamma \qquad (7.1)$$

For the polynomial f from Section 5 it can be proven numerically that

Lemma 5. *The function $H_4(y, \alpha)$ attains its maximum $h_4 = 12.4849$ at $\alpha = 72.4112^\circ$, and y with $\phi_1 = \phi_2 = \phi_3 = \phi_0,\ \ \phi_4 = 18.6172^\circ$.*

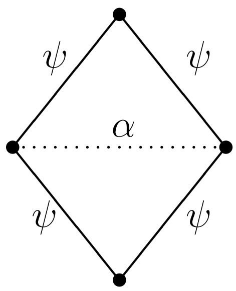

Fig. 7

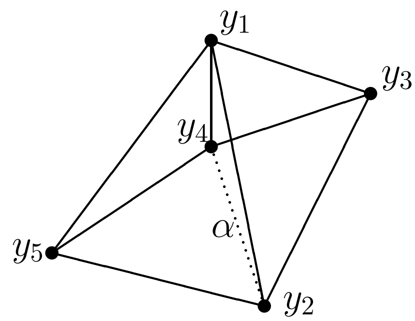

Fig. 8

When $n = 4$, $m = 5$, we also have an 1-parametric family as in (7.1). The degree of any vertex of Γ_5 is not less than 3. This implies that at least one vertex of Γ_5 has degree 4. Indeed, if all vertices of Γ_5 are of degree 3, then the sum of the degrees equals 15, i.e. is not an even number. There exists only one type of Γ_5 with these conditions (Fig. 8). Therefore, we have the 1-parametric family $\Delta_5(\alpha)$ on $\mathbf{S}^3$. Let $H_5(y,\alpha) = f(1) + f(-y \cdot y_1(\alpha)) + \ldots + f(-y \cdot y_5(\alpha))$. Then

$$h_5 = \max_{y,\alpha} H_5(y,\alpha), \quad y \in \mathbf{S}^3, \quad y \cdot y_k(\alpha) \geq -t_0, \quad 1 \leq k \leq 5, \quad \psi \leq \alpha \leq \gamma \tag{7.2}$$

Now we consider this case for the polynomial $f(t)$ from Section 4, where $z = 1/2$ and $t_0 = -0.60794$ ($f(t_0) = 0$). Here we have $\psi = 60^\circ$ and $\gamma = 90^\circ$, i.e. $60^\circ \leq \alpha \leq 90^\circ$.

Let the vertices of $\Delta_5(\alpha) \subset \mathbf{S}^3$ have coordinates

$$y_1(\alpha) = (0,0,0,1), \quad y_2(\alpha) = (p(\alpha), 0, q(\alpha), 1/2), \quad y_3(\alpha) = (0, r(\alpha), s(\alpha), 1/2),$$

$$y_4(\alpha) = (-p(\alpha), 0, q(\alpha), 1/2), \qquad y_5(\alpha) = (0, -r(\alpha), s(\alpha), 1/2), \quad \text{where}$$

$$p(\alpha) = \sqrt{(1-a)/2}\,, \quad q(\alpha) = \sqrt{(2a+1)/4}\,, \quad r(\alpha) = \sqrt{(3a+1)/(4a+2)}\,,$$

$$s(\alpha) = 1/\sqrt{8a+4}\,, \quad \text{and} \quad a = \cos\alpha.$$

If $y \in \mathbf{S}^3$ has coordinates (u_1, u_2, u_3, u_4), then $u_4 = \sqrt{1 - u_1^2 - u_2^3 - u_3^2}\,$. Let $\widetilde{H}(u_1, u_2, u_3, \alpha) = H_5(y, \alpha)$, $60^\circ \leq \alpha \leq 90^\circ$, then consider the following optimization problem for the variables u_1, u_2, u_3, α:

$$\text{maximize} \quad \widetilde{H}(u_1, u_2, u_3, \alpha) \quad \text{subject to}$$

$$60^\circ \leq \alpha \leq 90^\circ, \quad y \cdot y_k(\alpha) \geq -t_0, \quad k = 1, 2, 3, 4, 5. \tag{7.3}$$

For the polynomial f from Section 4 this optimization problem was solved numerically by using the Nelder-Mead simplex method (see [**?**, **?**]).

Lemma 6. *For the polynomial f from Section 4 the function $\widetilde{H}(u_1, u_2, u_3, \alpha)$ achieves its maximum $h_5 = 24.6856$ at $\alpha = 60^\circ$ and y with $\phi_1 = 42.1569^\circ$, $\phi_2 = \phi_4 = 32.3025^\circ$, $\phi_3 = \phi_5 = \phi_0$.*

Let us briefly explain how the optimization problem (7.3) can be solved without using numerical methods. The equation $u_4 = \omega$ gives the hyperplane (3-space) in $\mathbf{R}^4$. Let $S(\omega)$ is the intersection of this hyperplane and $\mathbf{S}^3$. Note that $S(\omega)$ is a 2-sphere. For $1 > \omega \geq 1/2$ the intersection of $S(\omega)$ and $\Delta_5(\alpha)$ gives the rhomb $\Delta(\omega, \alpha)$.

Consider the function $\widetilde{H}(u_1, u_2, u_3, \alpha)$ on $\Delta(\omega, \alpha)$. $\widetilde{H}(u_1, u_2, u_3, \alpha)$ is a polynomial of degree $d = 9$ in the variable u_3. This function is a monotone decreasing on the interval $u_2 = const\, u_1$ in $\Delta(\omega, \alpha)$ and achieves its minimum at the center $y_c(\omega, \alpha)$ of $\Delta(\omega, \alpha)$. From this follows that $\widetilde{H}$ has its maximum on $\Delta(\omega, \alpha)$ at the point with the maximal u_3. Note that the equation $u_3 = const$ gives the circle of center $y_c(\omega, \alpha)$ in $S(\omega)$. The function $\widetilde{H}$ achieves its maximum on this circle when (i) $u_2 = 0,\ \alpha = 60^\circ$ or (ii) $u_1 = u_2,\ \alpha = 90^\circ$. The constraints in (7.3) determine u_3 and give that the maximum achieves in the case (i). Thus (7.3) can be reduced to an polynomial optimization problem in one variable.

For the last case ($n = 4,\ m = 6$) we have found a simpler proof that $h_6 < h_2$. However, let us still consider this case briefly. There are two types of Γ_6 (Fig. 9). If Γ_6 contains graph of Type I, then it could be proven that $\Lambda_6 = \emptyset$. If Γ_6 is of Type II, then this defines the 3-parametric family $\Delta_6(\alpha,\beta,\gamma)$ on $\mathbf{S}^3$. Thus

$$h_6 = \max_{y,\alpha,\beta,\gamma} H_6(y,\alpha,\beta,\gamma), \quad y \in \mathbf{S}^3, \quad y \cdot y_k(\alpha,\beta,\gamma) \geq -t_0.$$

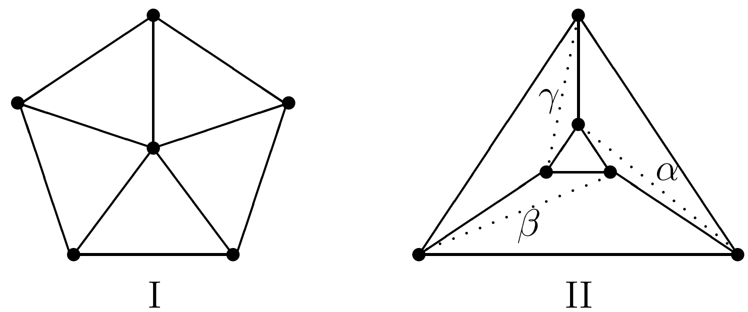

Fig. 9

For the polynomial f from Section 4 the calculations show that $H_6(y,\alpha,\beta,\gamma)$ attains its maximum $h_6 = 22.5205$ at $\alpha = \beta = \gamma = 60°$, when $\Delta_6(\alpha,\beta,\gamma)$ is a regular spherical octahedron.

Now let us prove that

Lemma 7. *For the polynomial f from Section 4 $h_6 < h_2$.*

Proof. In the converse case there exists a set $Y_6 = \{y_0, y_1, \ldots, y_6\}$ such that

$$h(Y_6) = f(1) + f(-\cos\phi_1) + \ldots + f(-\cos\phi_6) \geq h_2.$$

If $Y_5 = \{y_0, y_1, \ldots, y_5\}$, then $h(Y_5) \leq h_5$. Suppose $\phi_1 \leq \ldots \leq \phi_5 \leq \phi_6$, whence

$$f(-\cos\phi_6) \geq h_2 - h_5 = 0.1788.$$

Therefore,

$$\phi_6 \leq \alpha_0 = \cos^{-1}\left(-f^{-1}(0.1788)\right) = 48.3787°.$$

This implies that all $\phi_k \leq \alpha_0$.

But $\phi_6 \geq 45°$. That can be proven as Corollary 1. We have

$$\varphi_3(6) = 90° \text{ (see [?])}, \quad \rho(1/2, 45°) = 90°.$$

By assumption $\phi_k \le \phi_6$ for $1 \le k \le 6$. If $\phi_6 < 45^\circ$, then $\rho(1/2, \phi_6) > 90^\circ$. This is in contradiction with Lemma 2.

Actually, we have the same case as for $m = 5$ (see (7.2)).

$$h(Y_5) \le h_5(\alpha_0) = \max_{y,\alpha} H_5(y, \alpha), \quad y \in \mathbf{S}^3, \quad y \cdot y_k(\alpha) \ge \cos\alpha_0, \quad \gamma_0 \le \alpha \le 90^\circ,$$

where $\cos 2\alpha_0 = -\cos\gamma_0/(1 + 2\cos\gamma_0)$. In the same way as in the case $m = 5$, where t_0 is replaced by $-\cos\alpha_0$, we obtain $h_5(\alpha_0) = 23.5389$. Therefore,

$$h(Y_6) = h(Y_5) + f(-\cos\phi_6) \le h_5(\alpha_0) + f(-\cos 45^\circ) = 23.5389 + 0.4533 < h_2.$$

This contradiction concludes the proof. $\square$

8. Concluding remarks

The algorithm in Appendix I can be applied to other dimensions and spherical z-codes. If $t_0 = -1$, then the algorithm gives the Delsarte method. E is an estimation of h_{max} in this algorithm.

Direct application of the method developed in this paper, presumably could lead to some improvements in the upper bounds on kissing numbers in dimensions 9, 10, 16, 17, 18 given in [C-S]. ("Presumably" because the equality $h_{max} = E$ is not proven yet.)

In 9 and 10 dimensions Table 1.5 gives:
$306 \le \tau_9 \le 380, \quad 500 \le \tau_{10} \le 595.$
The algorithm gives:
$n = \ 9: \ \deg f = 11, \ E = h_1 = 367.8619, \ t_0 = -0.57;$
$n = 10: \ \deg f = 11, \ E = h_1 = 570.5240, \ t_0 = -0.586.$
For these dimensions there is a good chance to prove that $\tau_9 \le 367$, $\tau_{10} \le 570$.

From the equality $\tau_3 = 12$ follows $\varphi_3(13) < 60°$. The method gives $\varphi_3(13) < 59.4°$ ($\deg f = 11$). The lower bound on $\varphi_3(13)$ is 57.1367° [FeT]. Therefore, we have $57.1367° \le \varphi_3(13) < 59.4°$.

The method gives $\varphi_4(25) < 59.81°$, $\varphi_4(24) < 60.5°$. (This is theorem that can be proven by the same method as Theorem 1.) That improve the bounds:

$$\varphi_4(25) < 60.79°, \ \ \varphi_4(24) < 61.65° \ [Lev] \ (\text{cf. } [Bo]); \ \ \varphi_4(24) < 61.47° \ [Bo];$$

$$\varphi_4(25) < 60.5°, \quad \varphi_4(24) < 61.41° \ [AB2].$$

Now in these cases we have

$$57.4988° \ < \varphi_4(25) < 59.81°, \quad 60° \le \varphi_4(24) < 60.5°.$$

Appendix I. Algorithm for f.

Let us have a polynomial f represented in the form $f(t) = 1+\sum\limits_{k=1}^{d} c_k G_k^{(n)}(t)$. We have the following constraints for f in (3.3): (C1) $c_k \geq 0,\ \ 1 \leq k \leq d$; (C2) $f(a) > f(b)$ for $-1 \leq a < b \leq t_0$; (C3) $f(t) \leq 0$ for $t_0 \leq t \leq z$.

When $m \leq n,\ h_m = \max H_m(y),\ y \in \Lambda_m$. We do not know y where H_m attains its maximum, so for evaluation of h_m let us use y_c − the center of Δ_m. All vertices y_k of Δ_m are at the distance of R_m from y_c, where $\cos R_m = \sqrt{(1+(m-1)z)/m}$.

When $m = 2n-2,\ \Delta_m$ presumably is a regular $(n-1)$-dimensional spherical octahedron. (It is not proven yet.) In this case $\cos R_m = \sqrt{z}$.

Let $I_n = \{1,\dots,n\} \bigcup \{2n-2\},\ \ m \in I_n,\ \ b_m = -\cos R_m$, whence $H_m(y_c) = f(1) + mf(b_m)$. If F_0 is such that $H_m(y) \leq E = F_0 + f(1)$, then (C4) $f(b_m) \leq F_0/m,\ \ m \in I_n$. A polynomial f that satisfies (C1-C4) and gives the minimal E (note that $E = F_0 + 1 + c_1 + \ldots + c_d = F_0 + f(1)$ will become a lower estimate of h_{max}) can be found by the following

Algorithm.

Input: $n,\ z,\ t_0,\ d,\ N$.

Output: $c_1,\dots,c_d,\ F_0,\ E$.

First replace (C2) and (C3) by a finite set of inequalities at the points $a_j = -1 + \epsilon j,\ \ 0 \leq j \leq N,\ \ \epsilon = (1+z)/N$:

Second use linear programming to find $F_0, c_1, \dots, c_d$ so as to minimize $E - 1 = F_0 + \sum\limits_{k=1}^{d} c_k$ subject to the constraints

$$c_k \geq 0, \quad 1 \leq k \leq d; \qquad \sum_{k=1}^{d} c_k G_k^{(n)}(a_j) \geq \sum_{k=1}^{d} c_k G_k^{(n)}(a_{j+1}), \quad a_j \in [-1, t_0];$$

$$1+\sum_{k=1}^{d} c_k G_k^{(n)}(a_j) \leq 0, \quad a_j \in [t_0, z]; \quad 1+\sum_{k=1}^{d} c_k G_k^{(n)}(b_m) \leq F_0/m, \quad m \in I_n.$$

Appendix II. On another proof.

Here is presented a sketch of another proof of Theorem 1. This was my first proof. The proof in Section 4 was found while a paper with the first proof was in process.

Suppose X is a spherical 1/2-code in $\mathbf{S}^3$ with size $|X| = \tau_4$. Let us prove that $|X| < 25$. Assume the converse, then $|X| = 25$.

Step 1. *The graph* Γ_{25}.

Let us consider the graph Γ_{25} on the sphere $\mathbf{S}^3$ formed by joing every pair of X whose distance apart is less than $72°$. Danzer proved that $\varphi_3(11) = \varphi_3(12)$, where $\cos\varphi_3(12) = 1/\sqrt{5}$. From other side, we have $\rho(1/2, 72°) = \varphi_3(11)$. Similarly, as in Lemma 2, it could be proven that the degree d_i of any vertex x_i of Γ_{25} is at most 10, i.e. $d_i \leq 10$.

Step 2. *Constraints for* Γ_{25}.

Let $f = c_0 + c_1 U_1 + \ldots + c_9 U_9$, where $c_0 = 1.1797$; $c_1 = 3.7875$; $c_2 = 5.6792$; $c_3 = 5.4997$; $c_4 = 3.4008$; $c_5 = 1.2302$; $c_6 = c_7 = 0$; $c_8 = 0.0917$; $c_9 = 0.1836$. Here as above shown to 4 decimal places.

The function $f(t)$ satisfies the following properties:
(i) $f(t) \leq 0$ if $t \in [-1, t_0]$, $f(t_0) = 0$, where $t_0 = \cos 72° = (\sqrt{5} - 1)/4 = 0.3090$;
(ii) $f(t)$ is a monotone increasing function on the interval $[t_0, 1/2]$, $f(1/2) = 1$.

From (2.2) and (2.4) follows $\sum F_i(X) \geq 625 c_0$. Let $f_i = F_i(X) - f(1)$. Then

$$\sum f_i \geq 625 c_0 - 25 f(1) > 211 \qquad (C_1)$$

From $(i), (ii)$ follow $f_i \leq d_i$. Therefore, $f_i > 8$ if $d_i = 9$ or $d_i = 10$.

Step 3. If $d_i = 10$, then $f_i < 8$.

Let r_{10} is defined by the equation: $\rho(1/2, r_{10}) = \varphi_3(10)$, where $\varphi_3(10) = 66.1468°$ [**?**]. Denote by S_{10} the sphere in $\mathbf{S}^3$ of center x_i and radius r_{10}. Then at most 9 points $x_j \in X$, $j \neq i$ can lie inside S_{10}.

Let $y_1, \ldots, y_{10}$ are neighbors of x_i in Γ_{25}. Then any y_j is a vertex of $\Delta_{10} = \text{conv}\{y_1, \ldots, y_{10}\}$. Suppose $f_i \geq 8$. Then there are at most two vertices of Δ_{10} (let it be y_1 and y_2) that can lie outside of S_{10}. By detailed consideration of the cases (y_1 and y_2 are neighbors in Δ_{10}, only y_1 lies outside of S_{10}) it can be proven that y_j can be shifted in such way that all of them lie inside S_{10}.

Step 4. *Constraints for Q.*

Let $Q = \{x_i \in X : f_i > 8\}$. If $f_i > 8$, then $d_i = 9$. From C_1 follows $|Q| = m \geq 12$.

Let $h = c_0 + c_1U_1 + \ldots + c_5U_5$, where $c_0 = 0.4269$; $c_1 = 1.1462$; $c_2 = 0.8454$; $c_3 = c_4 = 0$; $c_5 = 0.1654$. Then $h(t) \leq 0$ for $t \in [-1, -1/8]$, $h(-1/8) = 0$; $h(t)$ is a monotone increasing function on the interval $[-1/8, 1/2]$, and $h(1/2) = 1$.

Let $h_i = F_i(Q) - h(1)$. In the same way as in Step 2 we have

$$\sum h_i \geq c_0m^2 - h(1)m = 0.4269m^2 - 2.5839m \qquad (C_2)$$

Step 5. *There exists $x_i \in Q$ with $f_i + h_i > 11.4559$.*

Up to permutations of labels we have $Q = \{x_1, \ldots, x_m\}$. If $w_i = f_i - 8$, then (C_1) implies $\sum w_i > 11$. For $i > m$ we have $w_i \leq 0$, thus $w_1 + \ldots + w_m > 11$. If $u_i = h_i + w_i$, then from (C_2) follows $\sum u_i > c_0m^2 - h(1)m + 11$. This implies that $\min\limits_{m \geq 12} \max\limits_i u_i > 3.4559$.

Now we show that there are no points as in Step 5. In the converse case there exists $Y = \{y_0, y_1, \ldots, y_9, \ldots, y_n\} \subset \mathbf{S}^3$, where for all i $t_i = \cos\phi_i = y_0 \cdot y_i \leq 1/2$; $t_i > \cos 72^\circ$ for $1 \leq i \leq 9$; and $\cos 72^\circ \geq t_i > -1/8$ for $10 \leq i \leq n$. Furthermore, $f_0 + h_0 > 11.4559$, where $f_0 = f(t_1) + \ldots + f(t_9) > 8$, $h_0 = h(t_k) + \ldots + h(t_n)$. Here $d_i = 9$ for $i = 0, k, \ldots, n$. We assume also that Y is an optimal set (as in Section 3), i.e. no y_i that can be shifted for increasing of $f_0 + h_0$.

We did not find short (analytical) prove for this contradiction (Steps 6, 7) and applied a geometrical approach.

Step 6. $n < 12$.

Let $\Delta_9 = \operatorname{conv}\{y_1, \ldots, y_9\}$. It is easy to see that each y_i is a vertex of Δ_9. The graph of edges G_9 of Δ_9 is uniquely defined by inequalities:
$f_0 > 8$,
$t_i > \cos 72°$. (If all faces of Δ_9 are triangles, then G_9 has 5 vertices of degree 4 and 4 vertices of degree 5.) Then the subgraph Γ_9 of G_9 with edges lengths are less than 72° can be obtained from G_9 by removing 0, 1, or 2 edges.

Suppose that converse holds: $n \geq 12$. We have Δ_9 and three points y_{10}, y_{11},
y_{12} at the distances are less than $\cos^{-1}(-1/8)$ from y_0. If Y is an optimal, then for any y_i, $i > 9$, there are at least three points of Y at the distances 60° from y_i. From detailed analysis of the types of Γ_9 and locations of y_i follow that it is impossible.

Step 7. $n \geq 12$.

We have $8 < f_0 \leq 9$, and $f_0 + h_0 > 11.4559$. $(*)$.
$n \neq 9$. From $(*)$ follows $k \leq 7$, i.e. among of y_i, $1 \leq i \leq 9$, there are at least three points with $d_i = 9$ and $f_i > 8$. It is not hard to prove, that these points can be at most two. (This prove based on the analysis of Γ_9 as in Step 6.)
$n \neq 10$. In the same way as in Step 3, $(*)$ implies that y_i can be shifted inside the sphere of center y_0 and radius r_{10}.
$n \neq 11$. Similarly, y_i can be shifted inside the sphere of center y_0 and radius $r_{11} = 72°$.

The last contradiction concludes the proof of Theorem 1.