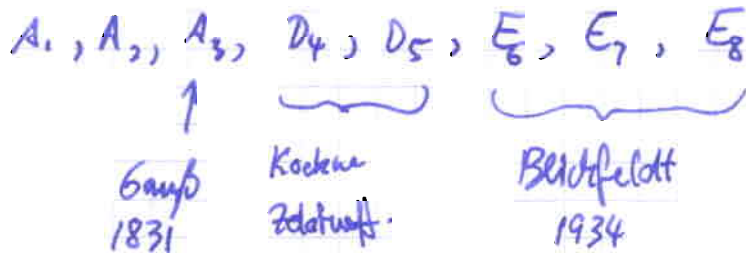


H. Cohen (with Kumar)

Thm: Λ_{24} is the unique densest lattice in \mathbb{R}^{24}
 \uparrow
 Leech lattice

History: previously known densest lattice for $n \leq 8$:



Proof Based on [Cohen-Elkies: LP bounds for sphere packings in \mathbb{R}^n]

$n = 8, 24$
 E_8, Λ_{24}

e.g. Take $f: \mathbb{R}^n \rightarrow \mathbb{R}$
 $f(0) = \hat{f}(0) = 1$
 $f(x) \leq 0$ for $|x| \geq r$
 $\hat{f}(t) \geq 0 \quad \forall t$
 decent f .

\Rightarrow density $\leq \frac{\pi^{n/2}}{(n/2)!} \left(\frac{r}{2}\right)^n$

(If $\text{vol}(\mathbb{R}^n/\Lambda) = 1$, then min vector length $\leq r$)

(3)

Lemma:

assume $\text{vol}(\mathbb{R}^n/\Lambda) = 1$

Poisson summation:

$$\sum_{x \in \Lambda} f(x) = \sum_{t \in \Lambda^*} \hat{f}(t)$$

if $\forall x \in \Lambda \setminus \{0\} : |x| \geq r$

$$1 = f(0) \geq \sum_{x \in \Lambda} f(x) = \sum_{t \in \Lambda^*} \hat{f}(t) \geq \hat{f}(0) = 1$$

for an exact function

$r=2$

$\rightarrow f(x) = 0 \quad \forall x \in \Lambda \setminus \{0\}$.

if radially symmetric, f has roots at all the vector lengths.

Naire approach: $|\Lambda'| = 1$, Λ' optimal in \mathbb{R}^{24} ,

f approximately optimal ($r = 2 + 10^{-30}$)

$\Rightarrow \Lambda'$ has short vector lengths near those

of Λ_{24}

\Rightarrow nearly even unimodular

not nearly close enough.

Lemma: Λ_{24} is local optimum.

Way around. Use

- (1) symmetry group CO_2 acts transitively on pairs of min vectors at a given angle.
- (2) G -class association scheme.
- (3) spherical 11-design. (exact interpolation up to degree 11)

Prove

- (a) Prove exactly 196560 nearby min vectors.
(related to kissing numbers)
- (b) approximate spherical 10-design (did not need 11)
from antipodal
i.e., $g: \mathbb{R}^{24} \rightarrow \mathbb{R}$, $\deg(g) \leq 10$
(average over sphere) - (average over code) $\leq \text{tiny } |g|_2$
- (c) association scheme (exact).
with intersection numbers as in Λ_{24} .
↳ unique

(d) Gram matrices nearly the same.

(not within known optimality of Λ_{24})

(e) technical stuff to increase accuracy
to get into local optimality

Remarks:

- (a) relies on having lattices
- local optimality relies on lattices.
- everything works in \mathbb{R}^3 with much less effort.