H. Cohn (with Kumar)

Thm: Λ_{24} is the unique densest lattice in $\mathbb{R}^{24}$

↑ Leech lattice

History: previously known densest lattice for $n \leq 8$:

$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

- A_3: Gauss 1831
- D_4, D_5: Korkine, Zolotareff.
- E_6, E_7, E_8: Blichfeldt 1934

Proof Based on [Cohn-Elkies: LP bounds for sphere packings in $\mathbb{R}^n$]

$n = 8, 24$

$E_8 \quad \Lambda_{24}$

e.g. Take
$f: \mathbb{R}^n \to \mathbb{R}$
$f(0) = \hat{f}(0) = 1$
$f(x) \leq 0$ for $|x| \geq r$
$\hat{f}(t) \geq 0 \quad \forall t$
decent f.

$\Rightarrow$ density $\leq \frac{\pi^{n/2}}{(n/2)!}\left(\frac{r}{2}\right)^n$

(If $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$, then min vector length $\leq r$)

Construct f by computer search.

How small can r be?

Conjecture: $r = \sqrt{2}$ in $\mathbb{R}^8$
$r = 2$ in $\mathbb{R}^{24}$

[CE] : density within 0.07 %
[CK] 10^{-30}

Ansatz: $f(x) = p(2\pi |x|^2) e^{-\pi(|x|^2)}$

↖ polynomial $\deg(p) = 803$

rational coefficients with
common denominator 10^{-3500}

"functions seem to converge"

<u>Lemma:</u>

assume $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$

Poisson summation:

$$\sum_{x\in\Lambda} f(x) = \sum_{t\in\Lambda^*} \hat{f}(t)$$

if $\forall x \in \Lambda\setminus\{0\} : |x| \geq r$

$$1 = f(0) \geq \sum_{x\in\Lambda} f(x) = \sum_{t\in\Lambda^*} \hat{f}(t) \geq \hat{f}(0) = 1$$

for an exact function $r=2$

$\rightarrow f(x) = 0 \quad \forall x \in \Lambda\setminus\{0\}$.

if radially symmetrize, f has roots at all the vector lengths.

<u>Naive approach:</u> $|\Lambda'| = 1$, Λ' optimal in $\mathbb{R}^{24}$,

f approximately optimal ($r = 2 + 10^{-30}$) ← not nearly close enough.

$\Rightarrow \Lambda'$ has short vector lengths near those of Λ_{24}

$\Rightarrow$ nearly even unimodular

<u>Lemma:</u> Λ_{24} is local optimum.

Way around. Use

(1) symmetry group Co_0 acts transitively on pairs of min vectors at a given angle.

→ (2) 6-class association scheme.

(3) spherical 11-design. (exact interpolation up to degree 11)

Prove

(a) Prove exactly 196560 nearby min vectors. (related to kissing numbers)

(b) approximate spherical 10-design (did not need 11) from antipodal

i.e., $g : \mathbb{R}^{24} \to \mathbb{R}$, $\deg(g) \leq 10$

(average over sphere) − (average over code) ≤ tiny $|g|_2$

(c) association scheme (exact).

with intersection numbers as in Λ_{24}.

↳ unique

(d) Gram matrices nearly the same.

(not within known optimality of Λ_{24})

(e) technical stuff to increase accuracy to get into local optimality

Remarks:

- (a) relies on having lattices
- local optimality relies on lattices.
- everything works in $\mathbb{R}^8$ with much less effort.