Periodicity and Sphere Packings in Hyperbolic Space

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Euclidean Space

- A sphere packing is a collection of nonoverlapping congruent balls in Euclidean space $\mathbb{E}^n$.

- What are the densest sphere packings?

$$density(P) = \lim_{R \to \infty} density(P \text{ in a radius-}R\text{ ball centered at } p)$$

Well-Posed

- $density(P)$ is independent of origin.
- The maximum density is attained.
- The maximum density is independent of radius.

Periodicity

- A periodic packing P is one whose symmetry group $G_P < Isom(\mathbb{E}^n)$ is cocompact.

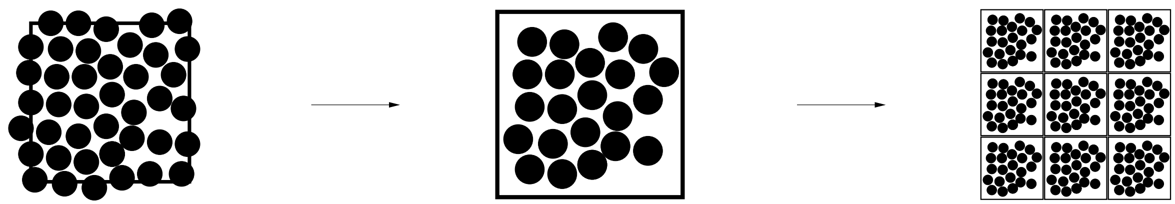


Figure 1: maximum density = sup density over periodic packings.

- Is this attained by periodic packings? (not known, except in dim $2, 3$).

Hyperbolic Space $\mathbb{H}^n$

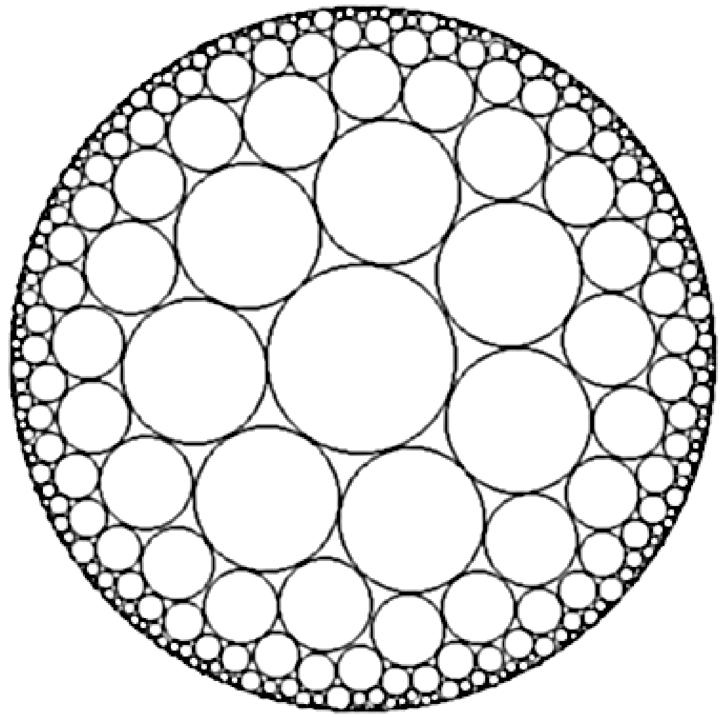


Figure 2: A densest circle packing in the hyperbolic plane (pic by Ken Stephenson)

Is the sphere-packing problem well-posed in $\mathbb{H}^n$?

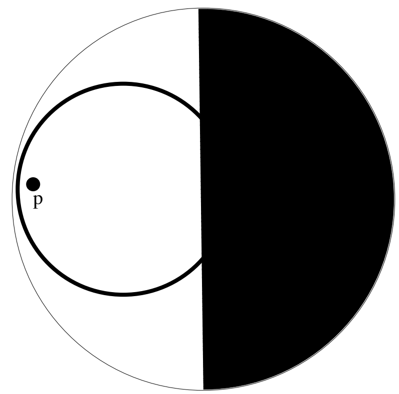


Figure 3: Density depends on origin

- Do densest packings exist? (more on this later).

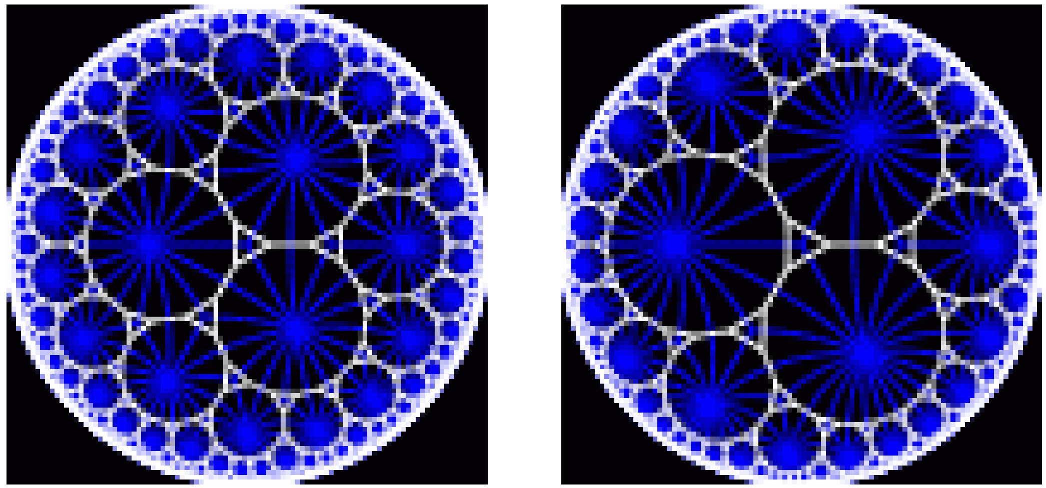


Figure 4: “maximum density” depends on radius (pics by Don Hatch).

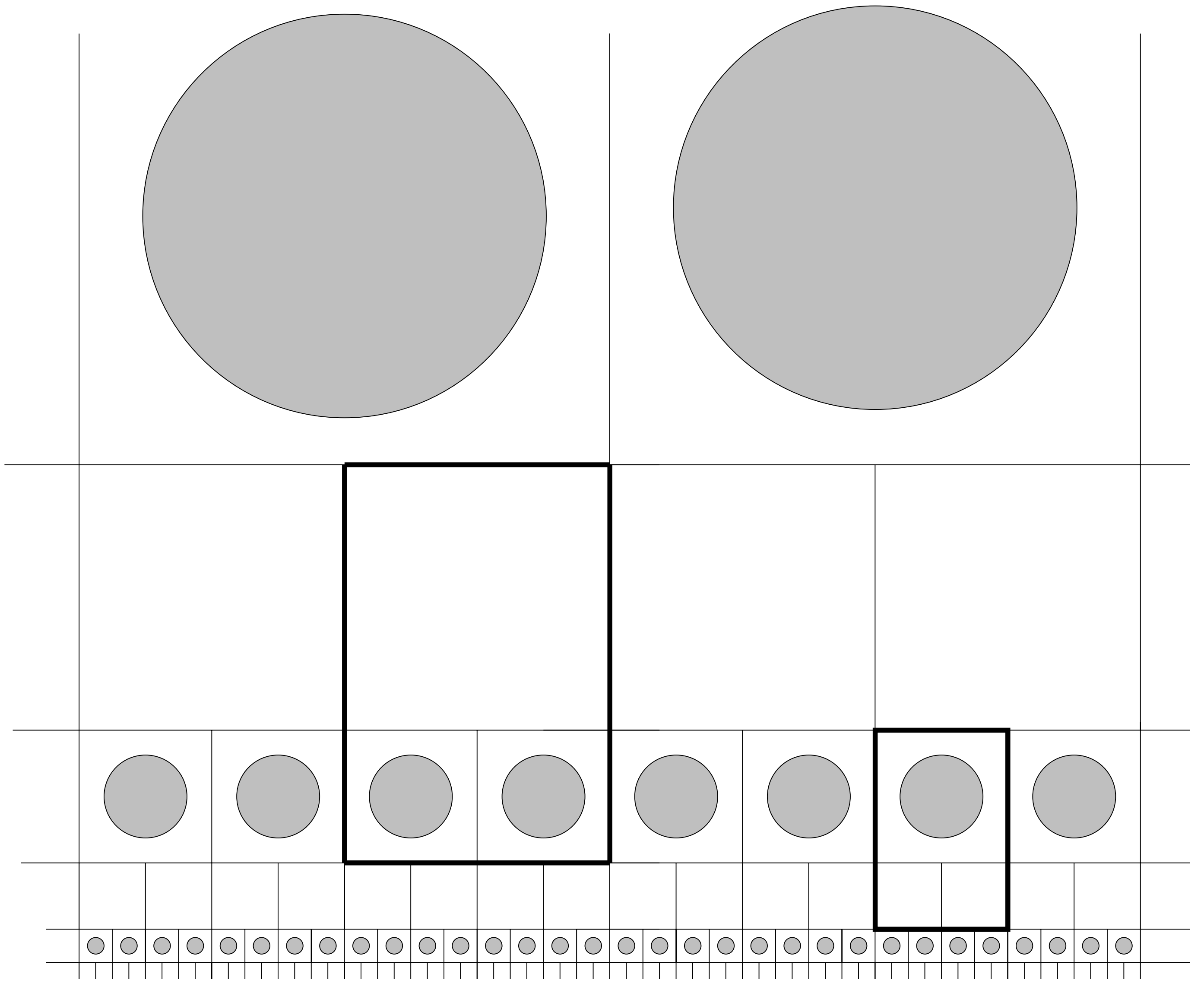


Figure 5: K. Böröczky's packing

Periodic Packings

- A periodic packing P is one whose symmetry group $G_P < Isom(\mathbb{H}^n)$ is cofinite.
- For a periodic packing, density exists independent of origin.
- For most radii r, there are no densest periodic packings by balls of radius r (Charles Radin and I).

Limits of Periodic Packings ?

- Naive approach fails.
- $\mu_P :=$ the uniform (probability) distribution on congruent copies of the packing P.
- $\{\mu_P\}_P$ has a limit point.
- "density" exists and has continuity properties.

Measures

(joint with Charles Radin, inspired by Oded Schramm)

- $\mathcal{P}_r :=$ space of radius r-sphere packings;
- $Isom(\mathbb{H}^n)$ acts on $\mathcal{P}_r$ in the natural way;
- $M_r :=$ space of probability measures on $\mathcal{P}_r$ invariant under $Isom(\mathbb{H}^n)$;
- For $\mu \in M_r$, $density(\mu) := \mu(\{$ packings P that cover the origin $\})$.

Tools

- μ is ergodic if it is not a nontrivial convex sum of two measures in M_r.
- If μ is ergodic then μ-almost all packings P have dense orbit in $supp(\mu)$.
- If μ is ergodic then for μ-almost any packing P, $density(P)$ exists independent of origin and equals $density(\mu)$.
- Ergodic theory results rely on work of A. Nevo and E. Stein.

More Tools

- M_r is compact in the weak* topology.

- $density : M_r \to [0, 1]$ is continuous.

- Let $D(r) = max_{\mu \in M_r} density(\mu)$. "optimal density function"

- $\mu \in M_r$, $density(\mu) = D(r) \leftrightarrow \mu$ is "optimally dense".

- An "optimally dense packing" P is a packing in the support of an optimally dense measure such that $density(P) = density(\mu)$ for every choice of origin.

Questions

1. What are the optimally dense measures??

2. How do the optimally dense measures vary with radius?

3. Is $D(r)$ the sup density over periodic radius r- sphere packings?

Continuity

Is there a continuous path of optimally dense measures parametrized by radius?

Theorem 1. *(continuity) The optimal density function D is continuous (in every dimension).*

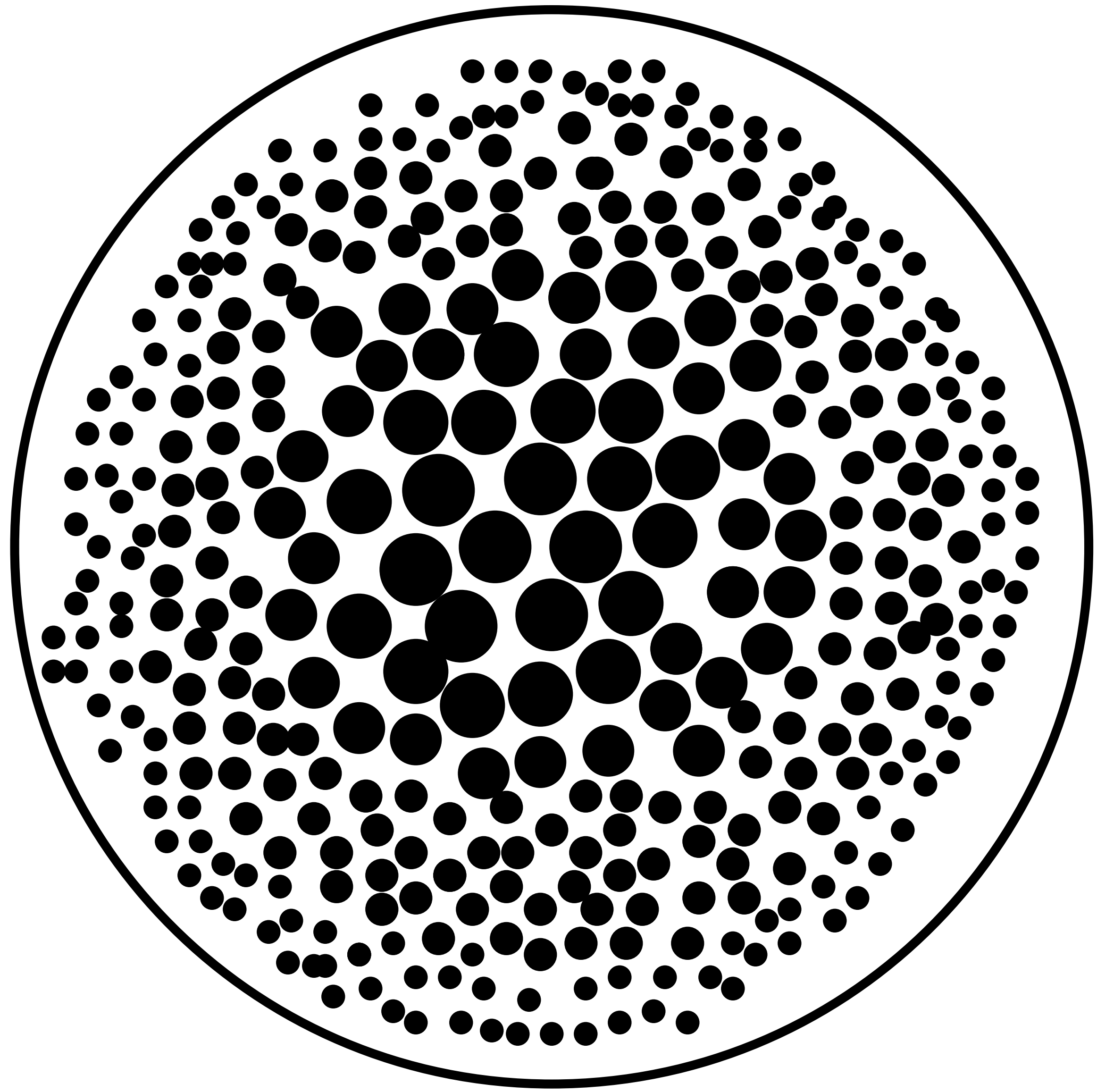


Figure 6: thick pea soup

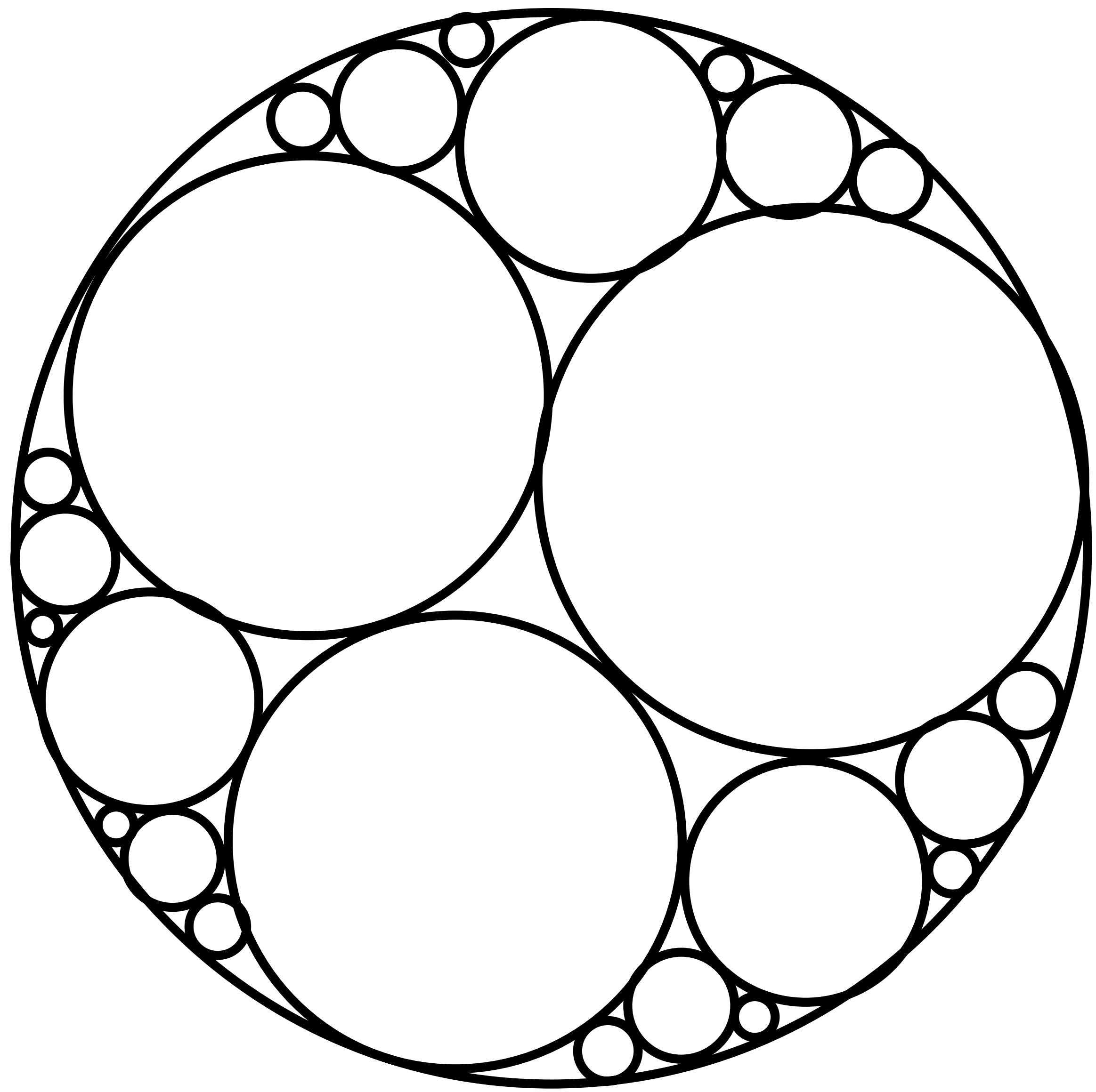
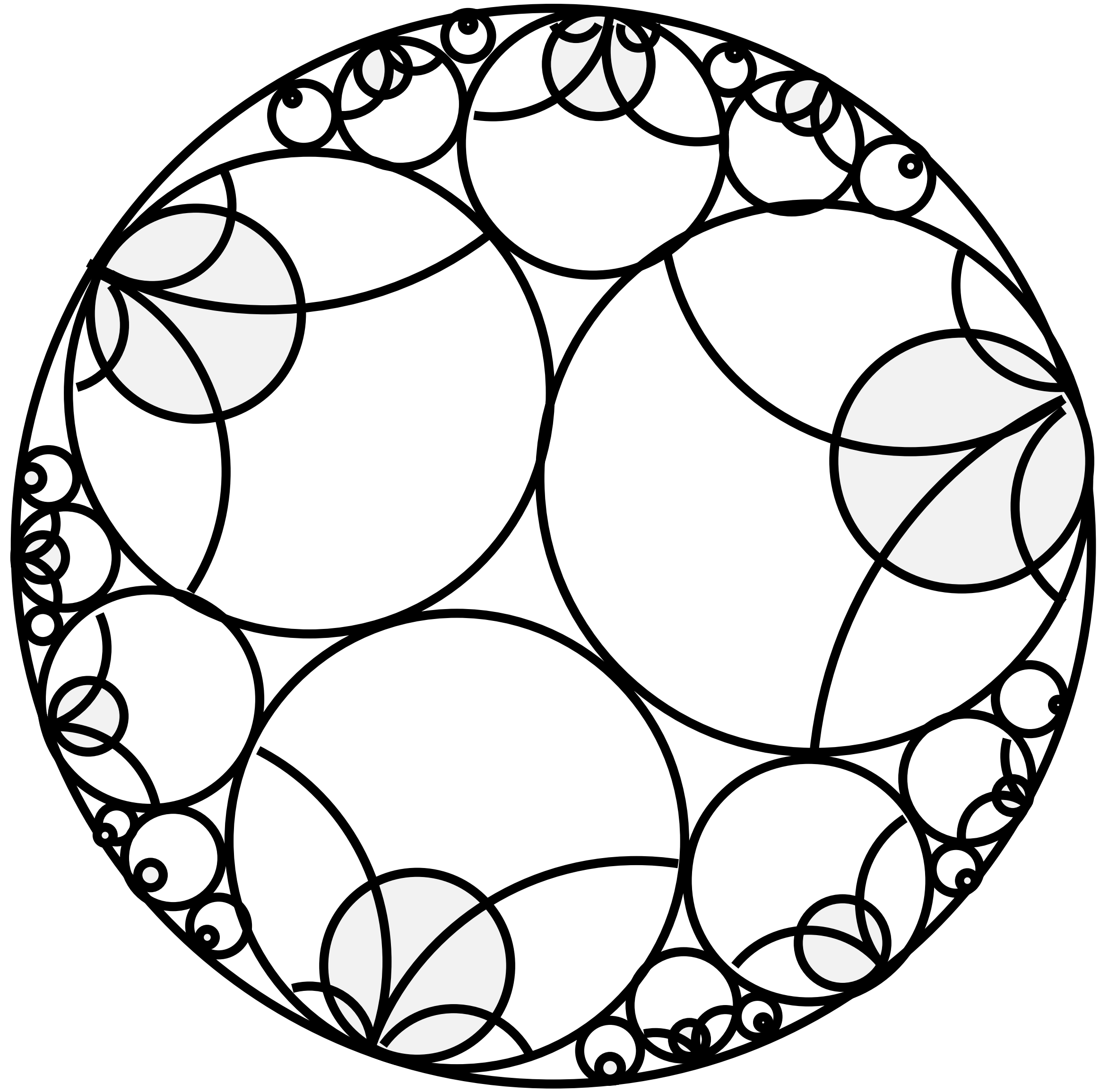


Figure 7: a horoball packing



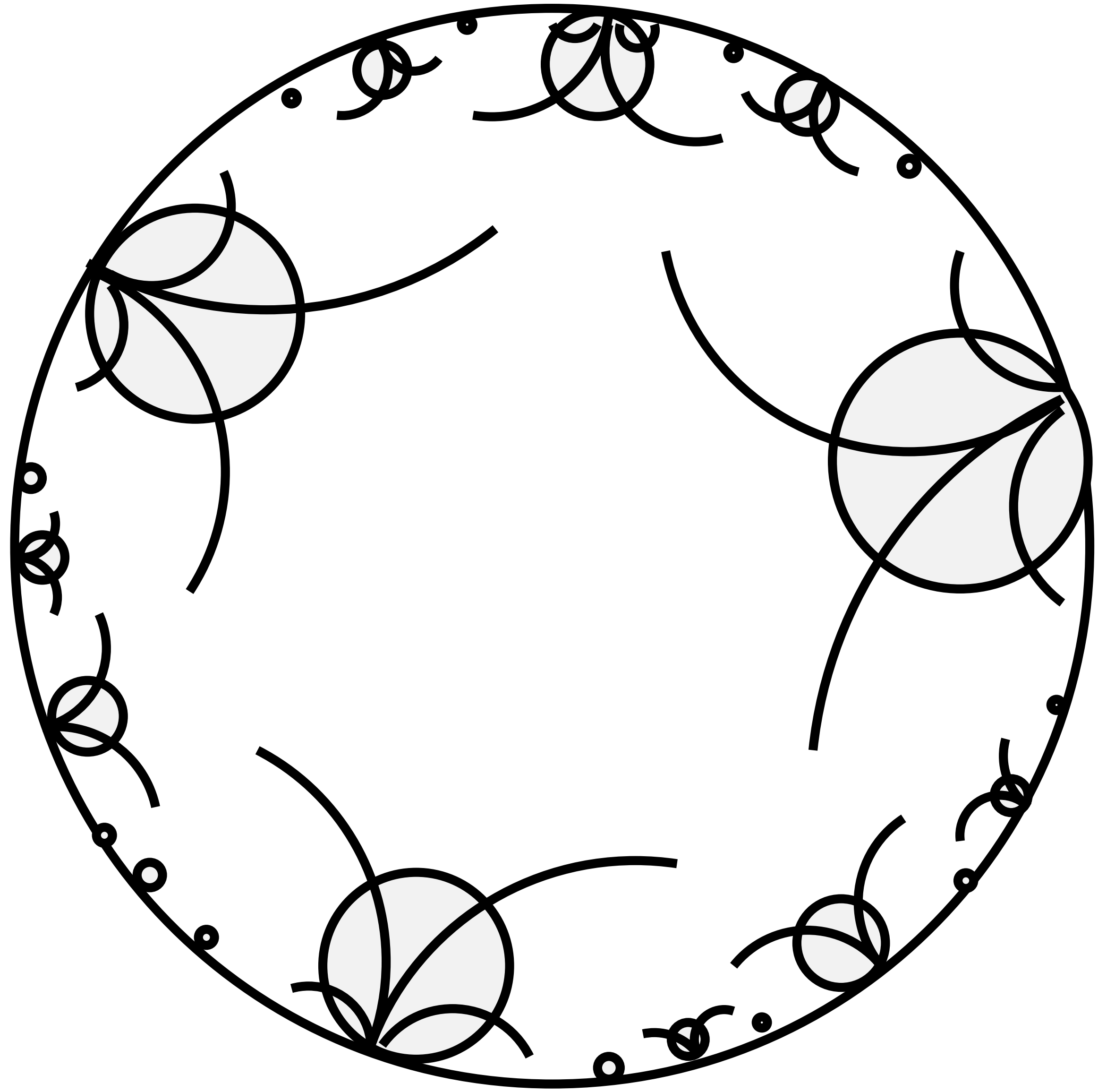


Figure 8: the region to be removed

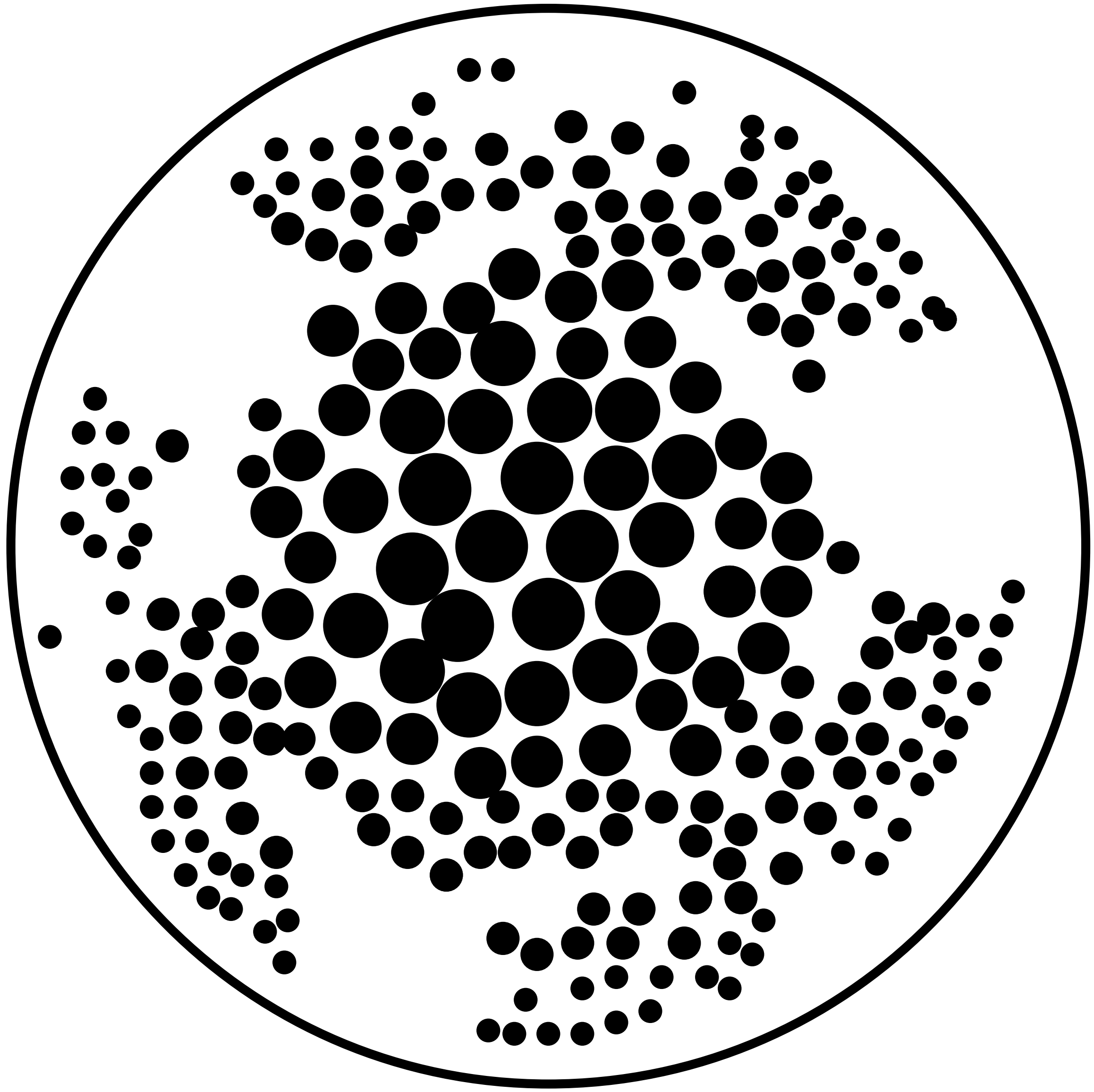


Figure 9: peas removed

Is $D(r)$ the sup over periodic packings?

Theorem 2. *Yes in dim* $= 2, 3$.

- Euclidean approach fails because of linear isoperimetric inequality in $\mathbb{H}^n$.

- Instead of sphere packings, consider packings by a set of polyhedra. If they have bounded diameter, then it's still true in Euclidean space. It's not true in $\mathbb{H}^n$; only known counterexample uses infinitely many polyhedra.

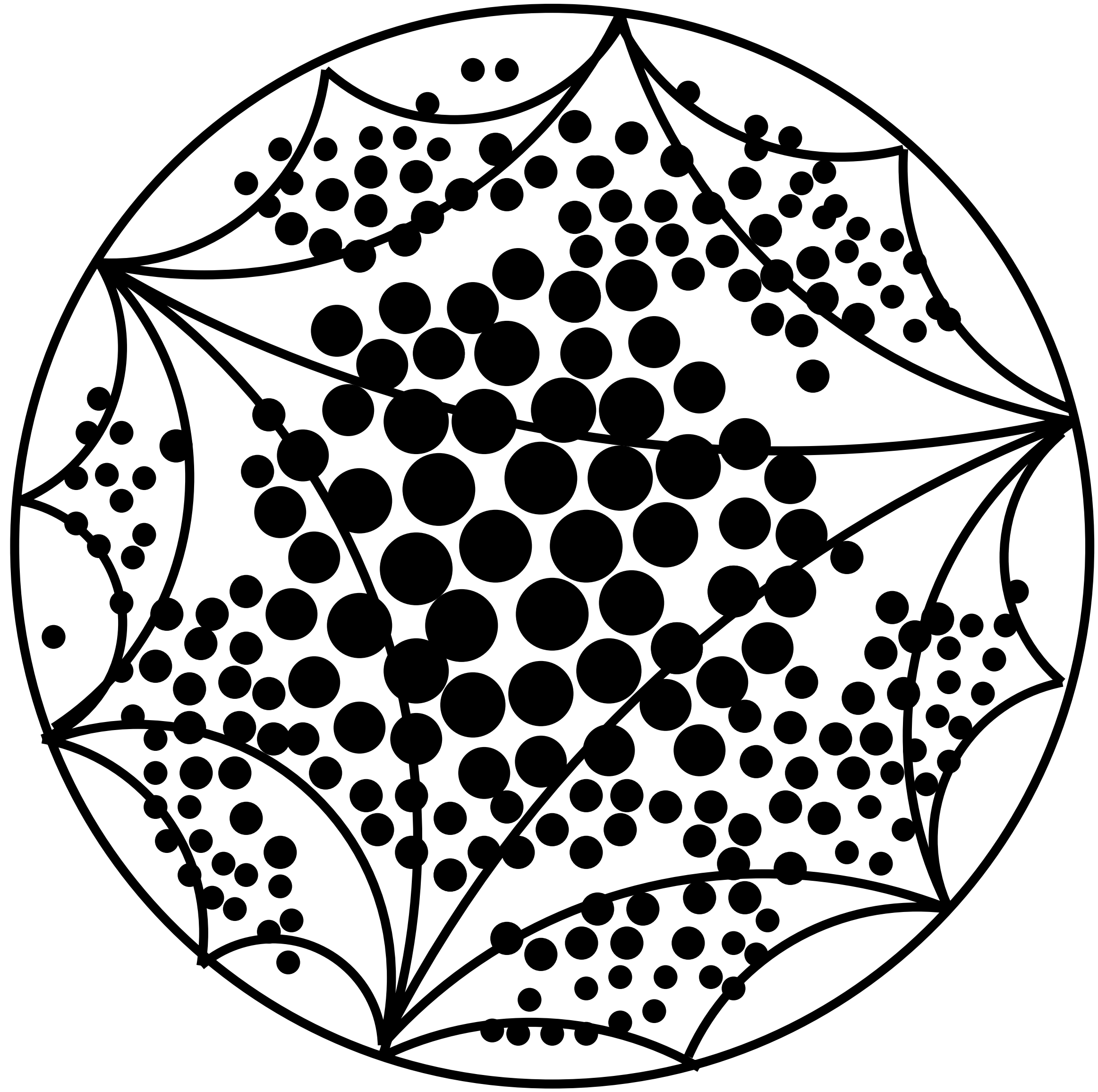


Figure 10: move peas a little so that they intersect each ideal triangle in only finitely many configurations

Symbolic Dynamics

- Let G be a group, K a finite set of colors.
- Let $\Sigma(G, K) = K^G$, the set of all colorings of G = the full shift in symbolic dynamics.
- G acts on $\Sigma(G, K)$ in the natural way.
- Let $M(G, K)$ be the space of G-invariant probability measures on $\Sigma(G, K)$

The Periodic Approximation Property

- $\mu \in M(G, K)$ is periodic $\leftrightarrow$ support(μ) is finite.
- G has the periodic approximation property $\leftrightarrow$ the set of periodic measures is dense in $M(G, K)$ (for all K).

Theorem 3. *If there exists a discrete cofinite group $G < Isom(\mathbb{H}^n)$ that has the periodic approximation property (PAP) then the answer is yes in dimension n.*

Groups with the Periodic Approximation Property

- $\mathbb{Z}^n$
- free groups $\rightarrow$ periodic approximability in dim. 2
- a mapping torus over a group with PAP $\rightarrow$ periodic approx. in dim. 3
- subgroups of groups with PAP $\rightarrow$ surface groups.
- It's not known if there are any cofinite groups in $Isom(\mathbb{H}^n)$ with PAP for $n > 3$.

Packings by polygons

- If we have a set of polygons with optimal density 1, what is the sup density over periodic packings?

- Assume they are based on a right-angled pentagon.

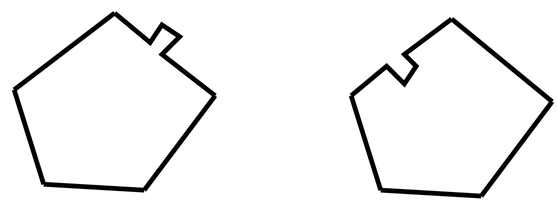


Figure 11: pentagons with bumps and dents

- If there are finitely many, then the sup density over periodic packings is 1.