APOLLONIAN PACKINGS:

GEOMETRY AND GROUP THEORY

JEFF LAGARIAS

AT&T LABS

FLORHAM PARK, NJ

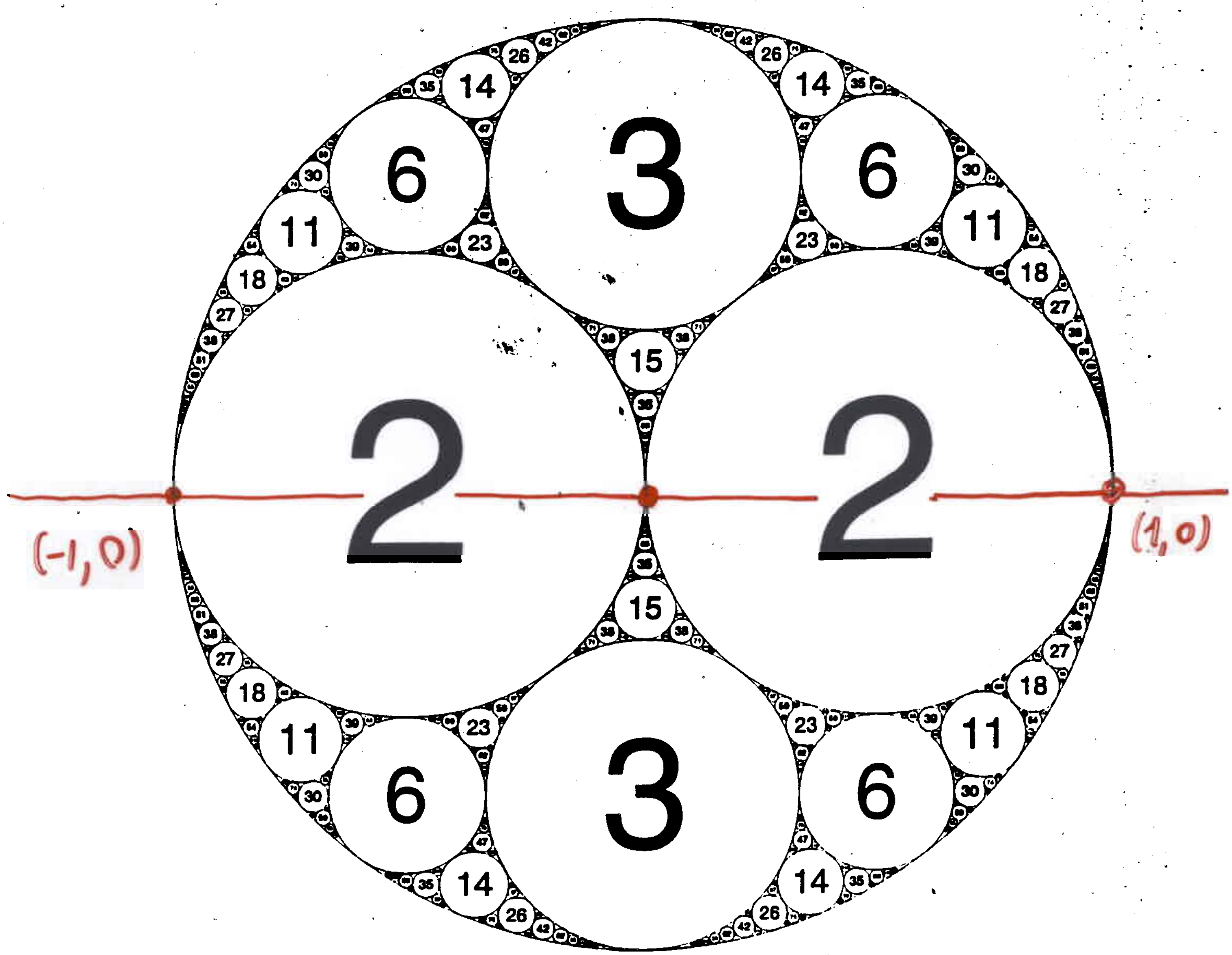
Joint work with:

RON GRAHAM [UCSD]

COLIN MALLOWS
ALLAN WILKS
} statisticians AT&T LABS

CATHERINE YAN [TEXAS A&M]

^Integral^ An Apollonian Circle Packing



- Integer Curvatures
- Rational Centers
- Curvature × Center in $\mathbb{Z}^2$

Observations:

- All curvatures in this Apollonian packing are integers.

- When origin (0,0) located at tangency point of circles labelled 2, then (it appeared) all circle centers are rational, and (in fact)

$$(\text{curvature}) \times (\text{center}) = \text{integers}$$

("Strongly integral Apollonian packing")

$\Longrightarrow$ Series of four papers by R.L. Graham, J.C. Lagarias, C.L. Mallows, A. Wilks and Catherine Yan (math.MG in 2000)

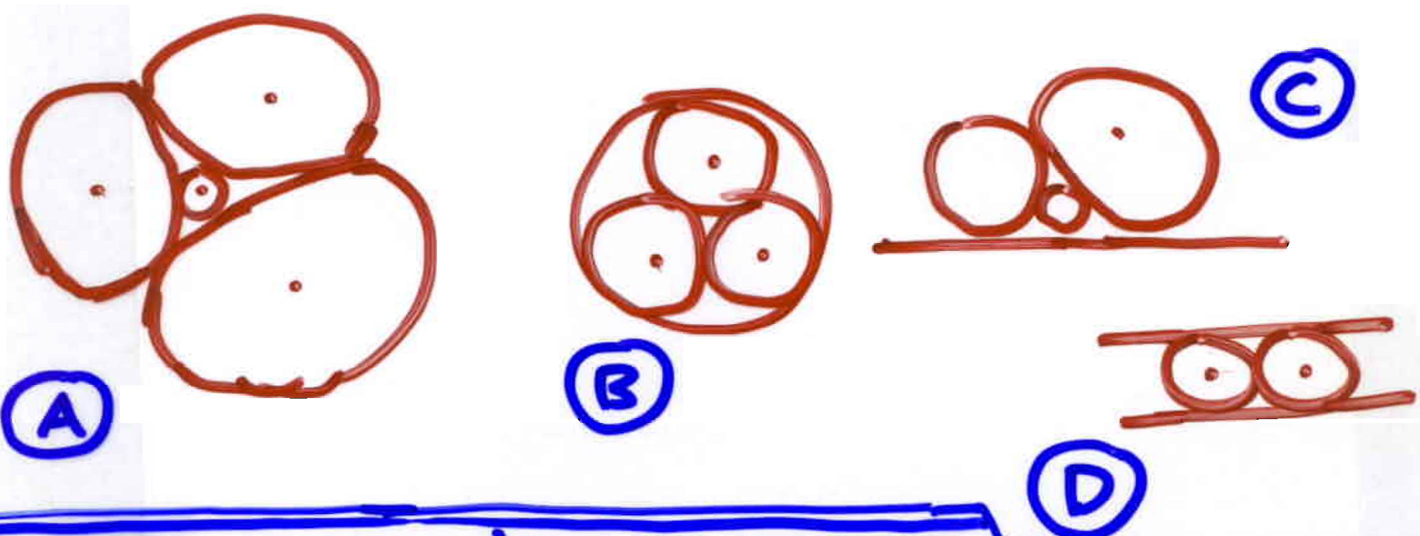
Topics

1. Descartes Circle Theorem
2. Apollonian Packings
3. Integral Apollonian Packings
4. Dual Apollonian Packings
5. Integral Apollonian Super-Gasket
6. Final Remarks

1. Descartes Circle Theorem

Descartes configuration.

Four mutually tangent circles in $\mathbb{R}^2$.



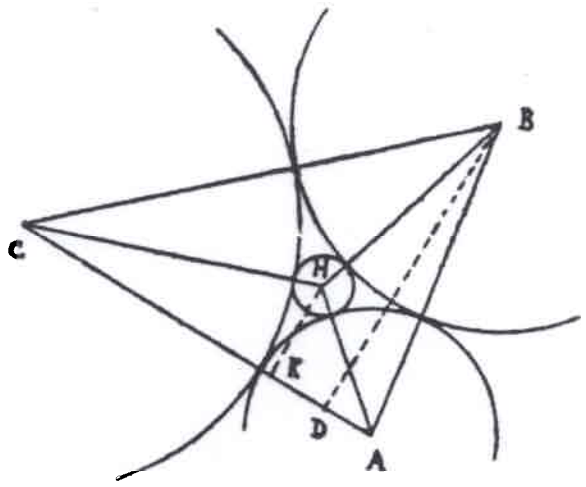
Descartes circle theorem (1643)

If $b_i = \frac{1}{r_i}$ is the curvature of i-th circle in configuration A, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2)$$

R. Descartes (1643)

trois AH, BH & CH, ce que n'ont pas les premieres. Et en ſuiuant le calcul auec ces ſix lettres, ſans les changer ny en adjoûter d'autres, par le chemin qu'a



pris Voſtre Alteſſe (car il eſt meilleur, pour cela, que celuy que i'auois propoſé), on doit venir à vne équation fort reguliere, & qui fournira vn Theoreme aſſez court. Car les trois lettres a, b, c, y ſont diſpoſées en meſme façon, & auſſi les trois d, e, f.

Mais, pour ce que le calcul en eſt ennuyeux, ſi Voſtre Alteſſe a deſir d'en faire l'eſſay, il luy ſera plus aiſé, en ſuppoſant que les trois cercles donnez s'entretouchent, & n'employant, en tout le calcul, que les quatre lettres d, e, f, x, qui eſtant les rayons des quatre cercles, ont ſemblable rapport l'vne à l'autre. Et, en premier lieu, elle trouuera

$$AK \propto \frac{dd + df + dx - fx}{d+f}, \quad \& \quad AD \propto \frac{dd + df + de - fe}{d+f},$$

où elle peut deſia remarquer que x eſt dans la ligne AK, comme e dans la ligne AD, pour ce qu'elle ſe trouue par le triangle AHC, comme l'autre par le triangle ABC. Puis enfin, elle aura cette équation[a],

$$\begin{array}{llll} & ddeeff & \infty & 2deffxx + 2deeffx \\ + & ddeexx & & + 2deefxx + 2ddeffx \\ + & ddffxx & & + 2ddefxx + 2ddeefx \\ + & eeffxx, & & \end{array}$$

de laquelle on tire, pour Theoreme, que les quatre ſommes, qui ſe produiſent en multipliant enſemble les quarrez de trois de ces rayons, font le double de ſix, qui ſe produiſent en multipliant deux de ces rayons l'vn par l'autre, & par les quarrez des deux autres; ce qui ſuffit pour ſeruir de regle à trouuer le rayon du plus grand cercle qui puiſſe eſtre décrit entre les trois donnez qui s'entretouchent. Car, ſi les rayons de ces trois donnez ſont, par exemple, $\frac{d}{2}\,\frac{e}{3}\,\frac{f}{4}$, i'auray 576 pour *ddeeff*, & 36*xx* pour *ddeexx*, & ainſi des autres. D'où ie trouueray

$$x \infty -\frac{156}{47} + \sqrt{\frac{31104}{2209}},$$

ſi ie ne me ſuis trompé au calcul que ie viens de faire.

Et Votre Alteſſe peut voir icy deux procedures fort differentes dans vne meſme queſtion, ſelon les differens deſſeins qu'on ſe propoſe. Car, voulant ſçauoir de quelle nature eſt la queſtion, & par quel biais on la peut ſoudre, ie prens pour données les lignes perpendiculaires ou paralleles, & ſuppoſe pluſieurs autres quantitez inconnuës, afin de ne faire aucune multiplication ſuperfluë, & voir mieux les plus courts chemins; au lieu que, la voulant acheuer, ie prens

a. Les ſignes + ſont omis devant les deux premières colonnes.

① J. Steiner, 1826 (Crelle J., vol I)

Es seien die drei Kreise M_1, M_2, M_3 (Fig. 36), welche einander be-
rühren, gegeben. Der Kreis m berühre sie äusserlich und der Kreis M
einschliessend. Die Radien der fünf Kreise M_1, M_2, M_3, M, m sollen
respective durch R_1, R_2, R_3, R, r bezeichnet werden.

②

Bemerkt man ferner, dass die Höhe eines Dreiecks durch die Seite desselben ausgedrückt werden kann, und dass die Seiten des Dreieck $M_1M_2M_3$ ihrer Grösse nach R_1+R_2; R_2+R_3; R_3+R_1 sind: so ha man z. B.

$$h_1 = \frac{\sqrt{2(R_1+R_2+R_3).2R_1.2R_2.2R_3}}{2(R_2+R_3)}$$
$$= \frac{2\sqrt{R_1R_2R_3(R_1+R_2+R_3)}}{R_2+R_3}.$$

Werden diese Werthe für h_1, h_2, h_3 in die obige Gleichung substituirt, so erhält man nach gehöriger Reduction folgende Gleichung:

$$(1) \qquad \frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + 2\sqrt{\frac{R_1+R_2+R_3}{R_1R_2R_3}}.$$

Diese Gleichung lehrt, wie man den Radius r desjenigen Kreises m, welcher drei gegebene, einander äusserlich berührende Kreise M_1, M_2, M_3, äusserlich berührt, aus den Radien R_1, R_2, R_3 der letzteren Kreise findet.

③

Um die Symmetrie zwischen den vier Grössen r, R_1, R_2, R_3, we in der Gleichung (1) vorkommen, leichter übersehen zu können, setzen

$$\frac{1}{r} = q; \quad \frac{1}{R_1} = q_1; \quad \frac{1}{R_2} = q_2; \quad \frac{1}{R_3} = q_3,$$

so dass nach Gleichung (1):

$$q = q_1+q_2+q_3+2\sqrt{q_1q_2+q_1q_3+q_2q_3}.$$

Daraus folgt:

$$(q-q_1-q_2-q_3)^2 = 4(q_1q_2+q_1q_3+q_2q_3),$$

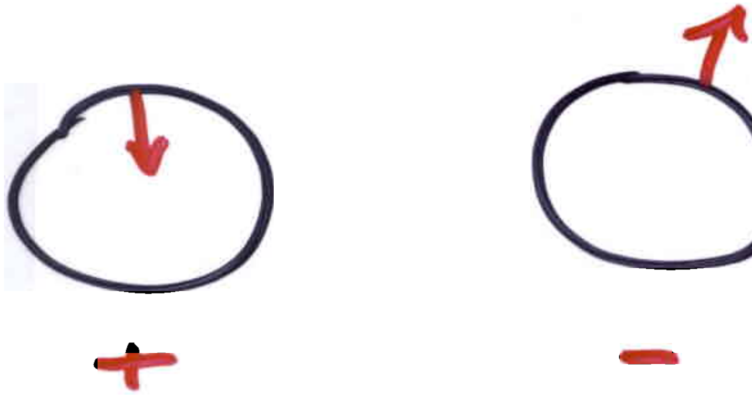
oder nach gehöriger Rechnung

$$\boxed{(8) \quad q^2+q_1^2+q_2^2+q_3^2-2(qq_1+qq_2+qq_3+q_1q_2+q_1q_3+q_2q_3) = 0.}$$

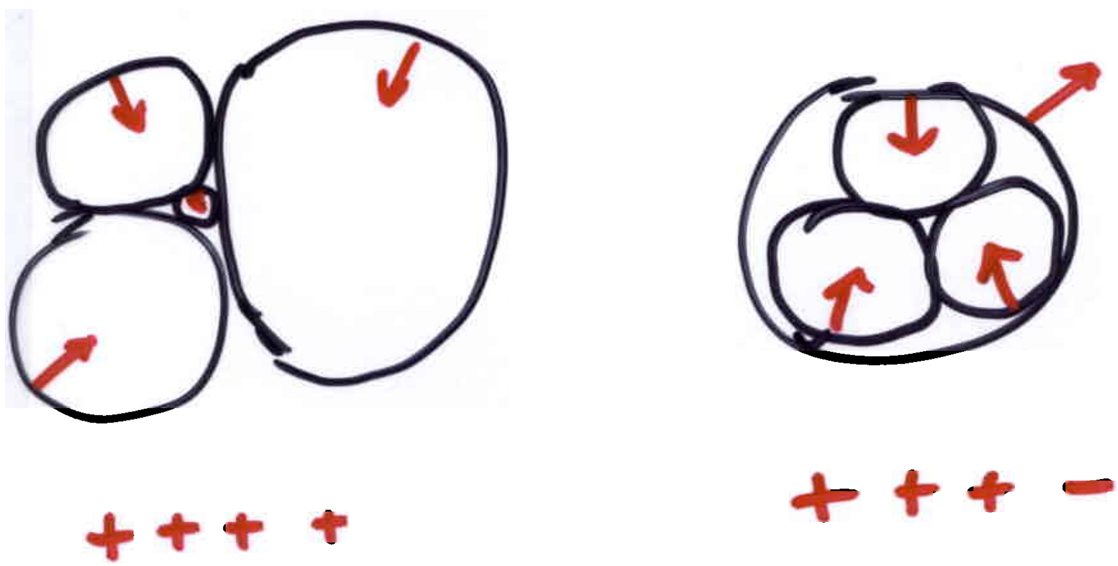
Setzt man ferner $\frac{1}{R}$ gleich Q, so findet man auf ähnliche Weise a der Gleichung (2) die folgende:

$$(9) \quad Q^2+q_1^2+q_2^2+q_3^2+2(Qq_1+Qq_2+Qq_3-q_1q_2-q_1q_3-q_2q_3) = 0$$

ORIENTED CIRCLES



ORIENTED DESCARTES CONFIGURATIONS



$\pm r$ = "oriented radius"

$b = \frac{1}{r}$ "oriented curvature"

DESCARTES CIRCLE THEOREM

Given an oriented Descartes configuration of four circles, having oriented curvatures ("bends")

$$b_i = \pm \frac{1}{r_i} \qquad r_i = \text{radius}$$

then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2$$

Soddy:

"The sum of the ~~bends~~ squares of the ~~squared~~ bends is half the square of the sum."

"Beyond Descartes":

- include the circle centers,

  not just the curvatures.

Circle Center: $z_j := x_j + i\, y_j \in \mathbb{C}$

complex number

COMPLEX DESCARTES CIRCLE THEOREM

Given an (oriented) Descartes configuration of four tangent circles in $\mathbb{R}^2$, then

$$(b_1 z_1)^2 + (b_2 z_2)^2 + (b_3 z_3)^2 + (b_4 z_4)^2$$
$$= \frac{1}{2}(b_1 z_1 + b_2 z_2 + b_3 z_3 + b_4 z_4)^2$$

Curvature - Center Coordinates

circle $C = \left(\frac{1}{r}, \underbrace{\frac{1}{r}x, \frac{1}{r}y}_{\text{curvature} \times \text{center}} \right)$

(signed) curvature

Form 4×3 matrix:

$b_i = \frac{1}{r_i}$ "signed curvature"

$$M = \begin{bmatrix} b_1 & b_1 x_1 & b_1 y_1 \\ b_2 & b_2 x_2 & b_2 y_2 \\ b_3 & b_3 x_3 & b_3 y_3 \\ b_4 & b_4 x_4 & b_4 y_4 \end{bmatrix}$$

Descartes Quadratic Form (matrix)

$$Q_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Theorem. (Extended Descartes Thm.)

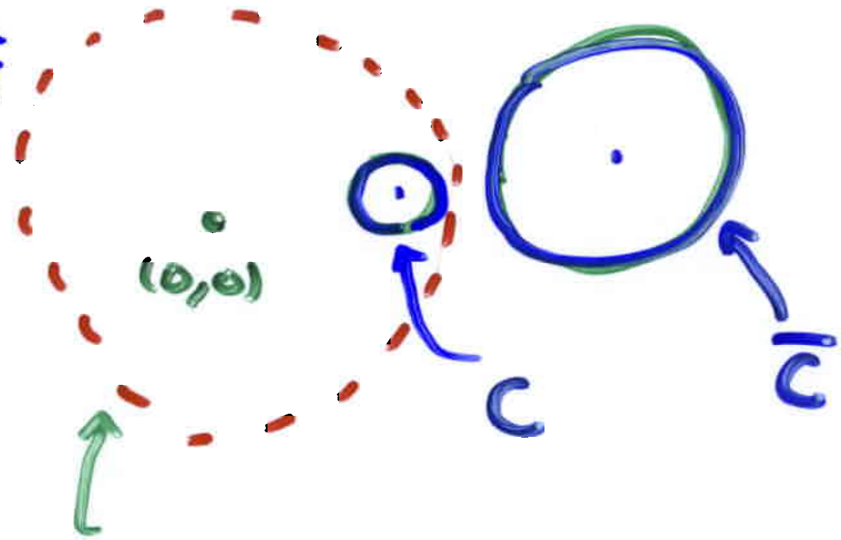
M is matrix of oriented Descartes config.

$$\Longleftrightarrow \quad M^T Q_2 M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Augmented Curvature – Center Coordinates

Inversion:

$$x \to \frac{x}{|x|^2} = \frac{1}{\bar{x}}$$



unit circle $|z| = 1$

r = radius (C)
(x_1, x_2) = center (C)

Augmented Coords. of C

$$\underline{m}(C) = \left(\frac{1}{\bar{r}}, \frac{1}{r}, \frac{x_1}{r}, \frac{x_2}{r} \right)$$

Coords of $\bar{C}$

$$\underline{m}(\bar{C}) = \left(\frac{1}{r}, \frac{1}{\bar{r}}, \frac{\bar{x}_1}{\bar{r}}, \frac{\bar{x}_2}{\bar{r}} \right)$$

$$= \left(\frac{1}{r}, \frac{1}{\bar{r}}, \frac{x_1}{r}, \frac{x_2}{r} \right)$$

Augmented Euclidean Descartes Theorem. To Descartes configuration $\mathcal{D} = \{C_1, C_2, C_3, C_4\}$ assign

$$M = \begin{bmatrix} \underline{m}(C_1) \\ \underline{m}(C_2) \\ \underline{m}(C_3) \\ \underline{m}(C_4) \end{bmatrix}$$

4×4 matrix

Then for

$$Q_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Descartes quadratic form

have identity:

$$\boxed{M^T Q_2 M = \begin{bmatrix} 0 & -4 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} =: \tilde{Q}_E \qquad (*)}$$

Conversely, any real M satisfying $(*)$ comes from a Descartes configuration.

Lorentz form

$$Q_{\mathcal{L}} = -w^2 + x^2 + y^2 + z^2$$

$$= \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Both Descartes form Q_2 and form Q_E are rationally equivalent to Lorentz form $Q_{\mathcal{L}}$.

- Descartes equation $\longleftrightarrow$ Light cone $-w^2+x^2+y^2+z^2=0$.

- Lorentz form $Q_{\mathcal{L}}$ has a large automorphism group of U with

$$U^T Q_{\mathcal{L}} U = Q_{\mathcal{L}}$$

Lorentz group $= O(3,1,\mathbb{R})$

6-diml. real Lie group.

Automorphisms of Descartes Configs.

$$M \longrightarrow U\, M\, V^{-1}$$

$U \in \mathrm{Aut}(Q_2)$

$V \in \mathrm{Aut}(Q_E)$

"Lorentzian" action

"Möbius" action

"conformal group"

$$\mathrm{Aut}(Q_2) \cong \mathrm{Aut}(Q_E) \cong O(3,1,\mathbb{R})$$

Group Actions on Descartes configurations:

- "Lorentz group" acts on left: mixes circles together

- Möbius group acts on right: circles treated separately

Descartes configuration: $M^T Q_2 M = Q_E$

$$M \longrightarrow U M V^{-1} =: M'$$

Lorentz action: $U^T Q_2 U = Q_2$

$V^T Q_E V = Q_E$ Möbius-action

- Both actions are transitive on Descartes configs!

Möbius group Möb(2)

2-dim'l conformal group

"double cover" of $PSL(2,\mathbb{C})$

6-dimensional real Lie group

$$z \longrightarrow \begin{cases} \dfrac{az+b}{cz+d} \\[1ex] \text{complex conjugate of this.} \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} \end{cases}$$

- acts on $\mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
  $\cong S^2$.

- takes circles to circles.

$$SL(2,\mathbb{C})/\pm I \xrightarrow{\cong} \text{Möb}(2)^+$$

$$\text{Möb}(2)^+ \cong O(3,1,\mathbb{R})^{\uparrow +}$$

Ex. Möbius group (4×4 repn.)

$$z \longrightarrow \frac{1}{z}$$

acts as

$$M \longmapsto M\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = MV^{-1}$$

[interchanges first & second columns of M]

$\text{Möb}(2) \cong$ isochronous Lorentz gp $\mathcal{L}_4^{\uparrow}$

$$\text{Möb}(2) = PSL(2,\mathbb{C}) \cup PSL(2,\mathbb{C}) \circ \bar{z}$$

$$\mathcal{L}_4^{\uparrow} = \underbrace{\mathcal{L}_4^{\uparrow +}}_{\boxed{\det = +1}} \cup \underbrace{\mathcal{L}_4^{\uparrow -}}_{\boxed{\det\ -1}}$$

$$\pm\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} \mathrm{Re}(a\bar{d}+b\bar{c}) & \mathrm{Im}(a\bar{d}+b\bar{c}) & \mathrm{R}(a\bar{b}-c\bar{d}) & \\ & & \mathrm{Re}(a\bar{b}+c\bar{d}) & \\ \mathrm{Im}(-a\bar{d}+b\bar{c}) & \mathrm{R}(a\bar{d}-b\bar{c}) & \mathrm{Im}(-a\bar{b}+c\bar{d}) & \\ & & \mathrm{Im}(-a\bar{b}-c\bar{d}) & \\ \mathrm{Re}(a\bar{c}-b\bar{d}) & \mathrm{Im}(a\bar{c}-b\bar{d}) & \frac{1}{2}\left(\begin{matrix}|a|^2-|b|^2 \\ -|c|^2+|d|^2\end{matrix}\right) & \frac{1}{2}(-\cdots \\ \mathrm{Re}(a\bar{c}+b\bar{d}) & \mathrm{Im}(a\bar{c}+b\bar{d}) & \frac{1}{2}\left(\begin{matrix}|a|^2+|b|^2 \\ -|c|^2-|d|^2\end{matrix}\right) & \frac{1}{2}(-\cdots \\ & & \frac{1}{2}\left(|a|^2+|b|^2 + |c|^2+|d|^2\right) & \end{bmatrix}$$

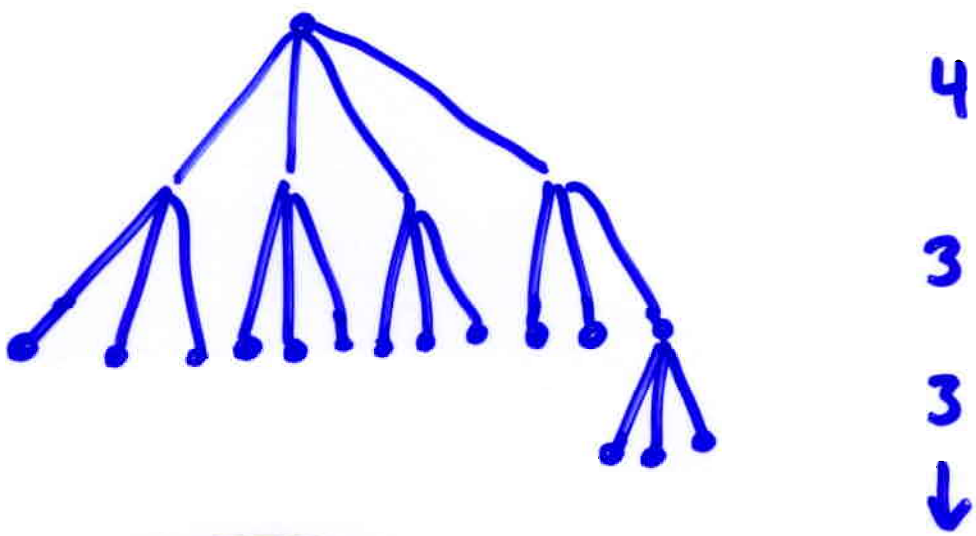
$PSL(2,\mathbb{C})$ $\qquad\qquad$ $\mathcal{L}_4^{\uparrow +}$

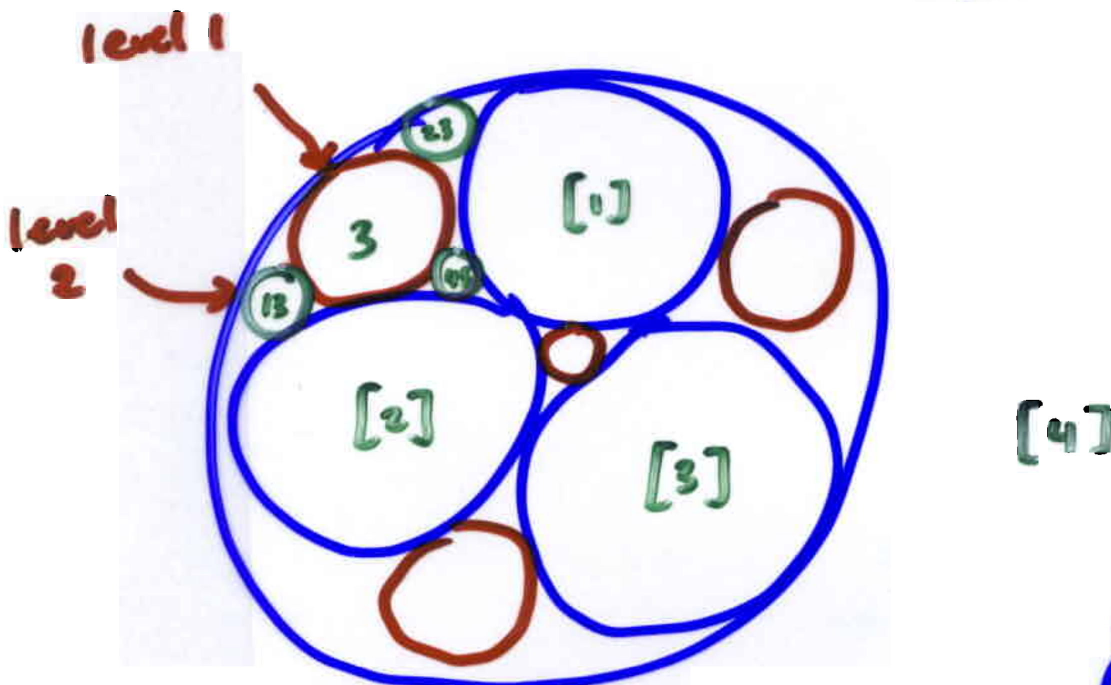
$$\bar{z} \longmapsto \mathrm{diag}[-1,-1,-1,+1]$$

Walker (1981) Theorem 10.

2. Apollonian Packings

- Start with (enclosing) Descartes configuration.
- Recursive construction
- Fill in tangent circles successively:





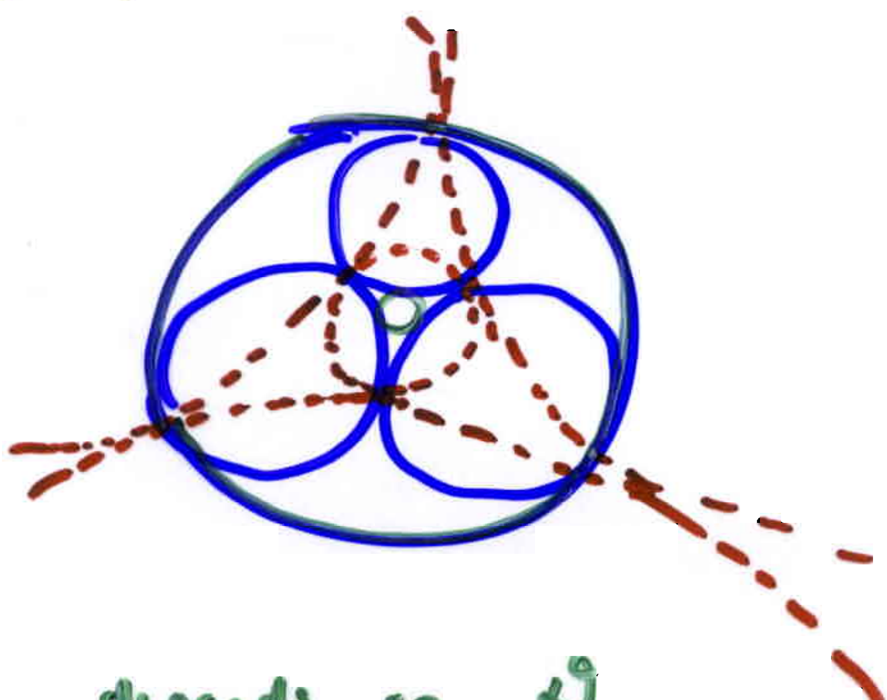
$\mathcal{P}$:= completed packing

Möbius group viewpoint.

An Apollonian packing $\mathcal{P}$ is the set of circles obtained from an intial Descartes configuration $\mathcal{D}$ by action of [discrete] group of [Möbius] inversions

$$G_{\mathcal{D}} := \langle \underset{\sim}{g_1}, \underset{\sim}{g_2}, \underset{\sim}{g_3}, \underset{\sim}{g_4} \rangle$$

in 4 circles going through three tangency points of $\mathcal{D}$



- $G_{\mathcal{D}}$ <u>depends</u> <u>on</u> <u>$\mathcal{D}$</u>.

Lorentz group viewpoint

The set of Descartes configurations in an Apollonian packing ("Apollonian gasket") is constructed from an initial configuration $\mathcal{D}$ by a fixed discrete subgroup Γ of "Lorentz group" $\mathrm{Aut}(Q_2)$ $\boxed{\text{"Apollonian Group"}}$

$$\Gamma := \left\langle \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix} \right\rangle.$$

- Γ responsible for "integer structure" on packings.

- **Integral Structure**

  comes from Apollonian group.

(1) Apollonian packing = an orbit of Apollonian group Γ,

$$\Gamma \cdot M_0$$

(2) $$\Gamma = \left\langle \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix} \right\rangle$$

$$M_0 = \begin{bmatrix} \bar{c}_1 & c_1 & c_1 x_{11} & c_1 x_{12} \\ \bar{c}_2 & c_2 & c_2 x_{21} & c_2 x_{22} \\ \bar{c}_3 & c_3 & c_3 x_{31} & c_3 x_{32} \\ \bar{c}_4 & c_4 & c_4 x_{41} & c_4 x_{42} \end{bmatrix}$$

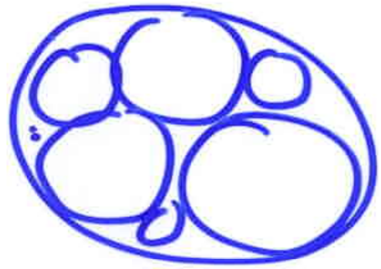
- If $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ integral $\Rightarrow$ all curvatures in packing integral

- If M_0 integral $\Rightarrow$ strongly integral packing

  $\boxed{\text{"everything integral"}}$

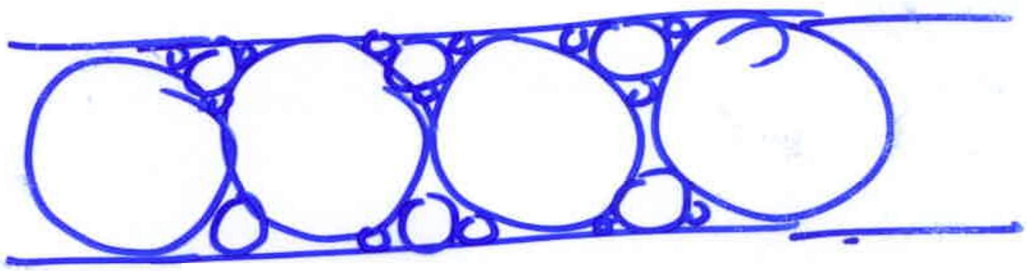
Apollonian Circle Packing : 4 Types

(A) Bounded



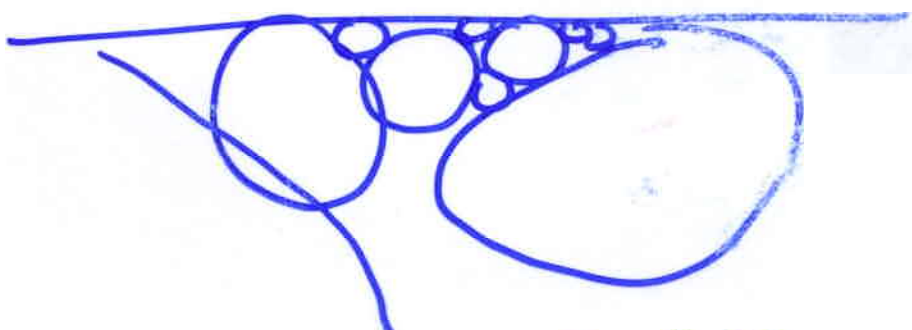
$a < 0$

(B) Double Semibounded



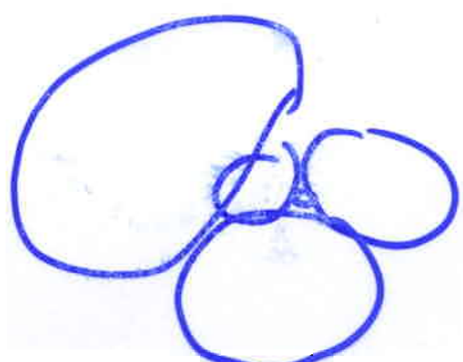
$a = b = 0$

(C) Semibounded



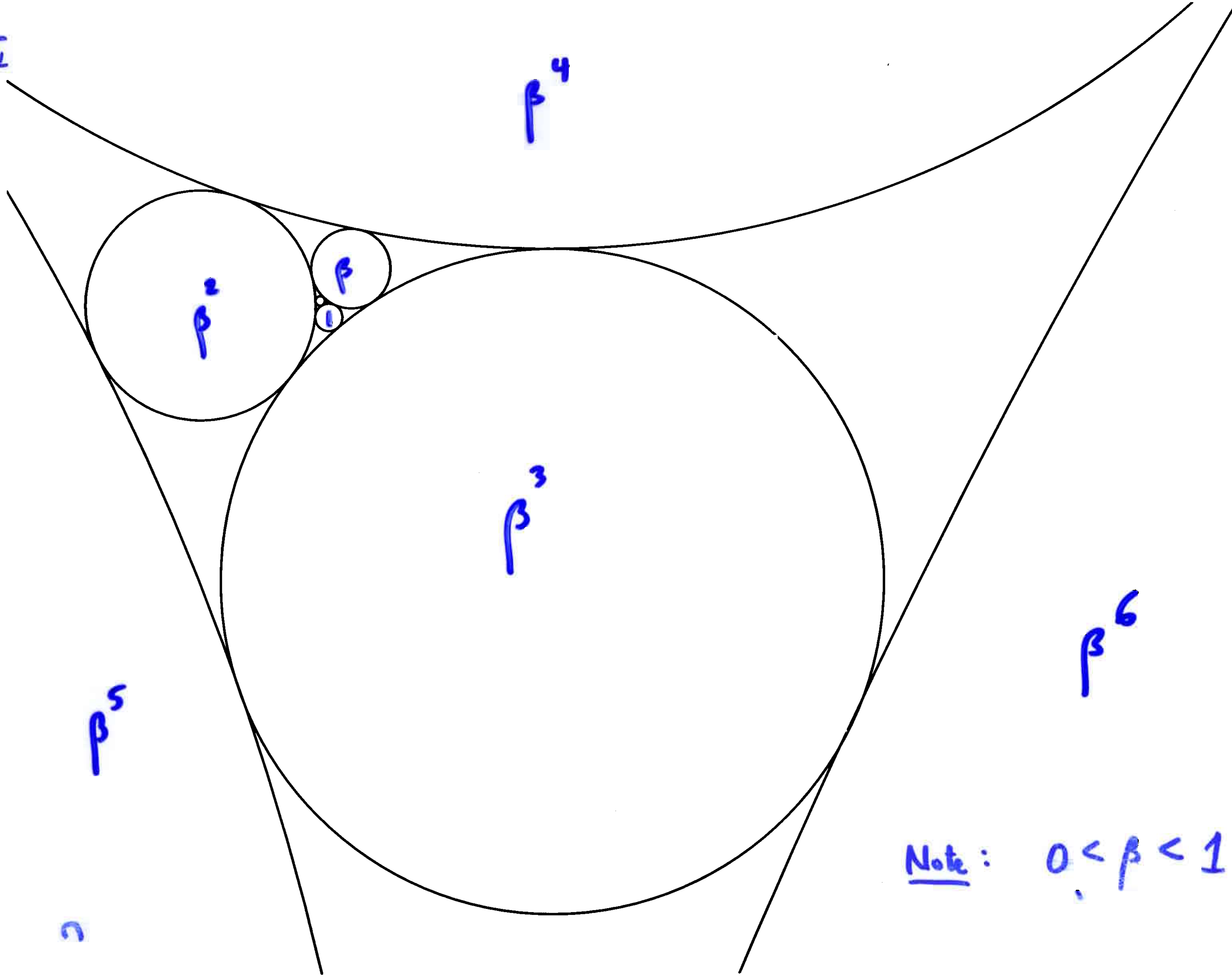
$a = 0 \quad b > 0$

(D) Unbounded



$a > 0$ always

$\lim a \to 0$



<u>Note</u>: $0 < \beta < 1$.

Hausdorff Dimension

Theorem. (David Boyd 1982)

The Hausdorff dimension $\bar{d}$ of the "uncovered set" of Apollonian circle packing is independent of packing. and

$$1.300 \leq \bar{d} \leq 1.3145$$

Physics: $1.3056\overline{86729}$

↑ Thomas & Dhar 1994

Dirichlet series

$$D(s) = \sum_{\substack{\text{circles} \\ r_i}} r_i^{-s}$$

Fact. $\bar{d}$ = abscissa of absolute convergence of $D(s)$.

Apollonian Group Encodes Hausdorff dim. $\bar{d}$.

Theorem. [GLMWY]

Let $\|M\| = \left(\sum_{i=1}^{4} \sum_{j=1}^{4} |M_{ij}|^2 \right)^{1/2}$.

measure size of 4 x 4 matrix.

Then

$$\#\{ \text{elts. } M \text{ of Apollonian group } \Gamma \text{ with } \|M\| < T \} = T^{\bar{d} + o(1)}$$

as $T \to \infty$, where

$\bar{d}$ = Hausdorff dim. of limit set.

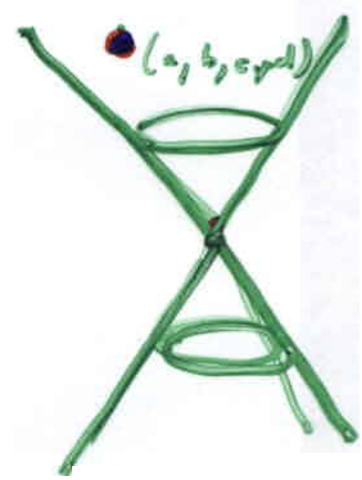
Geometric Pictures

1. Apollonian packing [2-diml.]

- Circle packing on $\mathbb{R}^2$
  $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$

2. Apollonian gasket [3-diml.]

- Point set (a, b, c, d) on Lorentzian null cone $-w^2 + x^2 + y^2 + z^2 = 0$



3. Super-gasket [6-diml.]

- Point set $\{M_i\}$ in: augmented curvature-center-coordinate space

$\left\{ \begin{array}{c} 4\times 4 \text{ real} \\ \text{matrices} \end{array} \right\} \supseteq O(3,1,\mathbb{R})^{\uparrow}$

3. Integral Apollonian Circle Packings

- Means: all curvatures are integers.

- Reduction algorithm always halts at a root quadruple

  $(0, 0, 1, 1)$

  or

  $(-n, a, b, c)$ $\quad 0 < a \leq b \leq c$

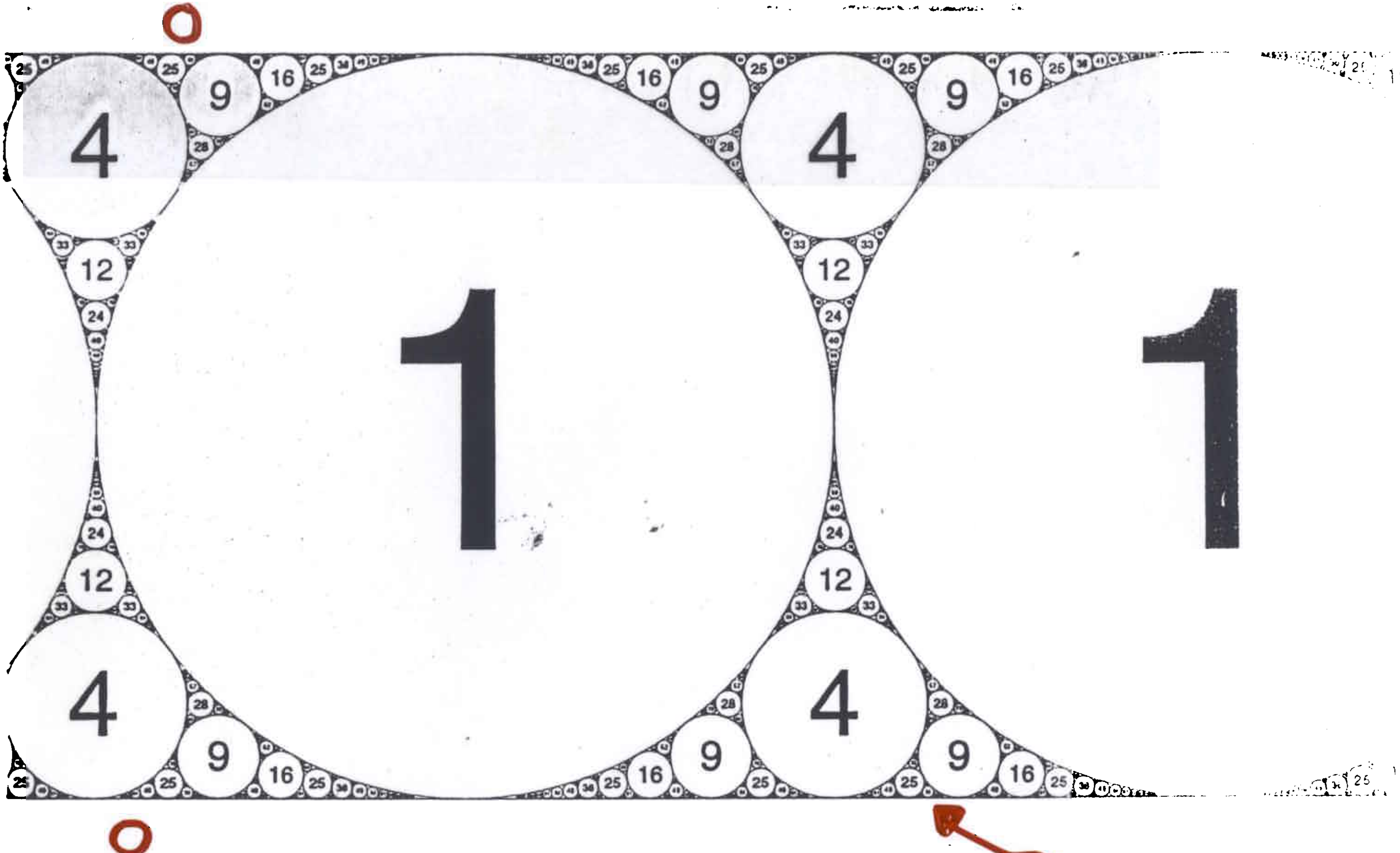
  with

  $-n < 0 < -n+a+b-c \leq -n+a \leq -n+b.$

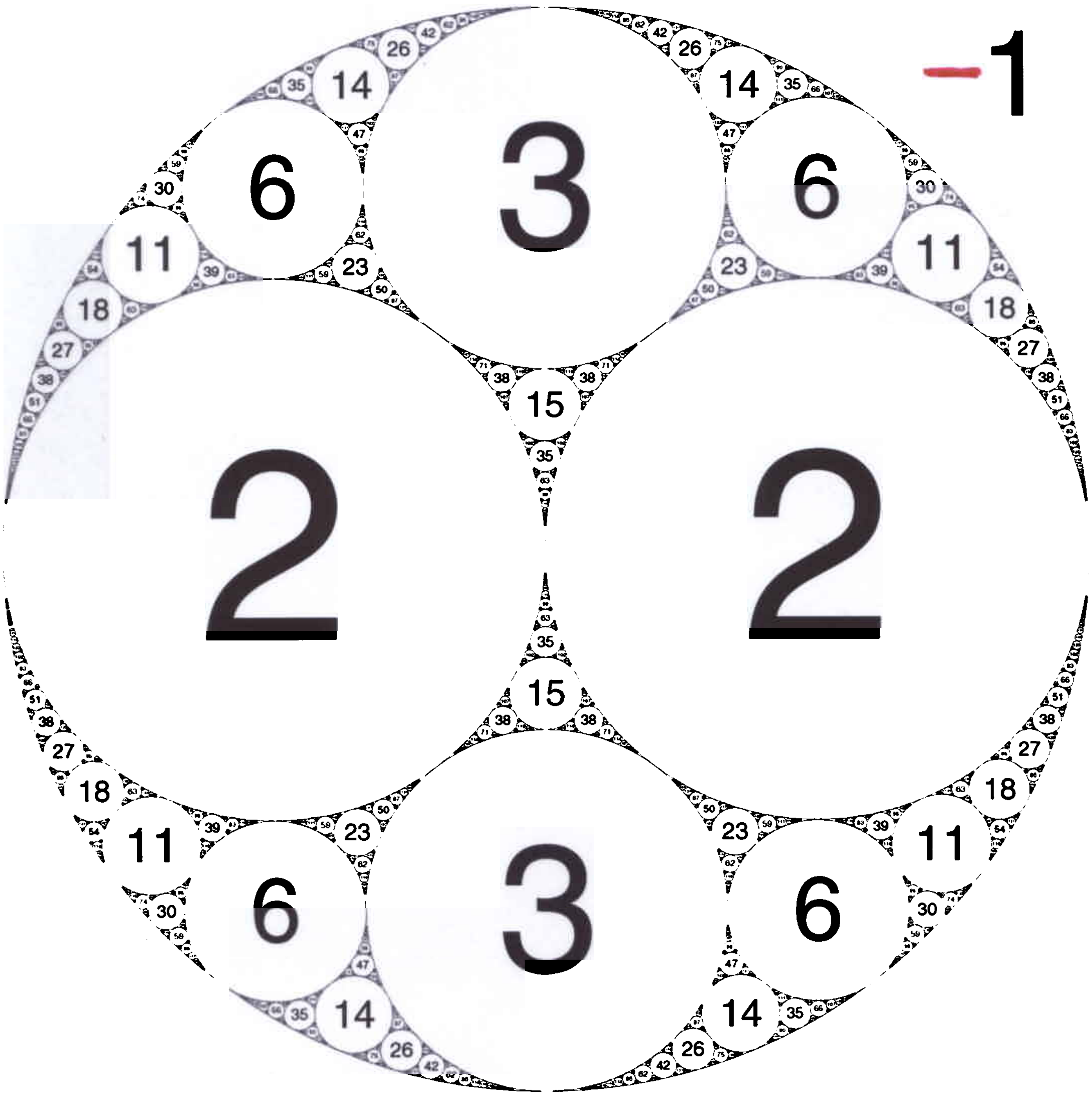
- All integer solns. of Descartes eqn.

  $$-a^2-b^2-c^2-d^2+2ab+2ac+2bc+2ad+2bd+2cd=0$$

  appear among these packings.

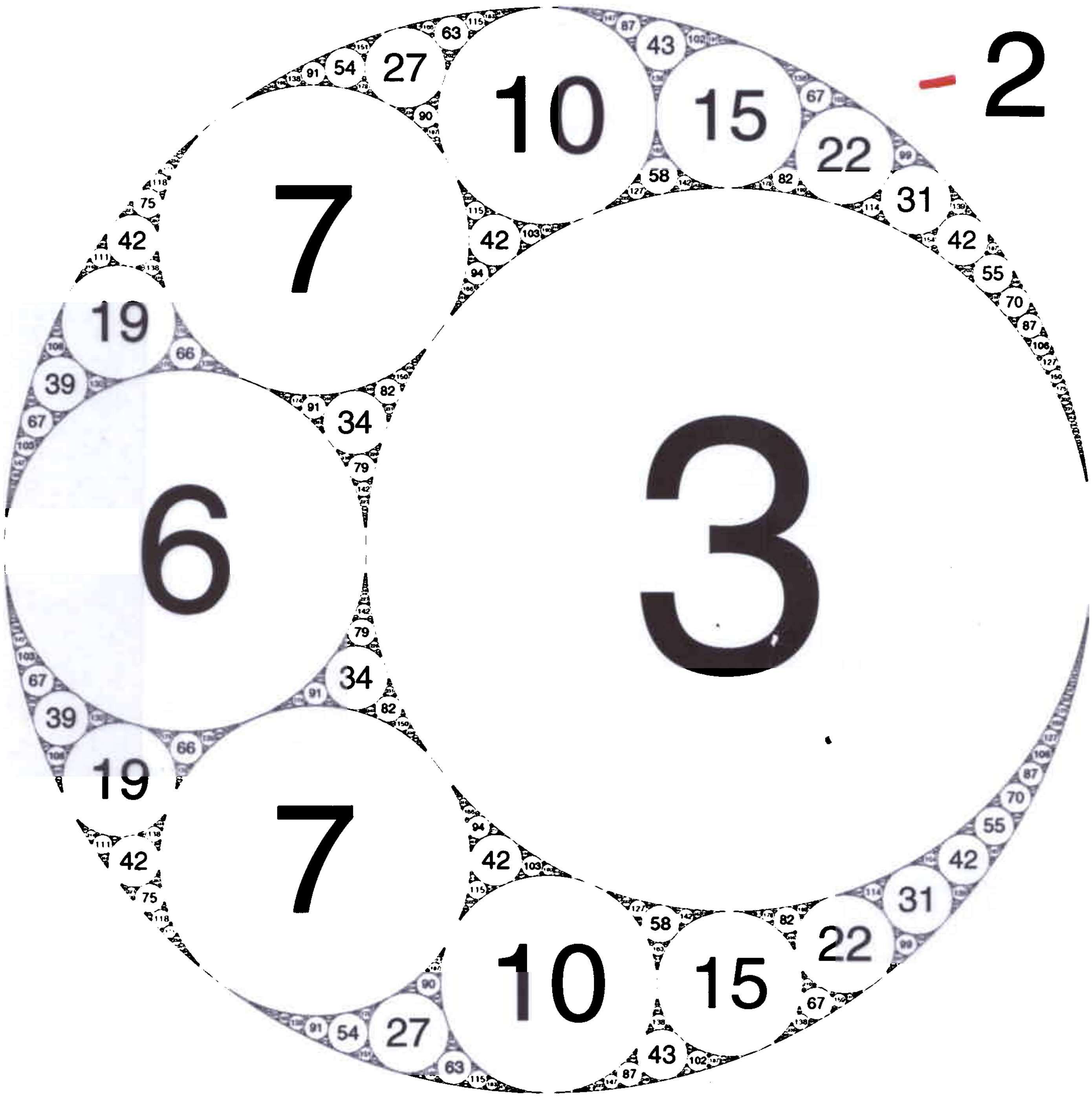


(0,0,1,1) PACKING ("Ford Circles")

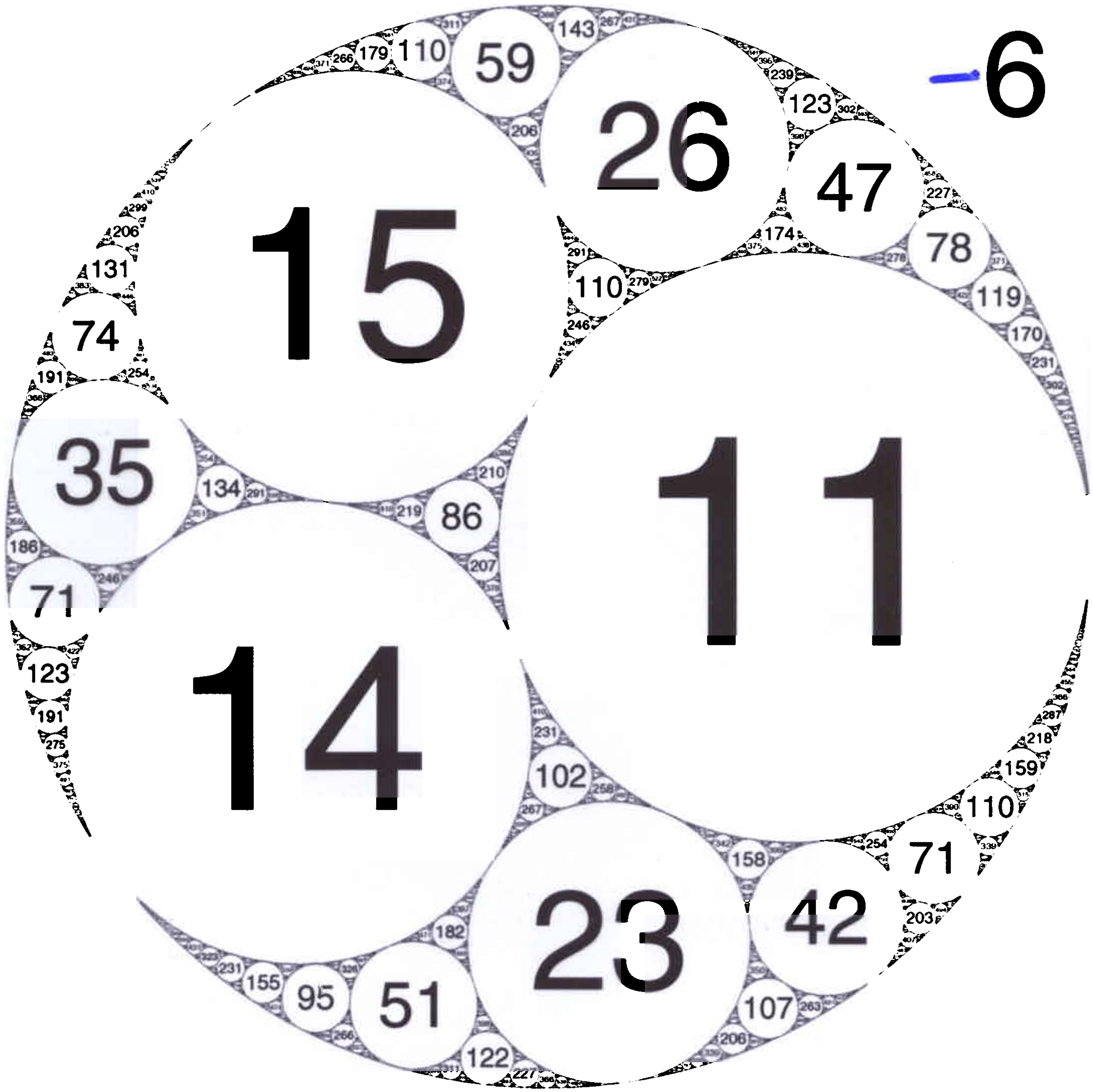

3
6
6
2
2
15
15
3
6
6
14
14
26
26
35
35
42
42
47
47
11
11
18
18
23
23
27
27
30
30
38
38
38
38
39
39
50
50
11
11
18
18
23
23
27
27
30
30
38
38
38
38
39
39
50
50
14
14
26
26
35
35
35
35
42
42
47
47

−1

(-1, 2, 2, 3)


-2
3
6
7
7
10
10
15
15
19
19
22
22
27
27
31
31
34
34
39
39
42
43
43
54
54
55
55
58
58
63
63
66
66
67
70
79
82
87
90
91
94
103
106
115

(-2, 3, 6, 7)



-6
15
11
14
26
23
35
47
59
78
42
51
86
74
71
95
102
107
110
119
122
123
131
134
143
155
158
159
170
174
179
182
186
191
203
206
207
210
218
219
227
231
239
246
254
258
266
267
275
287
291
302
311

(-6, 11, 14, 15)

<u>Theorem A.</u>

The number of primitive integer root quadruples with least elt $-n$ equals the <u>class number</u>

$h^{\pm}(-4n^2)$ = $GL(2,\mathbb{Z})$-equivalence classes of binary quadratic forms of discriminant $-4n^2$.

$h(-4n^2)$ = $SL(2,\mathbb{Z})$-equivalence classes of same

(<u>usual</u> class number)

$\Rightarrow$ ($GL(2,\mathbb{Z})$ = det ± 1.)

$$h^{\pm}(-4n^2) = \frac{1}{2}\left[h(-4n^2) + a(-4n^2)\right]$$

$a(-4n^2)$: #(ambiguous forms)

Which integers occur in Integer Apollonian packing?

- (Depends on packing)
- Congruence restrictions (mod 24).

[For (0,0,1,1)-packing

$$n \equiv 0, 1, 4, 9, 12, 16 \pmod{24}$$]

- Experimental Data:

  All "sufficiently large" integers not excluded by congruence conditions occur.

- (Hard) Conjecture: Positive fraction of integers occur.

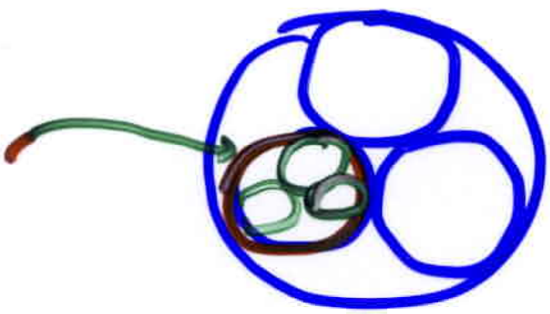
4. Dual Apollonian Packings

- Start with Descartes configuration $\mathcal{D}$

(1) Möbius form:

$$G^{\perp} = \langle \underset{\sim}{g_1^{\perp}}, \underset{\sim}{g_2^{\perp}}, \underset{\sim}{g_3^{\perp}}, \underset{\sim}{g_4^{\perp}} \rangle$$

inversions



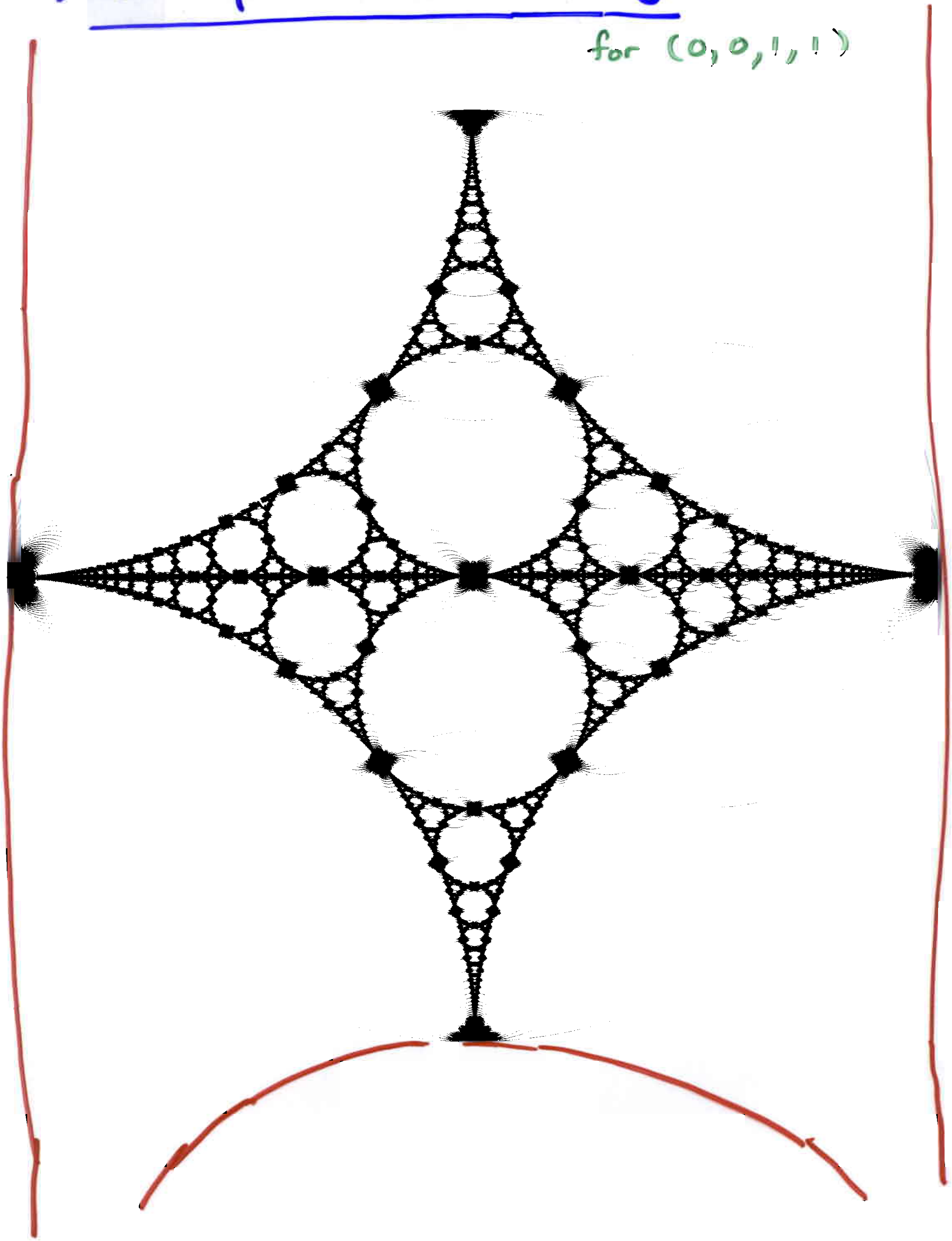
(2) Lorentz form

Dual Apollonian group $\Gamma^{\perp}$

$$\Gamma^{\perp} := \left\langle \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\rangle = \Gamma^{T}$$

Dual Apollonian Packing

for (0, 0, 1, 1)



5. Integral Apollonian Super-Gasket

- strongly integral packing $\iff$ curvature-center matrix M is integral

- Super-Apollonian Group

$$\mathcal{A}_S = \langle \Gamma, \Gamma^{\perp} \rangle$$

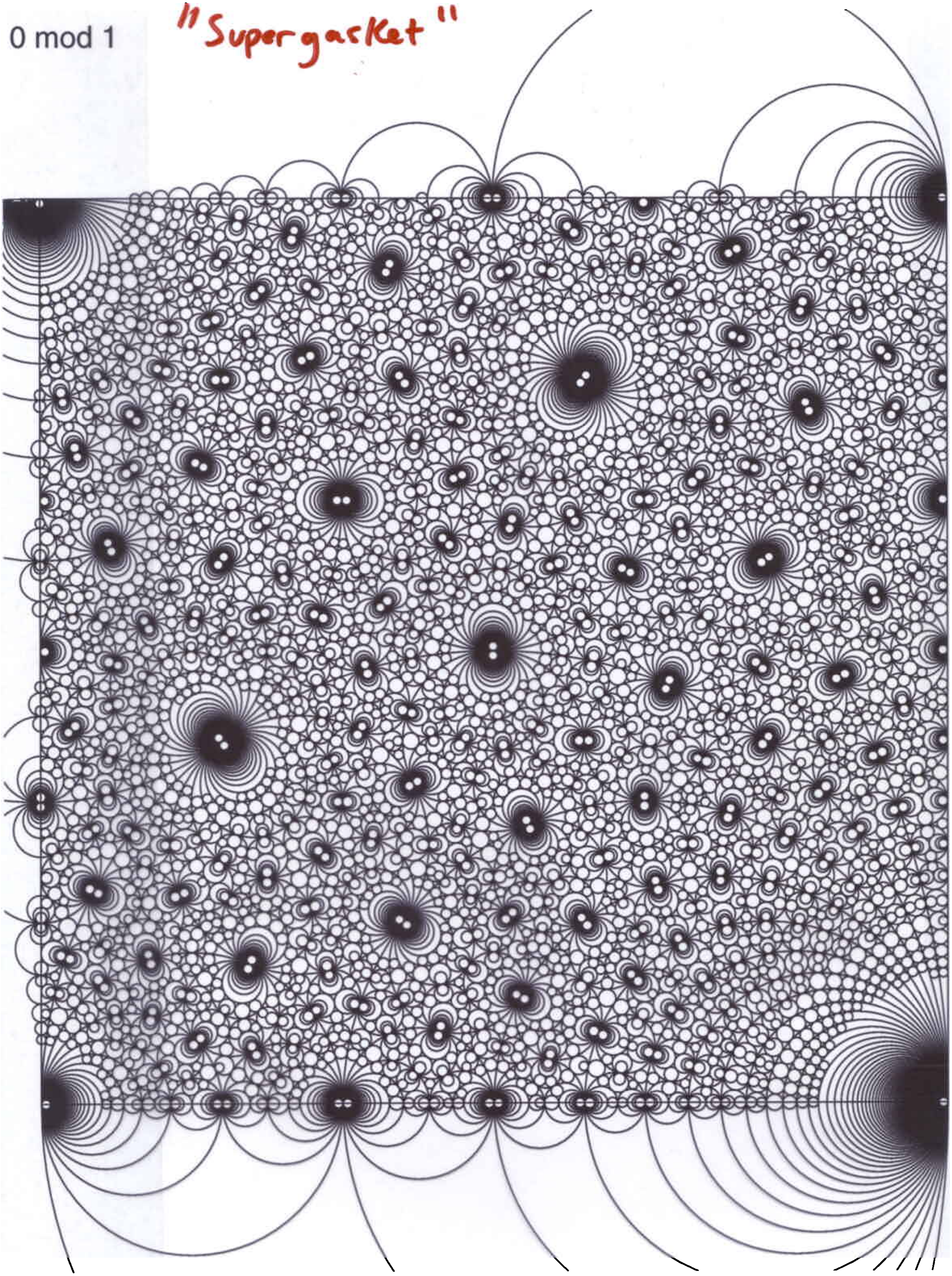
- integral supergasket = $\left\{ \begin{array}{l} \text{orbit of} \\ \text{strongly integral} \\ \text{packing} \\ \text{under } \mathcal{A}_S \end{array} \right\}$

[lives in 6-dim'l. space of matrices M

$$M^T Q_2 M = Q_E$$]

0 mod 1

"Supergasket"



Theorem [GLMWY]

(1) The group $\mathcal{A}_S$ is a (hyperbolic) Coxeter group

$$\mathcal{A}_S = \langle g_1, g_2, g_3, g_4, g_1^{\perp}, g_2^{\perp}, g_3^{\perp}, g_4^{\perp} \rangle$$

$$g_1^2 = g_2^2 = g_3^2 = g_4^2 = I$$

$$(g_1^{\perp})^2 = (g_2^{\perp})^2 = (g_3^{\perp})^2 = (g_4^{\perp})^2 = I$$

$$g_i g_i^{\perp} = g_i^{\perp} g_i \qquad 1 \le i \le 4.$$

(2) The group $\mathcal{A}_S$ "is" a finite index subgroup of $O(3,1,\mathbb{Z})$ of index 96.

[$24 = |S_4|$ = permuting elts of quadruple]

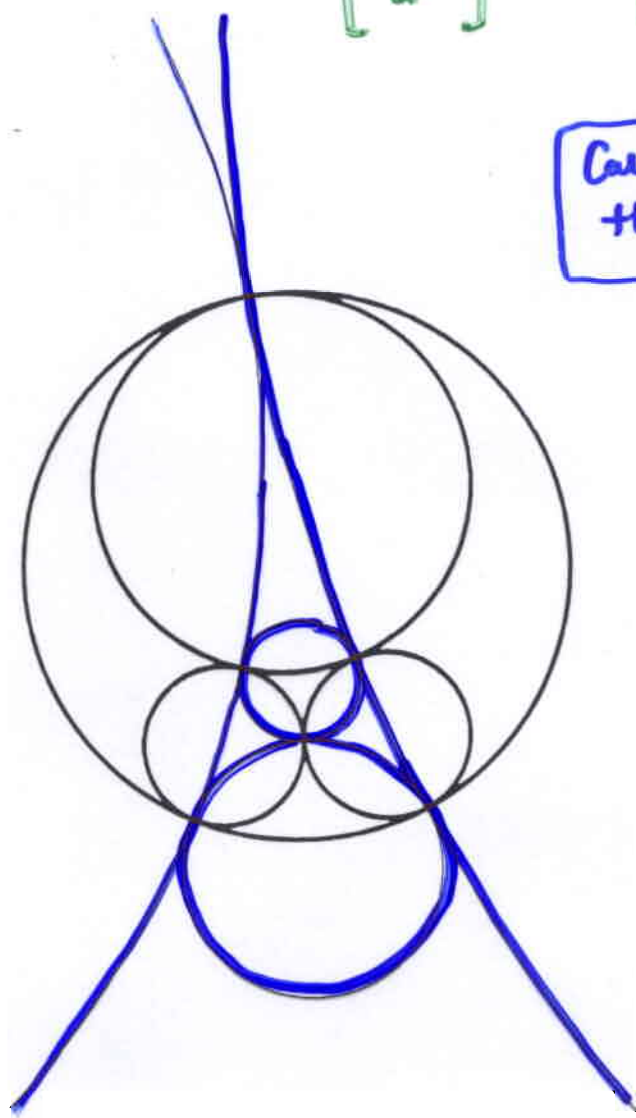
($\Rightarrow$ "96 supergaskets" (essentially only one)

"Dual" Circles

Descartes Configuration

$$\begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$\boxed{\text{Call this } D^*}$



black circles: a, b, c, d

blue circles: a', b', c', d'

[a' intersects b, c, d ; etc]

"Duality" is Outer Automorphism of Super-Apollonian Group

- $D M_i D = M_i^{\perp}$ $\quad [D^2 = I]$
- $D M_i^{\perp} D = M_i$

$$\Longrightarrow \quad D \Gamma D = \Gamma^{\perp}$$
$$D \Gamma^{\perp} D = \Gamma$$

- D is not in $A_S = \langle \Gamma, \Gamma' \rangle$ because it has half-integer entries.

6. Final Remarks

- N-dimensional case

- Lie groups & geometry

Facts.

(1) Descartes circle thm. generalizes to n dimensional space $\left\{\begin{array}{l}\text{Euclidean}\\ \text{spherical}\\ \text{hyperbolic}\end{array}\right.$

(2) Apollonian group generalizes.

(3) Dual Apollonian group generalizes.

(4) Integral structures partially survive.

N-dimensional case

- **Packings** extend to dimension 3 only
  (R. Boyd)
  1973

- **Descartes configurations** extend to **all** dimensions

n-diml. Apollonian group $\equiv$ { rational matrices are integer in dimensions 2 and 3 only }

$$A_n = \left\langle \begin{bmatrix} -1 & \frac{2}{n-1} & \frac{2}{n-1} & \cdots & \frac{2}{n-1} \\ 0 & & I_{n+1} & & \end{bmatrix} \cdots \right\rangle$$

But dual Apollonian group:

$$A_n^{\perp} = \left\langle \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & & & & \\ \vdots & & I_{n+1} & & \\ 2 & & & & \end{bmatrix} \right\rangle$$

is still integral.

Integral Dual Packings & Rational Apollonian Gaskets (n-dimensional)

Theorem. [GLMWY]

A rational Descartes $(n+2)$-tuple of curvatures exists in dimension n

$\iff$ $n = 2k^2$

or $n = (2k-1)^2$

Thus: $n = 1, 2, 8, 9, 18, 25, \ldots$

Open Problem. Which dimensions have an integral $(n+2)$-tuple?

LIE GROUPS & GEOMETRY

Klein's Erlangen Programm

real dimension

projective geometry

$\mathbb{R}^2 \cup \{\text{line at } \infty\}$

8 — $PGL(3, \mathbb{R})^+$

$\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\text{point at } \infty\}$

conformal group

inversive geometry

6 — $\text{Möb}(2) = O(3,1,\mathbb{R})^{\uparrow}$ $\triangleq SL(2,\mathbb{C})/(\pm I)$

4 — complex $ax+b$ group $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = AGL(1,\mathbb{C})$

3 — $E(2) = \mathbb{R}^2 \rtimes SO(1)$ — $SO(2,1) \triangleq SL(2,\mathbb{R})$ — $SO(3)$

Euclidean geometry — **hyperbolic geometry** — **spherical geometry**

Note: $\triangleq$ = "infinitesimally equivalent" Lie gps.

Real Lie Groups

$GL(3, \mathbb{R})$ real form of A_3

$SO(3, 1, \mathbb{R})$ real form of D_2

$\parallel\!\triangleright$ "an accident" $\parallel\!\triangleright$

$SL(2, \mathbb{C})$ real form of $A_1 \times A_1$

$SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$

$i : (M_1, M_2) = (M_2, M_1)$

real form : $(M, \bar{M})$

involution i acts as complex conjugation