The tropical rank of a matrix

(Joint work with F. Santos and B. Sturmfels)

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Ordinary linear algebra

One of the most important notions in ordinary linear algebra over a field is that of the rank of a matrix M. The following definitions are all equivalent in that setting.

Definition. *The rank of* M *is the largest number* r *such that* M *has a nonsingular* $r \times r$ *minor.*

Definition. *The rank of* M *is the dimension of the set of linear combinations of its rows (or columns).*

Definition. *The rank of* M *is the dimension of the smallest linear space containing its rows (or columns).*

Definition. *The rank of* M *is the smallest* r *for which* M *can be written as the sum of* r *rank-1 matrices.*

The equivalence of these four definitions is trivial. However, in a more generalized setting, these definitions diverge.

The tropical semiring

Our goal is to do linear algebra over a setting without additive inverses, in particular the tropical semiring.

Definition. *The tropical semiring is* $(\mathbb{R}, \oplus, \odot)$*, that is, the real numbers under the operations of tropical addition* $\oplus$ *given by* $a \oplus b = min(a, b)$ *and tropical multiplication* $\odot$ *given by* $a \odot b = a + b$*.*

Sometimes $\mathbb{R}$ is augmented by the element $+\infty$, in the obvious way. We will be working over the spaces $\mathbb{R}^n$, viewed as a semimodule over the tropical semiring:

$$k \odot (x_1, \dots, x_n) = (k + x_1, \dots, k + x_n)$$

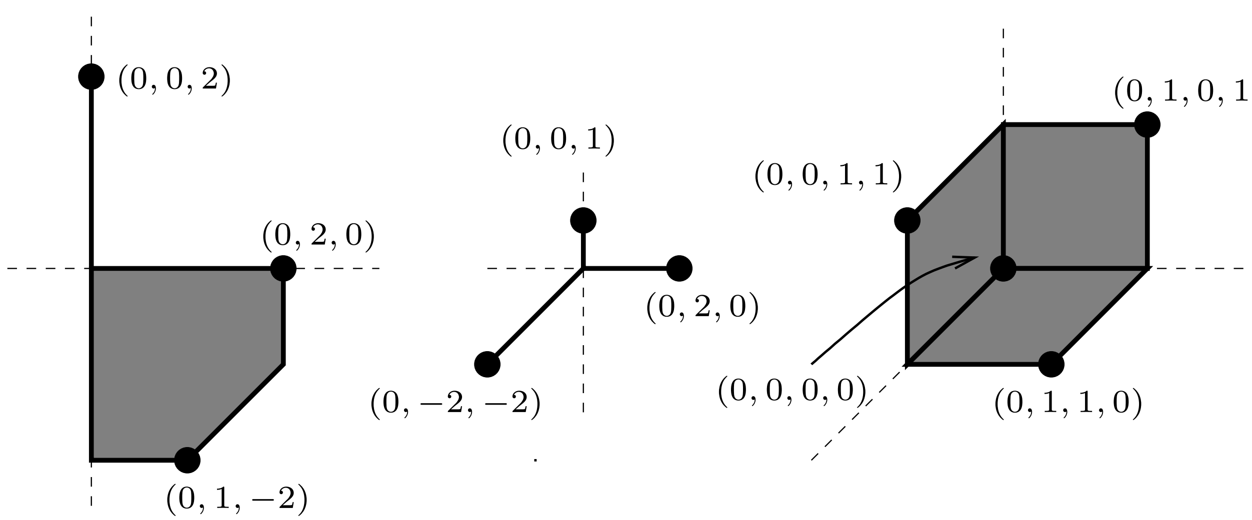
$$(x_1, \dots, x_n) \oplus (y_1, \dots, y_n) = (\min(x_1, y_1), \dots, \min(x_n, y_n))$$

Example. $6 \odot (3, 1, 3) \oplus 5 \odot (2, 3, 4) = (9, 7, 9) \oplus (7, 8, 9) = (7, 7, 9)$.

Tropical convexity

In ordinary linear algebra, the set of linear combinations of a given set of vectors is analogous to its convex hull rather than its linear hull.

Theorem (D. & Sturmfels, 2003). *Given a set of vectors* $V \subset \mathbb{R}^n$, *the set of tropical linear combinations of elements of* V *has lineality space* $(1, \ldots, 1)$, *and upon modding out by this vector, the resulting subset of* $\mathbb{TP}^{n-1} := \mathbb{R}^n/(1, \ldots, 1)$ *is bounded. We call this the tropical convex hull of* V.



Tropical rank

Definition. *The tropical rank of a matrix is the dimension of the convex hull of its rows (or columns), viewed as a subset of* $\mathbb{R}^n$*.*

Example. *The matrix below has tropical rank 3.*

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition. *The tropical rank of a matrix is the size of its largest tropically nonsingular minor.*

A matrix is tropically nonsingular if the tropical determinantal sum

$$\bigoplus_{\sigma \in S_n} \Big(\bigodot_{i \in [n]} M_{i\sigma(i)} \Big)$$

achieves its indicated minimum twice.

Algebraic geometry

Consider the field K of power series with discrete real exponents including a minimal one, i.e. $c_1 t^{a_1} + c_2 t^{a_2} + \cdots$ with $a_i \in \mathbb{R}$ and $a_1 < a_2 < \cdots$. (The c_i's are taken from some ground field k, usually $\mathbb{C}$.)

Let deg : $K \to \mathbb{R}$ be the degree map taking this element to a_1. Then we have $\deg(fg) = \deg(f) + \deg(g)$, while unless there is a cancellation of leading terms we also have $\deg(f+g) = \min(\deg(f), \deg(g))$. Thus the tropical operations are the images of the ordinary operations under the degree map.

If we have an ideal I in $K[x_1, \ldots, x_n]$, we can define a corresponding tropical variety $\mathcal{T}(I)$ as the image under the degree map of its ordinary vanishing set $V(I)$. For every element $f \in I$ and point $v \in \mathcal{T}(I)$, we must have the tropicalization of f achieve its minimum twice on v (so as to obtain a cancellation of leading terms); indeed, this is equivalent if k is algebraically closed. This is why the natural definition of a tropical polynomial f vanishing on a point v is that it achieves its minimum twice.

Kapranov rank

Definition. *The Kapranov rank of a tropical matrix M is equal to the dimension of the smallest tropical linear subspace ($\mathcal{T}(I)$ for a linear ideal I) containing the rows (or columns) of M.*

This is equivalent to two other interesting concepts.

Theorem. *For a real matrix $M = (m_{ij}) \in \mathbb{R}^{m \times n}$ the following statements are equivalent:*

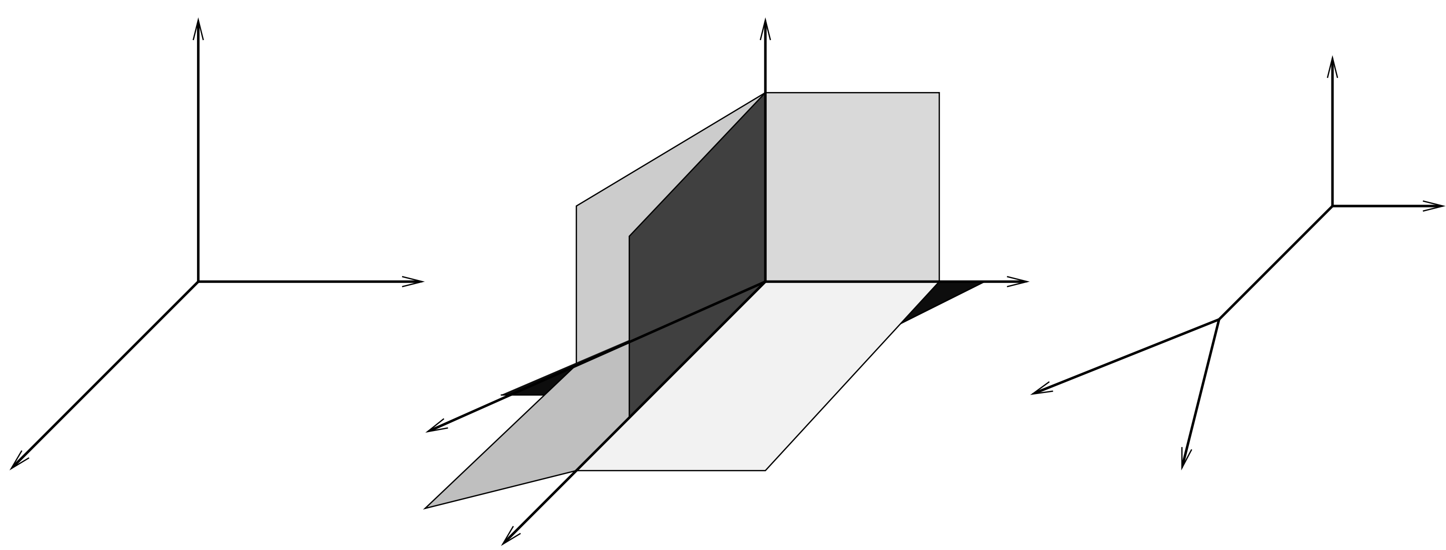
(a) *The Kapranov rank of M is at most r.*

(b) *There exists an $m \times n$-matrix $F = \big(f_{ij}(t)\big)$ with non-zero entries in the field $\tilde{K}$ such that the rank of F is less than or equal to r and $\deg(f_{ij}) = m_{ij}$ for all i and j.*

(c) *The matrix M lies in the tropical determinantal variety $\mathcal{T}(J_{r+1})$.*

Tropical linear spaces

The tropical linear spaces important in the definition of Kapranov rank are parametrized by the tropical Grassmannian. Hyperplanes are the tropical vanishing sets of a single tropical linear function $\bigoplus - a_i \odot x_i$; in $\mathbb{TP}^{n-1}$, this consists of all points two of whose coordinates are the corresponding a_i's, and the others of which exceed a_i. Tropical lines are trees with n leaves going off to infinity, where n is the dimension of the ambient space.



Barvinok rank

Definition. *The Barvinok rank of a matrix M is the smallest r for which M can be written as the tropical sum of r matrices of tropical rank one.*
Example. *The matrix*

$$\begin{pmatrix} 0 & 4 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 6 & 4 \\ 2 & 6 & 4 \end{pmatrix} \oplus \begin{pmatrix} 9 & 7 & 6 \\ 3 & 1 & 0 \\ 6 & 4 & 3 \end{pmatrix}$$

has Barvinok rank two.

It also has Kapranov rank two. A lift which demonstrates this is:

$$\begin{pmatrix} 1 & t^4 & t^2 \\ t^2 & t & 1 \\ t^2 + t^5 & t^4 + t^6 & t^3 + t^4 \end{pmatrix}$$

Its rows lie in the vanishing set of the linear function

$$2 \cdot x_0 \oplus (-1) \cdot x_1 \oplus 0 \cdot x_2.$$

More on Barvinok rank

Barvinok rank arises naturally in combinatorial optimization, where given a distance matrix of fixed Barvinok rank r, there exists a polynomial time to solve the corresponding traveling salesman problem. It also has connections to both tropical convexity and tropical algebraic geometry.

Proposition. *Let M be a real $m \times n$-matrix. The following properties are equivalent:*

(a) M has Barvinok rank at most r.

(b) The rows of M lie in the tropical convex hull of r points in $\mathbb{TP}^{n-1}$.

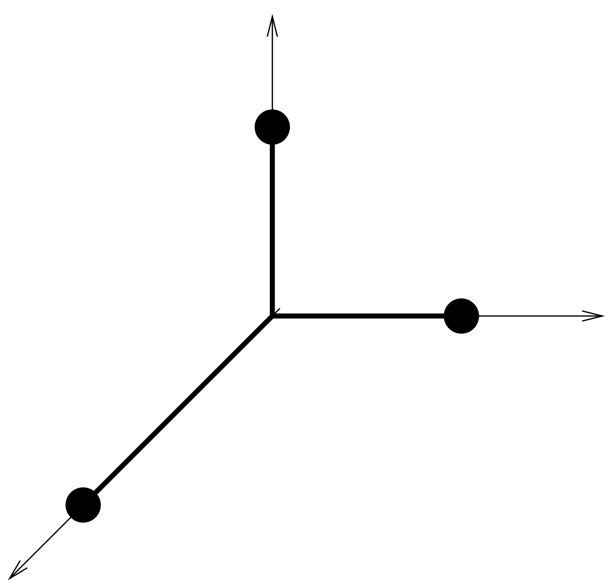
(c) M lies in the image of the following tropical morphism, which is defined by matrix multiplication:

$$\phi_r \; : \; \mathbb{R}^{m\times r} \times \mathbb{R}^{r\times n} \to \mathbb{R}^{m\times n} \,, \;\; (X,Y) \mapsto X \odot Y.$$

An example

Consider the example

$$C_3 \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



This has tropical and Kapranov ranks 2, but Barvinok rank 3. In general, the tropical and Kapranov ranks of C_n are 2 for all n, while the Barvinok rank grows as $\log_2 n$.

Comparison of ranks

Theorem. *The tropical rank of a matrix is always less than or equal to its Kapranov rank, which in turn is less than or equal to its Barvinok rank.*

The difference from ordinary linear algebra is that these inequalities can be strict. We have already seen that the Kapranov rank can be less than the Barvinok rank; however, the tropical rank can also be less than the Kapranov rank. In particular, there exists a set of points whose convex hull is 3-dimensional, but which do not lie in any 3-dimensional linear subspace.

To do this we use another connection between tropical rank and classical mathematics, this time via matroids.

Cocircuit matrix of a matroid

Given a matroid $\mathcal{M}$, we make the following definition.

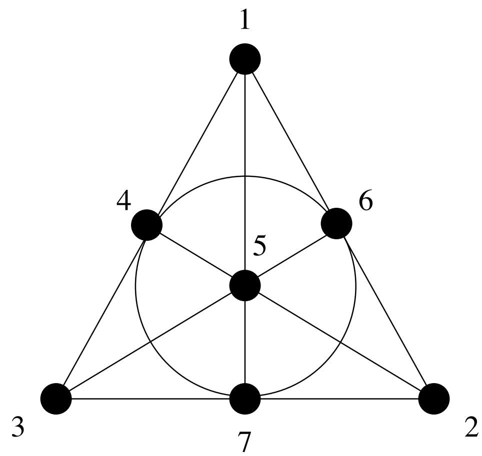
Definition. *The cocircuit matrix of a matroid* $\mathcal{M}$*, denoted* $\mathcal{C}(\mathcal{M})$ *has its rows indexed by the ground set of* $\mathcal{M}$ *and its columns indexed by the cocircuits of* $\mathcal{M}$*. It has a 0 in entry* ij *if element* i *is in cocircuit* j*, and a 1 otherwise.*

The cocircuit matrix is actually a well-known object associated to a matroid, in disguise.

Proposition. *The tropical convex hull of the columns of* $\mathcal{C}(\mathcal{M})$ *is the Bergman complex of* $\mathcal{M}$*.*

Theorem. *The tropical rank of* $\mathcal{C}(\mathcal{M})$ *is equal to the rank of* $\mathcal{M}$*. The Kapranov rank of* $\mathcal{C}(\mathcal{M})$ *over a field* k *is equal to the rank of* $\mathcal{M}$ *if and only if* $\mathcal{M}$ *is representable over* k*.*

An example: the Fano matroid



The Fano matroid has cocircuit matrix

$$\mathcal{C}(\mathcal{M}) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is the smallest known example both in dimension and number of points of a matrix whose Kapranov rank over $\mathbb{C}$ strictly exceeds its tropical rank.

Some more cases

Theorem. *If a matrix has Kapranov rank n, it has tropical rank n. If a matrix has tropical rank 2, it has Kapranov rank 2.*

We can consider the space of all $m \times n$ matrices of Kapranov rank 2. This is a polyhedral complex in $\mathbb{R}^{m \times n}$, which has some trivial components. When we remove these, we have the following results:

Proposition. *The space of $3\times n$ matrices of Kapranov rank two is a shellable simplicial complex, with top homology free of dimension $2^n - 3$.*

Proposition. *The space of 4×4 matrices of Kapranov rank two has top homology free of dimension 73 and no other homology.*

Are these spaces shellable in general?

The space of 4×4 matrices of Barvinok rank two, on the other hand, has homology $\mathbb{Z}/2$ in dimensions 1 and 2 and $\mathbb{Z}$ in dimension 4. What is going on here?