On simplices spanned by points in three-dimensional space

Peter Brass

Department of Computer Science

City College, CUNY

New York

Two problems

d-dimensional Heilbronn triangle problem

What is the smallest $f_d(n)$ such that any set of n points in the d-dimensional unit cube contains $d+1$ points that span a simplex of volume at most $f_d(n)$?

d-dimensional MinMax triangulation problem

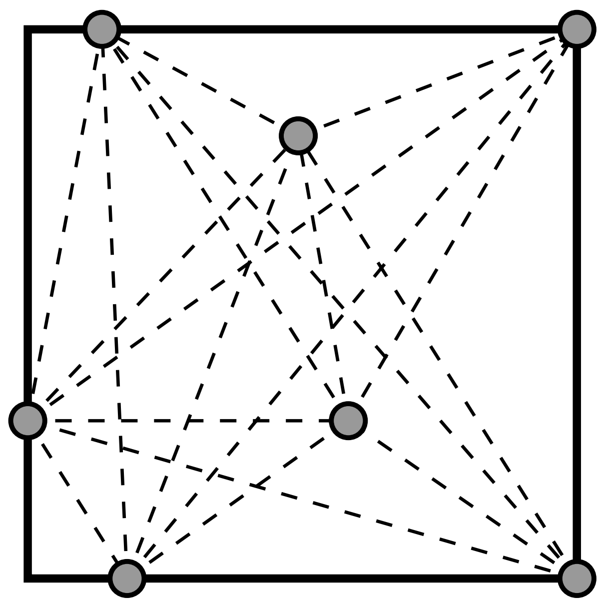
What is the largest $g_d(n)$ such that any set of n points in the d-dimensional space, in general position, allows some triangulation with at least $g_d(n)$ simplices ?

Connection: $$f_d(n) \leq \frac{1}{g_d(n)}$$

In the following look at $d = 3$, but all the arguments for the first problem generalize to all odd dimensions, and all for the second to all higher dimensions.

The Heilbronn triangle problem (1)

The original Heilbronn triangle problem is the planar version: what is the maximum area of the smallest-area triangle that can be reached by proper choice of n points in the unit square?



Such a set is as far as possible from containing three collinear points.

This problem was originally asked by P. Heilbronn, and studied in many papers by Roth, Schmidt, and Komlós, Pintz and Szemerédi. Exact values were determined for up to 7 points.

The Heilbronn triangle problem (2)

There is a simple lower bound of $\frac{1}{2n^2}$: select from the $n \times n$ integer grid square a subset of n points without three collinear points (e.g. by a modular quadratic curve: 'no-three-in-line problem'). Any triangle spanned by integer-coordinate points has an area which is a multiple of $\frac{1}{2}$. So if there are no three collinear points, then the area of each spanned triangle is at least $\frac{1}{2}$. Scaling down to the unit square gives a set of n points with all triangles of area at least $\frac{1}{2n^2}$.

There is a trivial upper bound: Any triangulation of the set has at least $n - 2$ triangles, which are disjoint and contained in the unit square, so there is one with area less than $\frac{1}{n-2}$.

Neither bound give the right order.

The d-dimensional Heilbronn problem (1)

The d-dimensional analogue of the Heilbronn triangle problem was first noticed by Barequet in 2000. It asks for the maximum over all choices of n points from the unit cube of the volume of the smallest spanned simplex. Again such a set is as much as possible in 'general position': as far as possible from having $d+1$ coplanar points.

The lower bound generalizes: one can select n points with no $d+1$ coplanar from the $n \times \cdots \times n$ integer lattice cube, which gives a lower bound of $\frac{1}{d!n^d}$.

This bound was improved by Lefmann to $\frac{c_d \log n}{n^d}$, which also gave a new proof of the famous planar Komlós-Pintz-Szemerédi lower bound.

The d-dimensional Heilbronn problem (2)

The upper bound also generalizes, we can triangulate any set of n points and obtain $\Omega(n)$ simplices. So the smallest simplex has volume $O\left(\frac{1}{n}\right)$. A nontrivial bound on $g(n)$ would improve this bound.

This is the only known upper bound, weaker than in the planar case.

Theorem: Any set of n point in the three-dimensional unit cube determines a simplex of volume $O(n^{-\frac{7}{6}})$.

This generalizes to give $O(n^{-(1+\frac{1}{2d})})$ in any *odd* dimension $d \geq 3$.

The lower bound is $\Omega(\frac{\log n}{n^3})$.

The d-dimensional Heilbronn problem (3)

Proof: For any $t > 1$ the maximum number of points in the unit cube with all pairwise distances larger $\frac{1}{t}$ is less than ct^3. So n points in the unit cube determine at least $c'\frac{n^2}{t^3}$ point pairs with distance smaller than $\frac{1}{t}$.

Each point pair determines a direction, which are all distinct. The minimum angular distance between them is at most $c''\left(\frac{n^2}{t^3}\right)^{-\frac{1}{2}}$, so one can select two edges which are

- short, length less than $\frac{1}{t}$, and
- almost parallel, angular distance less than $c''\left(\frac{n^2}{t^3}\right)^{-\frac{1}{2}}$.

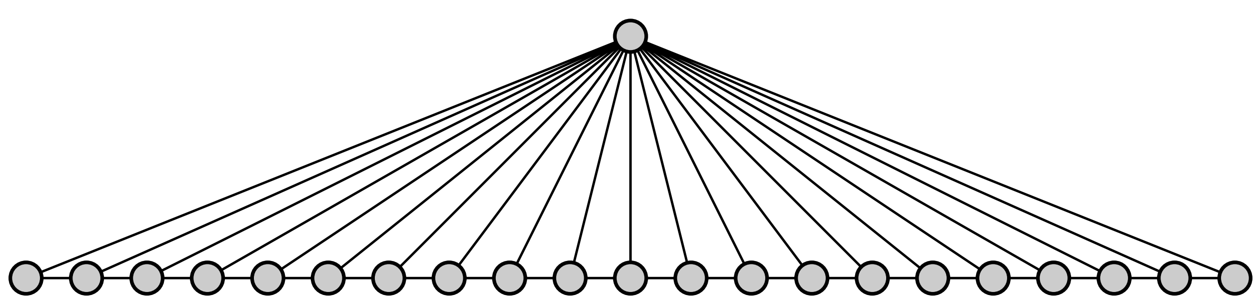
These two edges determine a simplex of volume less than $c'''\frac{1}{t^{\frac{1}{2}}n}$. Choosing t as large as possible, $t = (cn)^{\frac{1}{d}}$, proves the theorem.

Triangulations

Given a set X of n points in d-dimensional space, a *triangulation* $\mathcal{T}$ of X is a division of the convex hull $\text{conv}(X)$ into non-degenerate simplices such that

- the vertices of each simplex are points of X,
- each simplex contains no other points of X but its vertices,
- the interiors of the simplices are disjoint, and
- the union of the simplices is $\text{conv}(X)$.

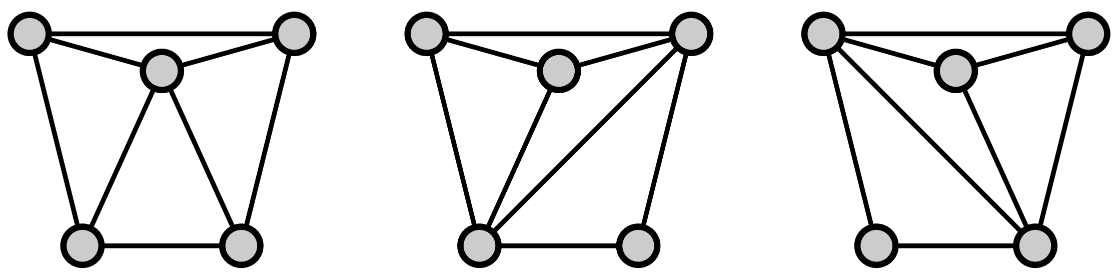
To avoid degeneracies, we will in the following always assume that the points are in general position: no $d + 1$ points on a hyperplane.



Otherwise there are sets with a unique triangulation.

The size of a triangulation (1)

A point set in general position has many triangulations



but in the plane, all triangulations of one set have the same number of triangles.

This is a consequence of the interior angle sum theorem: each triangle has an interior angle sum of π, and each point of X in the interior of $\mathrm{conv}(X)$ contributes an angle sum of 2π that has to occur in the adjacent triangles. If there are k points on the boundary of $\mathrm{conv}(X)$, they contribute an angle sum of $(k-2)\pi$, so the number of triangles is $2n-k-2$.

The size of a triangulation (2)

This does not hold in higher-dimensional space, since there is no analogue of the interior angle sum theorem. There are simplices with arbitrary small sums of vertex angles.

A classical example is that the cube can be divided in five or six simplices. If one cuts off alternating vertices, one gets a division into four outer and one central simplex; if one first halves the cube into two triangular prisms, one obtains six congruent simplices.

In fact it is a well-known open problem to bound the minimum number of simplices required in a triangulation of the d-dimensional cube; the maximum number is $d!$, since each non-degenerate simplex with integer vertex coordinates has volume at least $\frac{1}{d!}$.

This suggest that one should ask for the *size*, the number of simplices, of d-dimensional triangulations.

The size of a triangulation (3)

The minimal size of any triangulation of any set of n points in general position in d-dimensional space is $n - d$.

The maximal size of any triangulation of any set of n points in general position in d-dimensional space is $h_d(n+1, d+1) - (d+1) = \Theta(n^{\lceil \frac{1}{2}d \rceil})$, where $h_d(n+1, d+1)$ is the number of d-faces of the $d+1$-dimensional cyclic polytope on $n+1$ vertices.

B.L. Rothschild, E.G. Straus 1985

Does every set have a *small* triangulation?

Yes: $O(n)$ in general, $\leq 3n - 11$ for $d = 3$

Does every set have a *large* triangulation?

?

H. Edelsbrunner, F.P. Preparata, D.B. West 1990

MinMax Triangulations (1)

What is the minimum over all sets of n points of the maximum size of any triangulation of that set?

What is the largest $g_d(n)$ such that any set of n points in d-dimensional space, in general position, has some triangulation with at least $g_d(n)$ simplices?

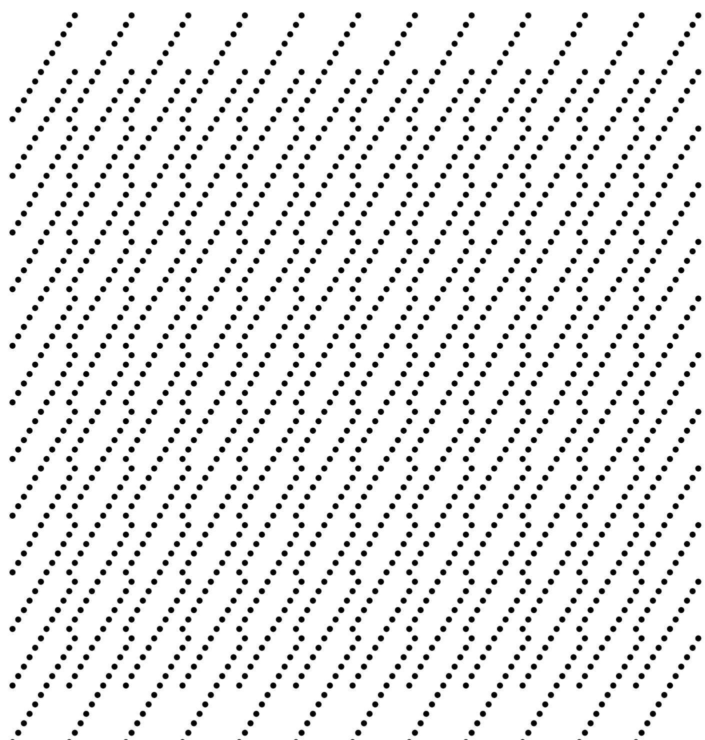
In the following we consider only the three-dimensional case.

Edelsbrunner et al. obtained only the trivial bounds: $g_3(n) = \Omega(n)$ and $g_3(n) = O(n^2)$.

Theorem: $g_3(n) = O(n^{\frac{5}{3}})$.

MinMax Triangulations (2)

The theorem claims there is a set of n points in three-dimensional space, in general position, that does not allow any triangulation with more than $O(n^{\frac{5}{3}})$ simplices.



The $n^{\frac{1}{3}} \times n^{\frac{1}{3}} \times n^{\frac{1}{3}}$ lattice cube is not in general position, but there are only $O(n)$ nondegenerate simplices in any triangulation of the lattice cube, since each nondegenerate simplex has volume at least $\frac{1}{6}$.

MinMax Triangulations (3)

If we perturb each point of the lattice cube by a small amount, we obtain a set in general position. Each simplex in any triangulation of this set belongs to one of three classes:

- big simplices: nondegenerate in the unperturbed set.
- flat simplices: planar in the unperturbed set.
- needle simplices: collinear in the unperturbed set.

There are only $O(n)$ big simplices, so we have to bound the numbers of flat and needle simplices. We do this by studying their preimages in the unperturbed lattice cube.

MinMax Triangulations (4)

Each flat simplex belongs in the unperturbed version of the point set to some lattice plane.

Consider such a plane containing a flat simplex, and its intersection with the big simplices. The flat simplices contained in this plane cannot intersect the interior of any big simplex, so they must be contained in the union of faces of big simplices that lie in this plane.

So flat simplices occur only in the planes spanned by faces of the big simplices, and the vertices of the flat simplices must be vertices of faces of big simplices in that plane.

MinMax Triangulations (5)

Let $(E_i)_{i\in I}$ be the planes that contain flat simplices, and

a_i be the number of faces of big simplices contained in E_i,

b_i be the number of points that are vertex of some flat simplex contained in E_i

c_i be the number of flat simplices contained in E_i. Then

- $\sum_{i\in I} a_i = O(n)$: there are only $O(n)$ faces on the $O(n)$ big simplices.
- $b_i = O(a_i)$: each face contributes at most three vertices to the plane.
- $b_i = O(n^{\frac{2}{3}})$: no plane contains more than $n^{\frac{2}{3}}$ lattice points.
- $c_i = O(b_i^2)$: The number of simplices spanned by b_i points is $O(b_i^2)$.

Together this implies $\sum_{i\in I} c_i = O(n^{\frac{5}{3}})$. The same argument works for the needle simplices.

Further problems

Problem: Find a nontrivial lower bound for $g_d(n)$.

Problem: Find a better upper bound for $f_d(n)$.

The angle-sum argument shows that in a triangulation with many simplices, almost all of them must have small interior angle sum, so they are almost flat.

So a point set in which any simplex spanned by the set has a large interior angle sum would give an upper bound for g.

Problem: What is the maximum over all sets of n points in d-dimensional space of the minimum interior angle sum of a simplex spanned by that set?