On the Betti numbers of semi-algebraic sets

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A stroll though Betti number bounds and their Applicatons in Discrete Geometry

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Outline of talk

1. Bounds from antiquaty
   (a) The Oleinik-Petrovski/Milnor/Thom bound ('49, '64, '65)
   (b) Warren bounds on the number of strict sign conditions ('68)
2. Applications
   (a) Upper bounds for the number of simple order types (Goodman-P '86, Alon'86)
   (b) Upper bounds for the number of simple polytopes (Goodman-P '86, Alon "86)
   (c) Upper bounds for the number of real matrois (Alon '86)
   (d) Upper bounds for the number weaving patterns (Pach-P '90)

3. Bounds on the number of connected components of sign condition (P-Roy '93)
4. Applications
   (a) Upper bounds for the number of Isotopy classes (P-Roy '93)
   (b) Universality theorems (Mnëv ('88), Richter-Gebert ('95), Kapovich-Millson ('99)
5. Bounds on the number of connected components of sign condition restricted to a variety (Basu-P-Roy '96)
6. Applications
   (a) Geometric Transversal Theory, in particular
   (b) Upper bounds for the number of Geometric Permutation induced by $k-$flat transversals (Goodman-P-Wenger '96)
7. Bounding the individual Betti numbers (Basu '01)
8. Putting things together (Basu-P-Roy '03)

The Oleinik-Petrovski/Thom/Milnor bound

Let $b(k,d)$ be the maximum of the sum of the Betti numbers of any algebraic set defined by polynomials of degree $\leq d$ in R^k. The Oleinik-Petrovski/Thom/Milnor bound is the following:

$$b(k,d) \leq d(2d-1)^{k-1}.$$

Moreover, if instead we let $b(k,d,s)$ be the maximum of the sum of the Betti numbers of any basic semi-algebraic set defined by s polynomials of total degree d in R^k then,

$$b(k,d) \leq s^k d(2d-1)^{k-1}.$$

The **order type** of the labelled points $\{x_1, \ldots, x_n\} \subset R^d$ is determined by the signs of the $\binom{n}{d+1}$ determinants

$$\left(\det \begin{pmatrix} 1 & x_{i_0}^1 & \cdots & x_{i_0}^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i_d}^1 & \cdots & x_{i_d}^d \end{pmatrix} \right)_{1 \le i_0 < \cdots < i_d \le n} .$$

Some sets in R^N describing geometric phenomena in R^2, R^3 and R^4.

(a) The realization space of an order type in the plane ($N = 2n$).

(b) The configuration space of the combinatorial type of a 4-polytope ($N = 4n$).

(c) The configuration space of a planar linkage (also molecular models).

Connected component of sign conditions on a variety

$$\mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X_1, \ldots, X_k]$$

$$\sigma \in \{-1, 0, +1\}^s$$

$$R(\sigma) = \{x \in V \mid \sigma = (\mathrm{sign}(P_1(x)), \ldots, \mathrm{sign}(P_s(x)))\}$$

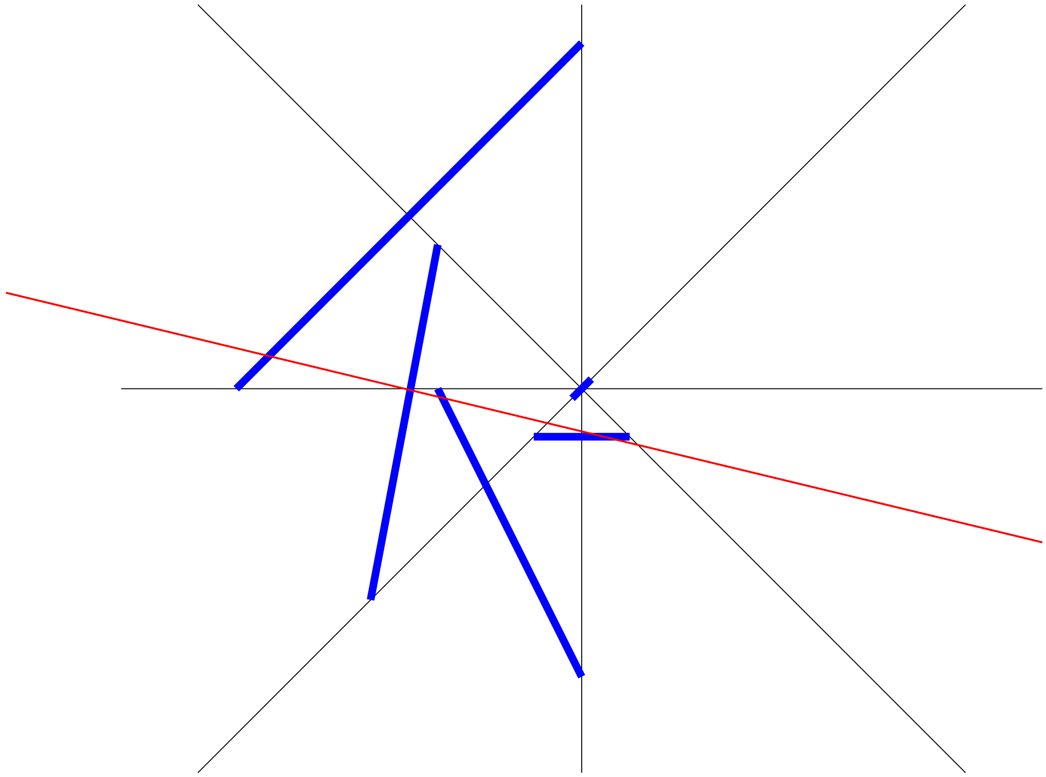
Let $|\sigma|$ denote the number of cells of $R(\sigma)$

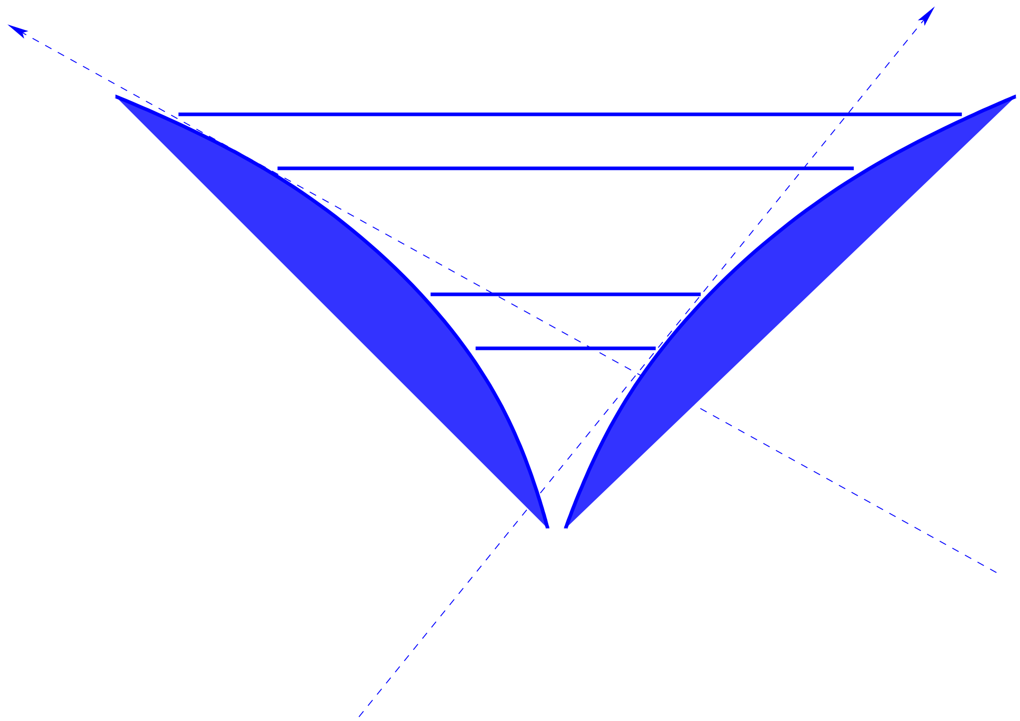
Theorem
If Q and all $P \in \mathcal{P}$ have degrees at most d and $V = Z(Q)$ is an algebraic variety of real dimension k'. Then,

$$\sum |\sigma| = s^{k'}(O(d))^k.$$

Geometric Transversal Theory

(a) Helly's theorem
(b) Vincensini's Problem
(c) Hadwiger's Transversal theorem
(d) Geometric permutations





Geometric permutations

(a) $g_1^2(n) = 2n - 2$ (Edelsbrunner-Sharir '90),

(b) $g_1^d(n) = \Omega(n^{d-1})$ (Katchalski-Lewis-Liu '92),

(c) $g_1^d(n) = O(n^{2d-2})$ (Wenger '90),

(d) $g_{d-1}^d(n) = O(n^{d-1})$ (Cappell-Goodman-Pach-P-Sharir-Wenger '94),

(e) $g_k^d(n) = O(k)^{d^2}\left(\binom{2^{k+1}-2}{k}\binom{n}{k+1}\right)^{k(d-k)}$ (or $g_k^d(n) = O(n^{k(k+1)(d-k)})$ for fixed k (Goodman-P-Wenger '96).

Bounding the topology of a semi-algebraic set on a variety

Sum of Betti numbers : (Basu '96)
If $S \subset R^k$ is closed, defined by s polynomials and contained in $Z(Q)$ of dimension k' and
$\deg P_i,\ \deg Q \leq d$ then,

$$\sum_i \beta_i(S) = s^{k'}(O(d))^k.$$

Putting things together

Theorem 1 *(Basu-P-Roy '03)*

$$b_i(d,k,k',s) \leq \sum_{0 \leq j \leq k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

Where $b_i(d,k,k',s)$ is the maximum of $b_i(\mathcal{Q},\mathcal{P})$ over all $\mathcal{Q},\mathcal{P}$ where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of of $\mathrm{R}[X_1,\ldots,X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = s$ (i.e. $\mathcal{P}$ has s elements) and the algebraic set $Z(\mathcal{Q}$ has dimension k'. With $b_i(\sigma)$ denoting the

i-th Betti number of $\mathcal{R}(\sigma, Z)$ and let

$$b_i(\mathcal{Q}, \mathcal{P}) = \sum_{\sigma} b_i(\sigma).$$