Locally plane graphs and excluded submatrices

Gábor Tardos
Rényi Institute, Budapest

GEOMGRAPH is k-locally plane if it has no self-intersecting path of length $\leq k$.

$=$ exclude all patterns of self-intersecting P_{k+1}

$= \frac{k}{2}$-neighborhood is plane

3-locally plane

ex(n, ) $= \Theta(n \log n)$

[J. Pach
R. Pinchasi
G. Tóth]

5-locally plane

$ex(n, \text{self-int. } P_6) = O\left(\frac{n \log n}{\log \log n}\right)$ ∈ [PPTT]

$\left(\begin{array}{l}\text{\# of edges} \\ \text{crossed by line} \\ \text{in } 2k\text{-locally plane}\end{array}\right) = O(n \log^{1/k} n)$

Argument for a linear bound for 10-locally plane graphs (false)

Assume G has min degree >1000
G 10-locally plane

Assume all edges $\approx$ same length ? ?

Find vertex v with 20 closest neighbors

Construction of locally plane graphs with more than linear # of edges

① Direct construction for 3-locally plane G

② Identify longer self-intersecting paths

③ Get rid of them by thinning G

① Project (middle layer of) ℓ-hypercube to plane

\# of vertices $n \approx 2^\ell$

\# of edges $e \approx \ell \cdot 2^\ell \approx n \log n$

Edges of hypercube project to

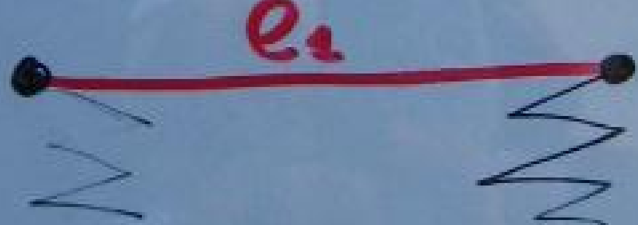
No self intersecting P_4

$j < i < k$

P = short self intersect. path in G_0

- first & last edges cross

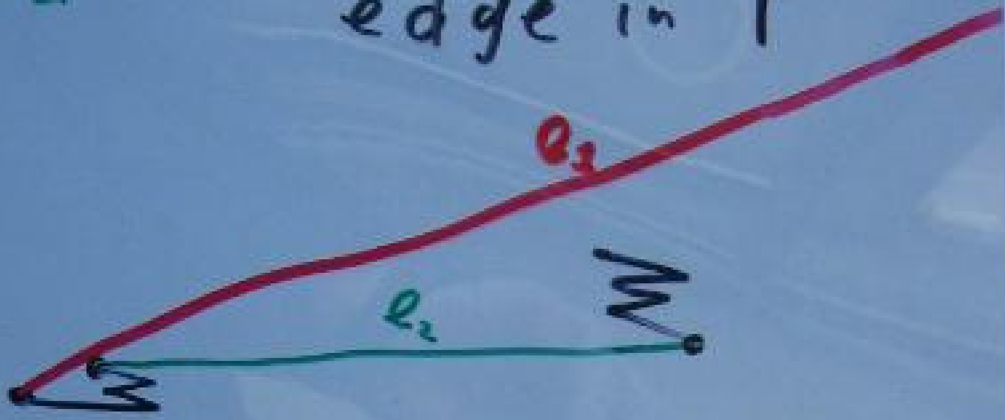
- $e_1 := $ Longest edge in P



- Other edges much shorter

$\Longrightarrow e_1$ first edge in P

e_2 := second longest edge in P



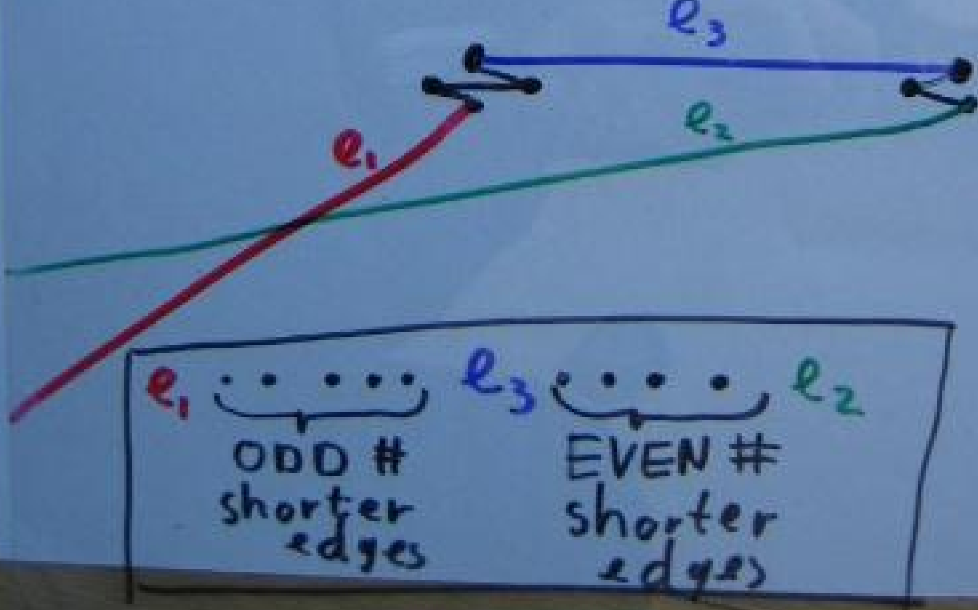
- other edges much shorter

$\Longrightarrow$ e_2 last edge in P

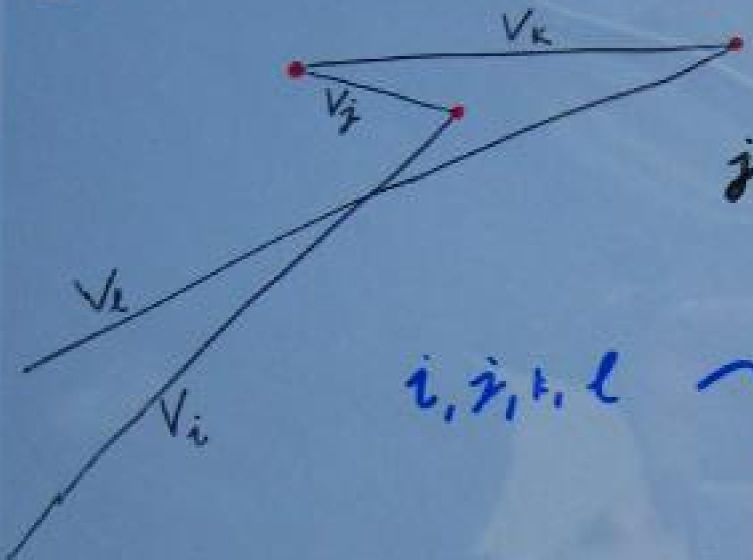
e_3 := third longest edge in P

– other edges much shorter

ONLY way for crossing:



② Identify all self-intersecting P_5



$j<k<l<i$

i,j,k,l ~ edge coloring

Need to get rid of

if $j<k<l<i$

3 THINNING LEMMA

G (abstract) graph

- edge-colored with $\{1, \ldots, d\}$

- Average degree $= \Theta(d)$

$\Longrightarrow \exists$ subgraph $G' \leq G$

no path

in G'

with $j < k < l < i$

- Average degree $= \Theta(\log d)$

- G' comes with nice edge-coloring with $\log d$ colors

1 + 2 + 3 Exist 4-locally plane graphs with $\Omega(n \log\log n)$ edges

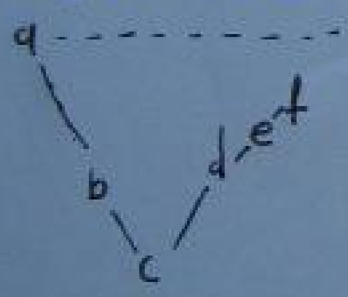
Luck: Graph constructed is

5-locally plane

$\Omega(n \log\log n)$ edges

Self-intersecting P_7

$a > b > c < d < e < f < a$



$$G \xrightarrow{\text{thinning}} G'$$

$n \log n$ edges

$\log n$ colors

3-locally plane

$n \log\log n$ edges

5-locally plane

REPEAT!

$$G' \xrightarrow{\text{thinning}} G''$$

$n \log\log n$ edges

$\log\log n$ colors

5-locally planar

$n \log\log\log n$ edges

7-locally planar

In general

$2k+1$-locally planar graphs

n vertices

$\Omega_k(n \log^{(k)} n)$ edges

0-1 matrices A, B

A **contains** B

if $\exists$(submatrix of A) $\geq$ B

delete rows/columns
don't PERMUTE
!ORDER!

has 1 where
B has 1

Extremal problem

$$f(n, B) = \left(\begin{array}{l}\text{max \# of 1} \\ \text{in } n\times n \text{ matrix} \\ \text{not containg } B\end{array}\right)$$

0-1 matrix

Example

$$\begin{bmatrix} 1 & & 1 & & & \\ & 1 & & 1 & 1 & \\ 1 & & & 1 & 1 & \\ & 1 & 1 & & 1 & 1 \\ 1 & & & & 1 & 1 \\ & 1 & & & 1 & 1 \end{bmatrix}$$

contains

$$\begin{bmatrix} 1 & 1 & & 1 \\ 1 & & 1 & \end{bmatrix}$$

$f\left(n, \boxed{\begin{smallmatrix} x & x & \\ x & & x \end{smallmatrix}}\right) = \Theta(n \log n)$ Z. Füredi

$f\left(n, \boxed{\begin{smallmatrix} & & & x \\ & x & & \\ x & & x & \end{smallmatrix}}\right) = \Theta(n\, \alpha(n))$ P. Hajnal - Z. Füredi

inverse Ackermann

$\boxed{\begin{smallmatrix} x & & x & \\ & x & & x \end{smallmatrix}}$

P permutation matrix

$f(n, P) = O(n)$ recent A. Marcus T.

$\Rightarrow$ Stanley-Wilf conjecture on permutations

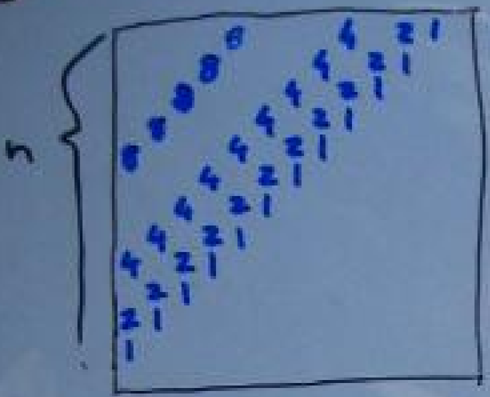
<u>Now:</u>

$f\left(n, \boxed{\begin{smallmatrix} x & x & \\ x & & x \end{smallmatrix}} \; \boxed{\begin{smallmatrix} x & & x \\ & x & x \end{smallmatrix}}\right) = \Theta(n \log\log n)$

Construction

1. direct for $f\left(n, \boxed{\begin{smallmatrix} x & x & \\ x & & x \end{smallmatrix}}\right)$
2. identify submatrices $\boxed{\begin{smallmatrix} x & & x \\ & x & x \end{smallmatrix}}$
3. remove them by THINNING

① Direct Construction



A

$\Theta(n \log n)$ entries

Doesn't contain

$$\begin{matrix} X & X & \\ X & & X \end{matrix}$$

or

$$\begin{matrix} X & X \\ X & \\ & X \end{matrix}$$

② $\begin{matrix} i & & j \\ & l & k \end{matrix}$ $\quad i > j > k < l < i$

G bipartite graph with adjecency matrix A

edge colored with Log n colors

$\begin{matrix} X & & X \\ & X & X \end{matrix} \approx$ path in G

$i > j > k < l < i$

$-$ $j < l$ $j > l$ possible

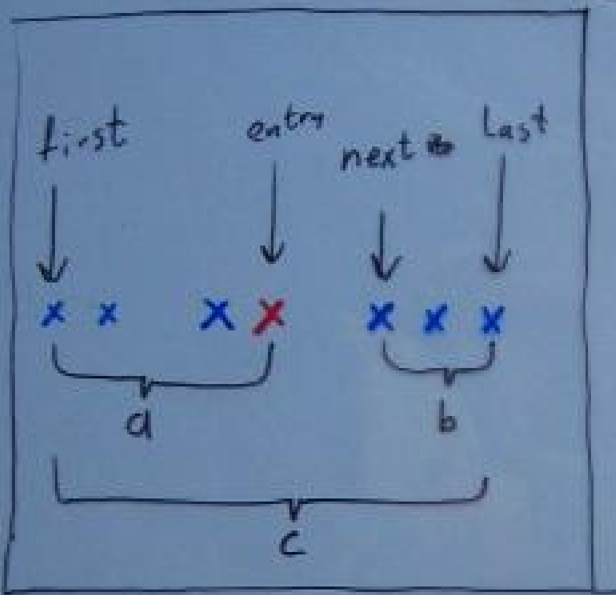
$+$ path starts on given SIDE

③ Very simmilar THINNING LEMMA

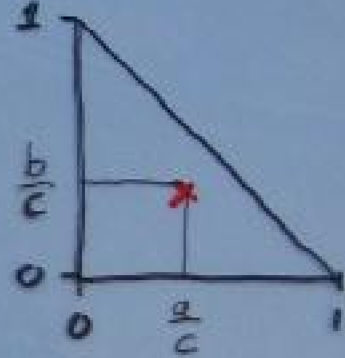
$\Rightarrow f($n,  $) = \Omega(n \log\log n)$

Tight result

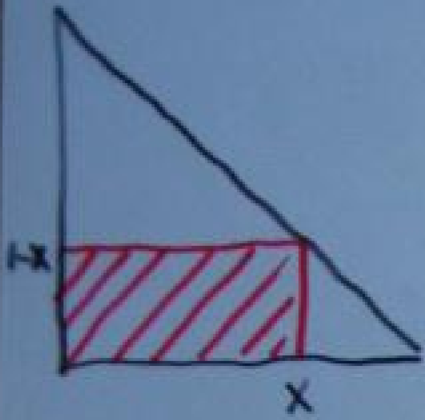
(not known for locally-plane construction) map each entry in matrix to point in $[0,1]^2$



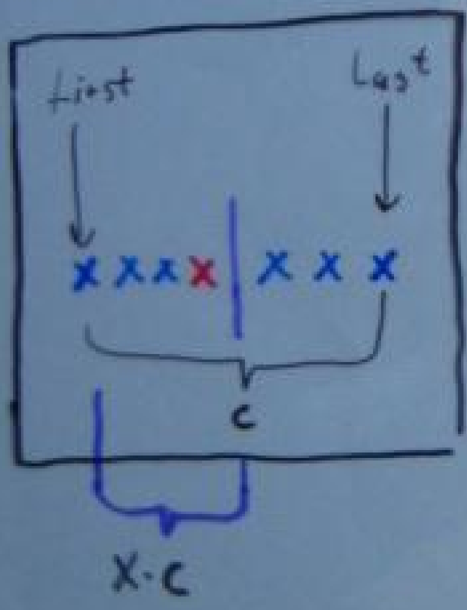
distances



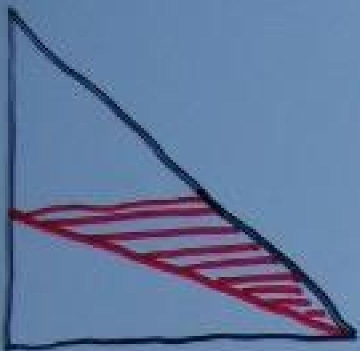
points with multiplicities OK



Exactly 1 entry
From each ROW
mapped to region



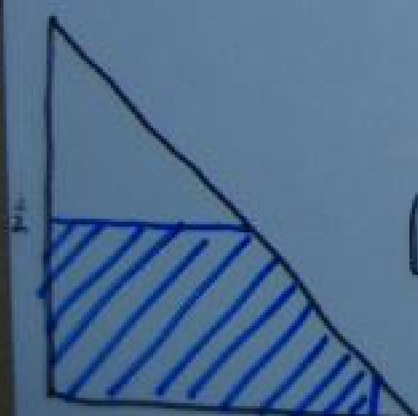
No excluded submatrices needed



At most
1 enty
from each column
mapped to region

if  and  excluded

Empty



Can be covered by
loglogn of each