Lifting Inequalities for Polytopes

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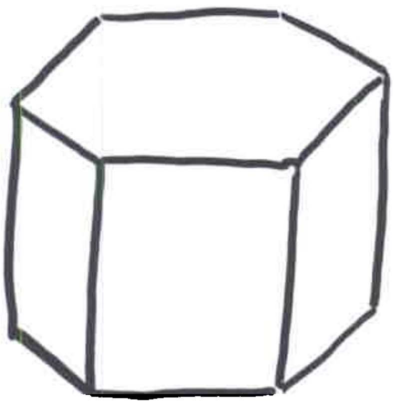
P = n-dim convex polytope

$S \subseteq \{0, 1, \ldots, n-1\}$

flag f-vector

$$f_S = \#(F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k)$$
$$\dim(F_i) = s_i$$

$$S = \{s_1 < s_2 < \cdots < s_k\}$$



S	f_S
ϕ	1
0	12
1	18
2	8
01	36
02	36
12	36
012	72

$$(f_S)_{S \subseteq \{0,1,\dots,n-1\}} \in \mathbb{R}^{2^n}$$

$$(f_S) \in GDSS_n \subseteq \mathbb{R}^{2^n}$$

the Generalized Dehn-Sommerville subspace

$$\dim(GDSS_n) = F_n$$

$= $ the nth Fibonacci number

[Bayer-Billera]

$$GDSS_n \cong \mathbb{R}\langle c,d\rangle_n$$

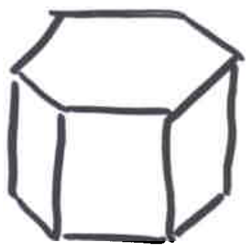
where

$$\mathbb{R}\langle c,d\rangle = \bigoplus_{n\geq 0} \mathbb{R}\langle c,d\rangle_n$$

non-commutative polynomials in c and d, $\deg(c)=1$ and $\deg(d)=2$.

$$(f_S) \longmapsto \Psi(P)$$

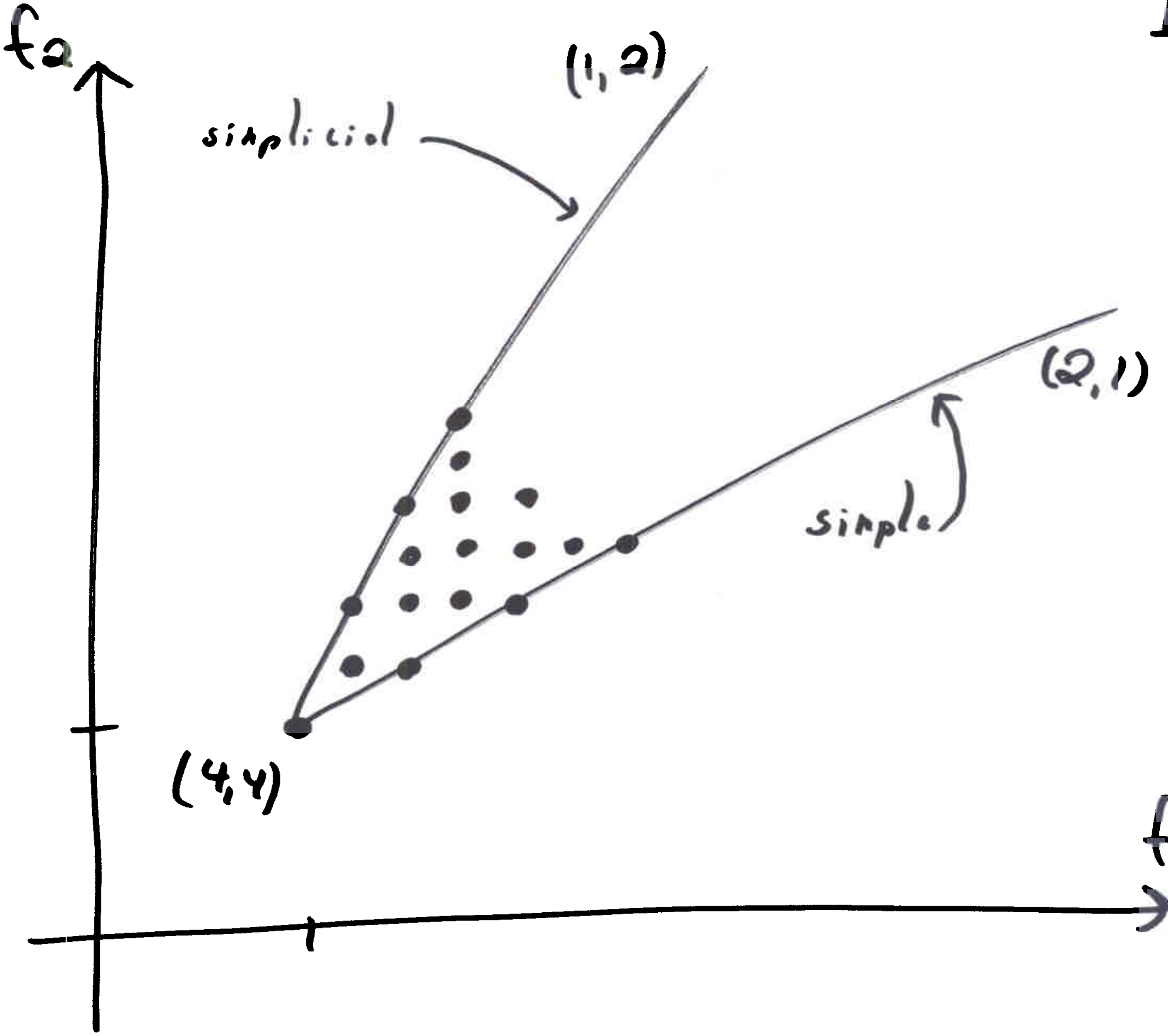
cd-index

$\Psi\Big($  $\Big) = c^3 + 10dc + 6cd$

One affine relation

$f_\phi = 1 \longleftrightarrow [c^n]\,\Psi(P) = 1$

Linear inequalities?



[Steinitz]

(0) $f_0 \geq n+1$

$$f_{n-1} \geq n+1$$

$$L \geq 0 \Rightarrow L^* \geq 0$$

where $L^*(P) = L(P^*)$

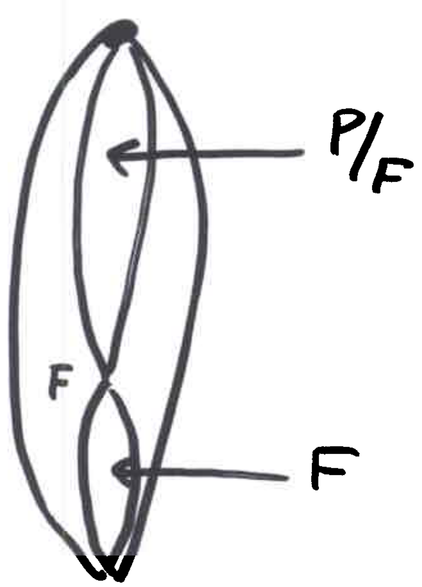
(1) Kalai convolution

M on m-dim polytopes
L on n-dim polytopes

Define

$$(M * L)(P) = \sum_{\substack{F \\ \dim(F) = m}} M(F) \cdot L(P/F)$$

P is $(m+n+1)$-dim



$M, L \geq 0$

$\Rightarrow$

$M * L \geq 0$

(2) [Stanley] For P rational

$$g_i(P) \geq 0$$

for $0 \leq i \leq \frac{n}{2}$

(3) [Kalai]

$$g_2(P) \geq 0$$

(4) [Karu]

$$g_i(P) \geq 0$$

(5) [Stanley]

$$\Psi(P) \geq 0$$

coefficient-wise

(6) [Billera-E]

$$\Psi(P) \geq \Psi(\Delta_n)$$

Δ_n = n-dim simplex

Define an inner product on

$\mathbb{R}\langle c,d\rangle$ by

$$\langle u|v\rangle = \begin{cases} 1 & \text{if } u=v \\ 0 & \text{otherwise} \end{cases}$$

for two cd-monomials u and v

Any linear inequality

$$L(P) \geq 0$$

for n-dim polytope P can be written as

$$\langle H \mid \psi(P) \rangle \geq 0$$

where $H \in \mathbb{R}\langle c,d \rangle_n$

Theorem Assume that the inequality

$$\langle H \mid \Psi(P) \rangle \geq 0$$

holds for all polytopes P. Then the inequality

$$\langle u \cdot H \cdot v \mid \Psi(P) \rangle \geq 0$$

holds for all polytopes P, where u and v are two cd-monomials such that u does not end with c and v does not begin with c.

Example

$$\Psi(n\text{-gon}) = c^2 + (n-2)\,d$$

$$\langle d - c^2 \mid \Psi(n\text{-gon}) \rangle = (n-2) - 1 \geq 0$$

Hence

$$\langle u\cdot(d - c^2)\cdot v \mid \Psi(P) \rangle \geq 0$$

Notation

For cd-monomial w of degree k let

$$\{w\} = w - \Delta_w c^k$$

where Δ_w is the coefficient of w in $\Psi(\Delta_k)$

$$\{w\} \geq 0 \iff [w]\,\Psi(P) \geq [w]\,\Psi(\Delta_k)$$

Theorem 2 *The* **cd**-*index (equivalently the* f-*vector) of a polygon* P *satisfies the inequality*

$$\{\mathbf{d}\} \geq 0 \tag{2.1}$$

Theorem 3 (Steinitz) *The* **cd**-*index (equivalently the f-vector) of a 3-dimensional polytope P satisfies the following two inequalities*

$$1 * \{\mathbf{d}\} \geq 0 \qquad \{\mathbf{d}\} * 1 \geq 0 \qquad (3.1)\ (3.1^*)$$

Theorem 4 (Bayer) *The* **cd**-*index (equivalently the flag f-vector) of a 4-dimensional polytope P satisfies the following list of six inequalities.*

$$
\begin{array}{llll}
\{\mathbf{dc}^2\} \geq 0 & \{\mathbf{c}^2\mathbf{d}\} \geq 0 & (4.1) & (4.1^*) \\
g_2^4 \geq 0 & & (4.2) & \\
1 * \{\mathbf{d}\} * 1 \geq 0 & & (4.3) & \\
\mathbf{c} * \{\mathbf{d}\} \geq 0 & \{\mathbf{d}\} * \mathbf{c} \geq 0 & (4.4) & (4.4^*)
\end{array}
$$

None of these inequalities are implied by any other currently known inequalities.

Theorem 5 *The* $\mathbf{cd}$*-index of a* 5*-dimensional polytope* P *satisfies the following list of* 13 *inequalities.*

$$
\begin{array}{rlrlll}
\{\mathbf{dc}^3\} & \geq 0 & \{\mathbf{c}^3\mathbf{d}\} & \geq 0 & (5.1) & (5.1^*) \\
1 * \{\mathbf{dc}^2\} & \geq 0 & \{\mathbf{c}^2\mathbf{d}\} * 1 & \geq 0 & (5.2) & (5.2^*) \\
1 * \{\mathbf{c}^2\mathbf{d}\} & \geq 0 & \{\mathbf{dc}^2\} * 1 & \geq 0 & (5.3) & (5.3^*) \\
1 * g_2^4 & \geq 0 & g_2^4 * 1 & \geq 0 & (5.4) & (5.4^*) \\
\mathbf{c} * \{\mathbf{d}\} * 1 & \geq 0 & 1 * \{\mathbf{d}\} * \mathbf{c} & \geq 0 & (5.5) & (5.5^*) \\
\mathbf{c}^2 * \{\mathbf{d}\} & \geq 0 & \{\mathbf{d}\} * \mathbf{c}^2 & \geq 0 & (5.6) & (5.6^*) \\
\{\mathbf{d}\} * \{\mathbf{d}\} & \geq 0 & & & (5.7) &
\end{array}
$$

Theorem 6 *The* **cd***-index of a* 6*-dimensional polytope* P *satisfies the following list of irreducible inequalities.*

$$\begin{array}{llllll}
\{\mathbf{dc}^4\} & \geq 0 & \{\mathbf{c}^4\mathbf{d}\} & \geq 0 & (6.1) & (6.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^2\} & \geq 0 & & & (6.2) & \\
\{\mathbf{dc}^2\} \cdot \mathbf{d} & \geq 0 & \mathbf{d} \cdot \{\mathbf{c}^2\mathbf{d}\} & \geq 0 & (6.3) & (6.3^*) \\
g_2^6 & \geq 0 & g_2^{6*} & \geq 0 & (6.4) & (6.4^*) \\
g_3^6 & \geq 0 & & & (6.5) &
\end{array}$$

Theorem 7 *The* **cd**-*index of a 7-dimensional polytope P satisfies the following list of eight irreducible inequalities.*

$$\begin{array}{llllll}
\{\mathbf{dc}^5\} & \geq 0 & \{\mathbf{c}^5\mathbf{d}\} & \geq 0 & (7.1) & (7.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^3\} & \geq 0 & \{\mathbf{c}^3\mathbf{dc}^2\} & \geq 0 & (7.2) & (7.2^*) \\
\{\mathbf{dc}^3\} \cdot \mathbf{d} & \geq 0 & \mathbf{d} \cdot \{\mathbf{c}^3\mathbf{d}\} & \geq 0 & (7.3) & (7.3^*) \\
g_2^7 & \geq 0 & g_2^{7*} & \geq 0 & (7.4) & (7.4^*)
\end{array}$$

Theorem 8 *The* **cd***-index of an* 8*-dimensional polytope* P *satisfies the following list of irreducible inequalities.*

$$\begin{array}{llllll}
\{\mathbf{dc}^6\} & \geq 0 & \{\mathbf{c}^6\mathbf{d}\} & \geq 0 & (8.1) & (8.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^4\} & \geq 0 & \{\mathbf{c}^4\mathbf{dc}^2\} & \geq 0 & (8.2) & (8.2^*) \\
\{\mathbf{c}^3\mathbf{dc}^3\} & \geq 0 & & & (8.3) & \\
\{\mathbf{dc}^2\mathbf{dc}^2\} & \geq 0 & \{\mathbf{c}^2\mathbf{dc}^2\mathbf{d}\} & \geq 0 & (8.4) & (8.4^*) \\
\{\mathbf{dc}^4\} \cdot \mathbf{d} & \geq 0 & \mathbf{d} \cdot \{\mathbf{c}^4\mathbf{d}\} & \geq 0 & (8.5) & (8.5^*) \\
\{\mathbf{dc}^2\} \cdot \mathbf{dc}^2 & \geq 0 & \mathbf{c}^2\mathbf{d} \cdot \{\mathbf{c}^2\mathbf{d}\} & \geq 0 & (8.6) & (8.6^*) \\
\{\mathbf{dc}^2\} \cdot \mathbf{d}^2 & \geq 0 & \mathbf{d}^2 \cdot \{\mathbf{c}^2\mathbf{d}\} & \geq 0 & (8.7) & (8.7^*) \\
\{\mathbf{c}^2\mathbf{d}\} \cdot \mathbf{dc}^2 & \geq 0 & \mathbf{c}^2\mathbf{d} \cdot \{\mathbf{dc}^2\} & \geq 0 & (8.8) & (8.8^*) \\
g_2^8 & \geq 0 & g_2^{8*} & \geq 0 & (8.9) & (8.9^*) \\
g_2^6 \cdot \mathbf{d} & \geq 0 & \mathbf{d} \cdot g_2^{6*} & \geq 0 & (8.10) & (8.10^*) \\
g_3^8 & \geq 0 & g_3^{8*} & \geq 0 & (8.11) & (8.11^*) \\
g_4^8 & \geq 0 & & & (8.12) &
\end{array}$$

n	2	3	4	5	6	7	8
$F_n - 1$	1	2	4	7	12	20	33
# facets of the polyhedron	1	2	6	13	29	60	119
# irreducible facets of the polyhedron	1	0	3	2	8	8	22

Open questions

Prove

$$\langle u \cdot H \cdot v \mid \psi(P) - \psi(\Delta_n) \rangle \geq 0$$

What is an inequality now?

$f_0 \geq n+1$

$\Updownarrow$

$\{d c^{n-2}\} \geq 0$

$f_{n-1} \geq n+1$

$\Updownarrow$

$\{c^{n-2} d\} \geq 0$

Are these sharp?

A 3-dim polytope has one 2-dim face that is either a triangle, square or a pentagon.

<u>Theorem</u> A 9-dim polytope has a 3-dim face that has less than 72 vertices or less than 72 faces.

Preprints:

http://www.ms.uky.edu/~jorge

or