"The Yang-Mills Flow on Kähler Surfaces"

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(joint with G. Daskalopoulos)

X is a Riemannian mfld

$E \to X$ is a Hermitian v.b.

$D : \Omega^p(X, E) \to \Omega^{p+1}(X, E)$

is a unitary connection

$F_D = D^2$ is the curvature

The Yang-Mills functional is:

$$YM(D) = \int_X |F_D|^2 \, d\mathrm{vol}$$

Critical points are Yang-Mills connections:

$$\begin{cases} DF_D = 0 & \text{(Bianchi)} \\ D^* F_D = 0 \end{cases}$$

Gauge symmetry:

$$g(D) = g \circ D \circ g^{-1}$$

where g is a unitary gauge transformation:

$$g \in \Omega^0(X, \operatorname{End} E) \qquad gg^* = I$$

Examples : (i) Flat connections

$$(F_D = 0)$$

(ii) In four dimensions:

(A)SD connections:

$$* F_D = \pm F_D$$

(iii) When X is Kähler:

Hermitian-Yang-Mills Conn.

$$\sqrt{-1}\, \Lambda F_D = \text{Const.}\, I$$

$$(\text{and } F_D^{0,2} = 0)$$

Gradient Flow:

$$\frac{\partial}{\partial t} D = -D^* F_D$$

$$D(0) = D_0$$

* In dim 2,3 we have long time existence and convergence to YM connections (Råde)

* In higher dimensions blow-up in finite time is possible

If X is complex, write:

$$D = D' + D''$$

D is integrable if $(D'')^2 = 0$

$$\left\{ \begin{matrix} \text{integrable} \\ \text{Connections} \end{matrix} \right\} \xleftrightarrow{H} \left\{ \begin{matrix} \text{Holomorphic} \\ \text{structures} \\ \text{on } E \end{matrix} \right\}$$

(unitary)

$$D'' = \bar{\partial}_E : \Omega^0(X, E) \longrightarrow \Omega^{0,1}(X, E)$$

Action of the complex gauge group

$$g(D) = (g(D''), H)$$

Assume X is Kähler:

(i) YM flow is contained in a complex gauge orbit

(ii) Alternative flow on metrics:

$$H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}\,\Lambda F_{(\bar\partial_E, H)}$$

$$H(0) = H_0$$

Thm: (Donaldson)

X compact Kähler, $(E, \bar\partial_E)$ holo'c

$\Downarrow$

Existence for all time

Question: What kind of convergence do we have as $t \to \infty$?

Answer: The limit is determined by the "stability" properties of the holomorphic bundle $(E, \bar{\partial}_E)$

<u>Fact</u>: ΛF_D is uniformly bounded along the flow

<u>Uhlenbeck compactness</u>:

X compact Kähler surface

Given $t_j \to \infty$ $\exists$ t_{j_k} and a finite set $Z \subset X$ such that

$$D(t_{j_k}) \longrightarrow D_\infty$$

weakly in $L^p_{1,loc.}(X - Z)$; $p > 4$

(D_∞ is on (E_∞, H_∞) locally isometric to (E, H))

Slope of E: $\mu(E) = \dfrac{\deg E}{\operatorname{rk} E}$

$$\deg E = \int_X c_1(E) \wedge \omega$$

<u>Def'n</u>: E is stable if

$$\mu(F) < \mu(E)$$

for all coherent subsheaves

$$F \subset E$$

$$0 < \operatorname{rk} F < \operatorname{rk} E$$

<u>Thm</u> (Donaldson, Uhlenbeck - Yau)

If $(E, \bar{\partial}_E)$ is stable then the YM flow with initial Condition $(\bar{\partial}_E, H)$ Converges to a Hermitian-Yang-Mills Connection on E.

More generally, we have the

<u>Harder</u>-<u>Narasimhan</u> <u>filtration</u>

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

where: E_{i+1}/E_i is (semi) stable

with slope μ_{i+1}

$$\mu_1 > \mu_2 > \cdots > \mu_\ell$$

$$Gr(E) = \bigoplus_{i=1}^{\ell} E_{i+1}/E_i$$

E: may not be subbundles

i.e. $Gr(E)$ may not be a bundle

related to bubbling

$Gr(E)^{**}$ <u>is</u> a vector bundle

Main Thm:

X compact Kähler surface

$(E, \bar{\partial}_E)$ holomorphic

D_t YM flow with initial condition $(\bar{\partial}_E, H)$

$D_{t_j} \longrightarrow D_\infty$ Uhlenbeck limit

Then:

(E_∞, D''_∞) is isomorphic to

$Gr(E, \bar{\partial}_E)^{**}$.

Step 1 : Harder - Nārasimhan type

$(E, \bar{\partial}_E) \rightsquigarrow \mu_1 > \mu_2 > \cdots > \mu_\ell$

HN type : $\vec{\mu} = (\mu_1, \ldots, \mu_R)$

(repeated with mult $= \mathrm{rk}\, E_{i+1}/E_i$)

Partial ordering :

$\vec{\mu} \leq \vec{\lambda}$ if $\sum_{i \leq k} \mu_i \leq \sum_{i \leq k} \lambda_i$

for all $k = 1, \ldots, R$.

$\vec{\mu}$ gives an absolute lower bound on YM in an orbit:

Prop: $(E, \bar{\partial}_E)$ holomorphic with HN type $\vec{\mu}$: then:

$$\sum_{i=1}^{r} \mu_i^2 \leq \frac{1}{2\pi} \int_X |\Lambda F_D|^2 \, d\mathrm{vol}$$

for all: $D = g(\bar{\partial}_E, H)$

g complex gauge transformation

Prop: $\vec{\mu}$ HN type of $(E, \bar{\partial}_E)$

$\vec{\lambda}$ HN type of (E_∞, D''_∞)

Then: $\vec{\mu} \le \vec{\lambda}$

Proof: $S \subset E$ holo. $\operatorname{rk} S = r$

By Chern-Weil:

$$\deg S = \frac{1}{2\pi}\int_X \operatorname{tr}(\sqrt{-1}\Lambda F_{D_j}\pi_j) - \frac{1}{2\pi}\int_X |D_j''\pi_j|^2$$

$$\leq \frac{1}{2\pi}\int_X \operatorname{tr}(\sqrt{-1}\Lambda F_{D_j}\pi_j)$$

$$\leq \frac{1}{2\pi}\int_X \operatorname{tr}(\sqrt{-1}\Lambda F_{D_\infty}\pi_j)$$

$$+ C\,\|\Lambda F_{D_j} - \Lambda F_{D_\infty}\|_{L^1}$$

By linear algebra: if $\sqrt{-1}\Lambda F_{D_\infty}$

has constant eigenvalues: $\lambda_1 \geq \cdots \geq \lambda_R$

$$\frac{1}{2\pi}\int_X \mathrm{tr}\left(\sqrt{-1}\Lambda F_{D_\infty}\pi_j\right) \leq \sum_{j\leq r} \lambda_j$$

Apply this to: $S = E_i$ to get:

$$\sum_{j\leq \mathrm{rk} E_i} \mu_j \leq \sum_{j\leq \mathrm{rk} E_i} \lambda_j$$

This suffices.

Step 2 : Find an approximate critical Hermitian structure !

Prop : Fix holo'c $(E, \bar\partial_E)$ with HN type $\vec{\mu}$. For any $\delta > 0$, $1 \leq p < \infty$ there is a Hermitian metric H s.t.

$$\left\| \sqrt{-1}\Lambda F_{(\bar\partial_E, H)} - \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_R \end{pmatrix} I \right\|_{L^p} \leq \delta$$

Easy when HN filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

is by subbundles.

* Resolve singularities to find L^p approximation for $p \sim 1$

* Apply a Hartman type "distance decreasing" argument to get the general result

<u>Corollary</u>: The HN types of $(E, \bar{\partial}_E)$ and the Uhlenbeck limit (E_∞, D_∞') <u>coincide</u>.

Step 3 : Convergence of projections

Prop: Let $\pi_j^{(i)}$ be projection to the i-th piece of the HN filtration of (E, D_j''). Then $\pi_j^{(i)} \rightharpoonup \pi_\infty^{(i)}$ weakly in $L^p_{2,loc.}$.

(weak holomorphic projection $\pi \in L^2_1$
$$(1-\pi)D''\pi = 0$$
)

Corollary: We may assume* that the HN filtration is preserved along the flow.

* i.e. up to gauge transformations

Step 4 : Construct holo$'$c maps

$$\mathcal{S} \hookrightarrow (E, \bar\partial_E) \xrightarrow[g_\delta]{} (E, D''_\delta)$$

Prop: The renormalized maps

converge to a nonzero map:

$$\mathcal{S} \xrightarrow[g_\infty]{} (E_\infty, D''_\infty)$$

Basic principle :

A nonzero holo'c map between stable bundles of the same rank and degree is an isomorphism.

The proof actually applies to minimizing sequences:

Thm: Let D_j be a sequence in the complex gauge orbit of $(E, \bar{\partial}_E)$

Assume:

$$\frac{1}{2\pi}\int_X |\Lambda F_{D_j}|^2 dvol \longrightarrow \sum_{i=1}^{R} \mu_i^2$$

Then: $D_j \longrightarrow D_\infty$ in $L^2_{loc.}$