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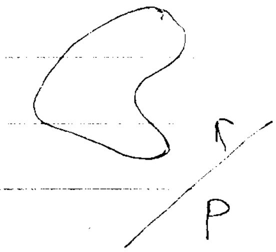
proof of Thm 1 (Not heavy either)



<u>Heuristic</u> : Geo Σ $\nabla u : S^2 \to R^3$

$$D^2 u = \begin{bmatrix} 0 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}$$

$\star \iff a\lambda_1 + b\lambda_2 = 0$, $\lambda < a, b < \lambda^{-1}$

principle curvatures of Σ $\left(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2}\right) = (k_1, k_2)$

Σ closed saddle surface in R^3 $\to \leftarrow$

Flow : $D^2 u = 0$, $\lambda_1 = \lambda_2 = 0$ may have singular pt at ∇u

Alexandrov 30s u Analytic ~~w~~/o Uniform ellipticity , $(0 < a, b)$ ok

HNY u $C^{2,\alpha}$ w/ uniform ellipticity (isolated) ok (shorter)

Pogorelov 48 u C^2 w/o Uniform ellipticity ~~ok~~

Martinez-Maure Counter-example (2001)

HNY u $W^{2,2}$ w/ uniform ellipticity ok (shorter)

<u>sketch of ~~the~~ proof</u>

Question: where does plane $P_{(1,0,0)}$ touch Σ ?

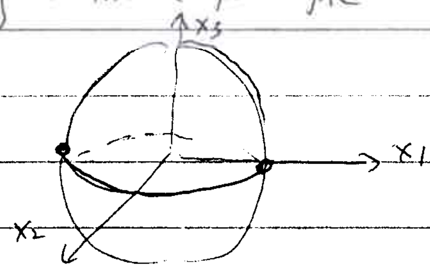
step 1. $\nabla u = (u_1, u_2, u_3) = \nabla u(x_1, x_2, 1) = (h_1, h_2, h - r h_r)$

$u = x_3 h\left(\frac{x_1, x_2}{x_3}\right)$

$$\sum_{ij=1}^{2} A_{ij}(x) D_{ij} h = 0$$ ~~satisfy~~ diverseace equation

Strong maximal principle max h_1 at ∞

or max u_1 along $x_3 = 0$



Also max u_1 along $x_2 = 0$

So $P_{(1,0,0)}$ touchs Σ at ± 1 .

step 2. P_γ touches Σ at $\nabla u(\pm\gamma)$

Now we can pick γ^* w/ $D^2 u(\gamma^*) \neq 0$ saddle pt

No touching $\to \leftarrow$ $\square$

Thm 2 (Nadirashvili - Y). Let $u(x) = |x|^d u(\frac{x}{|x|})$ be a viscosity sol to $F(D^2u) = 0$ w/ $F \in C^1$. Assume $d \neq 2$ if $n \geq 4$

$\lambda^{-1} I \leq F_{ij}(D^2u) \leq \lambda I$

Then u is harmonic, upto a change of variables, or

$$\sum_{i,j=1}^{n} F_{ij}(0) D_{ij} u = 0.$$

Consequently $u \equiv 0$ $\quad 0 > d > n-2$, or d non-integer

$u = h_d$ harmonic polynomial of deg d $\quad d$ otherwise

Rmk. Homogeneous deg $d < 2$ sol to Special Lagrangian Equation is trivial

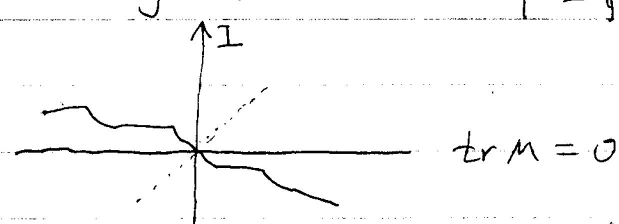
Rmk. In contrast to divergence system

$$\int_{R^3} F(\nabla U) \qquad U: R^3 \to R^5$$

Sverak - Yan (1999) $U(x) = |x|^{1-\varepsilon} U(\frac{x}{|x|})$ minimizer for strongly convex functional F.

4. Proofs

proof of Thm 2. (Light) $\qquad \Gamma = \{M \mid F(M) = 0\}$



$d \neq 2$. $\quad C = \{D^2u(x) = |x|^{d-2} D^2 u(\frac{x}{|x|}) \mid x \in R^n\}$ Cone

$x \to 0 \quad d > 2$

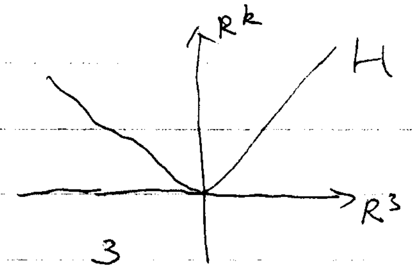
$\quad\quad\quad\quad\quad\quad\quad \Rightarrow \quad 0 \in C \subset \Gamma$

$x \to +\infty \quad d < 2$

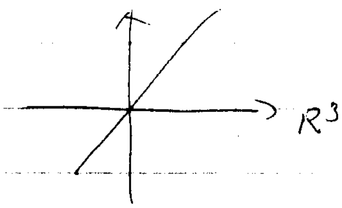
$C \perp (F_{ij}(0)) \qquad F \in C^1$

So $\displaystyle\sum_{i,j=1}^{n} F_{ij}(0) D_{ij} u = 0$ $\square$

• Minimal Cone 3-d graph-like cones ⟹ flat



⟹



$$\sum_{i,j=1}^{3} g^{ij}\partial_{ij} H = 0 \qquad H(x) = |x|\, H\left(\tfrac{x}{|x|}\right)$$

Thm 1 ⟹ H = linear   flat   (PDE proof)

Well-known result in 70s smooth case Fischer-Colbrie, Barbosa

• Fully nonlinear elliptic equations

$F(D^2u) = 0$ eg. Laplace $\Delta u = \lambda_1 + \cdots + \lambda_n = 0$ ok

$\lambda I < (F_{ij}(D^2u)) \le \lambda^{-1} I$ M-A $\log\det D^2u = \log\lambda_1 + \cdots + \log\lambda_n = 0$ ok

Special Lagrangian $\arctan\lambda_1 + \cdots + \arctan\lambda_n = 0$

Question: Schauder estimates?

$e \in R^n$ $\sum_{i,j=1}^{n} F_{ij}(D^2u)\, D_{ij} u_e = 0$, $u_e \in C^{\alpha}$

$n=2$ $u_e \in C^{1,\alpha}$, $u \in C^{2,\alpha}$ Yes

$n \geq 3$ $F_{ij} D_{ij} u_{ee} + \underline{F_{ij,kl} D_{ij} u_e D_{kl} u_e} = 0$

Krylov - Safonov not enough

Modest: homogeneous sols

$u(x) = |x|^2 u\left(\tfrac{x}{|x|}\right)$ $F(D^2u) = 0$

Question. Is $u \in C^2 / C^{2,\alpha}$?

Answer: $n=3$ Yes

$$\left.\begin{array}{l} \sum_{i,j=1}^{3} F_{ij}(D^2u)\, D_{ij} u_e \\ u_e \text{ homog. deg } 1 \end{array}\right] \xRightarrow{\text{Thm 1}} u_e \text{ linear}$$

Question: $u(x) = |x|^d u\left(\tfrac{x}{|x|}\right)$ sol to $F(D^2u) = 0$

what happens?

Answer: $n=3$ ε - regularity. Thm deg $1 \pm \varepsilon$ ok ⟹ $d = 2 \pm \varepsilon$ trivial

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Homogeneous Solutions to Elliptic Equations

1. Result $\sum_{i,j=1}^{3} a_{ij}(x) D_{ij} u(x) = 0$ ✡

$\lambda I \le (a_{ij}) \le \lambda^{-1} I$

$u(x) = |x|\, u\left(\frac{x}{|x|}\right)$

Thm 1 (Nadirashvili, Han, Y) Any homogeneous order one $W^{2,2}_{loc}$ strong sol to ✡ is linear.

2 Motivation.

Krylov - Safonov 79 $\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u = 0$

$\|u\|_{C^\alpha} \le C \|u\|_{L^\infty}$

Question: Can one improve $\alpha(n,\lambda) \in (0,1)$ to a larger one? (targeting nonlinear equations)

Answer: $n=2$ Yes $\|u\|_{C^{1,\beta}} \le C\|u\|_{L^\infty}$ Morrey - Nirenberg 50s

$n=3$ No Safonov 80s

$S(x) = |x|^\beta S\left(\frac{x}{|x|}\right)$ $0<\beta<1$

a sol to ✡ . Saddle surface $\left(\partial r - \frac{1}{r}\partial\theta \text{ plane}\right)$

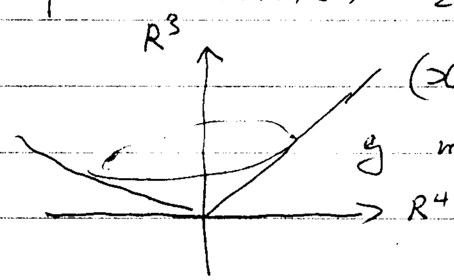
Question: $u(x) = |x|\, u\left(\frac{x}{|x|}\right)$ Lip sol NOT C^1/linear ?

$n=3$ NO Thm 1

$n=4$ Yes Lawson - Osserman (76)

$h(x) = \dfrac{|z_1|^2 - |z_2|^2}{|z|}$ sol to ✡$_4$ w/ $a_{ij}^h(x)$

Rmk. Hopf map $H(z_1, z_2) = \frac{\sqrt{5}}{2} \dfrac{\left(|z_1|^2 - |z_2|^2,\ 2 z_1 \bar{z}_2\right)}{|z|}$



$(x, H(x))$ minimal Cone

g metric induced

$\Delta_g H = 0$, harmonic coordinates $\Delta_g x = 0$

So $\sum_{i,j=1}^{4} g^{ij} \partial_{ij} H = 0$

3 Applications & Further Results