COHOMOLOGICAL CHARACTERISTICS OF REAL ALGEBRAIC VARIETIES

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In the work [K1], the author considered a space X with finite cohomology $H^*(X,\mathbb{Z}/2)$ and constructed a natural spectral sequence ${}^rH^*(X)$ which starts with $H^*(X;\mathbb{Z}/2)$ and converges to the cohomology $H^*(FX;\mathbb{Z}/2)$ of the fixed point set FX with respect to some natural filtration in $H^*(FX;\mathbb{Z}/2)$.

Virtual spectral sequence. . Denote by $K_{\mathbb{R}}$ the Grothendieck ring of the algebraic varieties over $\mathbb{R}$ (that is, reduced separated schemes of finite type over $\mathbb{R}$). It is the ring generated by symbols $[X]$, for X an algebraic variety over $\mathbb{R}$, with the relations $[X]=[X']$ if X is biregular isomorphic to X', $[X]=[X\setminus X']+[X']$, if X' is closed in X, and $[X\times X]=[X]\times[X']$.

This ring is closely related to the ring $K_0(\mathrm{Var}_{\mathbb{R}})$ from the work [PM]. The following theorem asserts that along with the virtual Betti numbers $\beta_i(\mathbb{R}X)$ constructed in [PM], there exists a *virtual spectral sequence* convergent to these numbers.

Theorem.

(1) *For all $1\le r\le+\infty$, $0\le i$ there exists a unique group isomorphism ${}^rh^i\colon K_{\mathbb{R}}\to\mathbb{Z}$ such that ${}^rh^i(X)=\dim {}^rH^i(X)$ for all compact non-singular variety X.*

(2) *For each $X\in K_{\mathbb{R}}$ we have*

$$\sum_{i\ge0}{}^{\infty}h^i(X)=\sum_{i\ge0}\beta_i(\mathbb{R}X).$$

(3) *For n-dimensional real algebraic variety X with the non-empty real part we have for all $1\le r\le\infty$*

$$\begin{aligned}{}^rh^i(X)=&0,\quad if\ \ i>n,\\ {}^rh^n(X)>&0\end{aligned}$$

This theorem about the virtual spectral sequence can be applied to calculate the usual spectral sequence discussed above.

Corollary. *Let $E \to X$ is a locally trivial fibration in the category of real algebraic varieties with a fiber F. Assume in addition that X and F are compact and non-singular. Then there exists isomorphisms*

(1) *of spectral sequences ${}^rH^*(E) \to {}^rH^*(X) \otimes {}^rH^*(F)$,*
(2) *of the filtered groups $H^*(\mathbb{R}E, \mathbb{Z}/2) \to H^*(\mathbb{R}X, \mathbb{Z}/2) \otimes H^*(\mathbb{R}F, \mathbb{Z}/2)$.*

Derived categories. The proof of the above theorem follows the scheme of the proof of existence for the virtual Betti numbers in [PS]. We consider three functors which associate to a good (having finite $\mathbb{Z}/2$-homology) space X with involution
(i) a filtered group $H^*(FX; \mathbb{Z}/2)$,
(ii) $\mathbb{Z}$-graded spectral sequence ${}^rH^*(X)$,
(iii) $\mathbb{Z}$-graded $\mathbb{Z}/2[t]$-module $H^*(X_G; \mathbb{Z}/2)$, where X_G is the total space of the Borel fibration.

The key observation in our proof is that these functors can be factorized through the bounded derived category of the ring $\mathbb{Z}/2[C_2]$, where C_2 is the group of order 2.

We observe that the main computations can be done in this category. The technique developed allows to calculate the spectral sequence ${}^rH^*(-)$ for the class of real algebraic varieties which includes non-singular complete intersections of even degree in products of projective spaces. Our calculation works as well for similar complete intersections in the products of arbitrary grassmannians and can be applied also to the varieties of k-degenerations of morphisms of vector bundles (under certain assumptions on their basespace). Note moreover, that if instead of a product we consider a single projective space, then the assumption on the degree of a complete intersection can be omitted by trivial reasons.

The latter results were published in [K2], although our arguments were presented in a bit different form. Using language of derived categories in the proof makes it more transparent.

References

[K1] I. Kalinin, *Cohomological characteristics of real projective hypersurfaces*, St. Petersburg Math. J., **3** (1992), no. 2, 313–332.

[K2] I. Kalinin, *Cohomology charactersitics of real algebraic manifolds*, St. Petersburg Math. J. **14** (2003), no. 5, 739–763.

[MP] C. McCrory, A. Parusinsky, *Virtual Betti numbers of real algebraic varieties*, arXive:math.AG/0210374, 2002.