A Weight Filtration for Real Algebraic Varieties

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Work of B. Totaro,
using methods of
F. Guillén + V. Navarro Aznar

HISTORY?

M. Wodzicki + others ....

Deligne's weight filtration

X complex algebraic variety

$H_c^i X = H_c^i(X;\mathbb{Q})$

$$0 \subset W_0^i X \subset W_1^i X \subset \cdots \subset W_i^i X = H_c^i X$$

NATURALITY: $f : X \to Y$ proper algebraic
$\Rightarrow f^* W_j^i Y \subset W_j^i X$.

MANIFOLD: X compact nonsingular
$\Rightarrow W_{i-1}^i = 0$.

NORMAL CROSSINGS: X divisor with normal crossings in a compact nonsingular $M \Rightarrow$ W is the filtration of the "Mayer-Vietoris" spectral sequence

$$X = X_1 \cup \cdots \cup X_r$$

$$X^{(p)} = \coprod_{i_0 < \cdots < i_p} X_{i_0} \cap \cdots \cap X_{i_p}$$

$$E_2^{pq} = H^q(X^{(p)}) \Rightarrow H^{p+q} X$$

(+ Dual property for X the complement of a divisor with normal crossings.)

- W is not a topological invariant (Steenbrink + Stevens)
- Consider the dual filtration on Borel-Moore homology $H_i X = H_i^{BM}(X;\mathbb{Q})$

$$H_i X = \hat{W}_i^0 X \supset \hat{W}_i^1 X \supset \cdots \supset \hat{W}_i^i X \supset 0$$

- $\alpha \in H_i X$ algebraic $\Rightarrow \alpha \in \hat{W}_i^i X$.
- $\hat{W}_i^i X = \mathrm{Im}\left[IH_i X \rightarrow H_i X \right]$ (X compact) (A. Weber)

Totaro's weight filtration
(ICM 2002)

X real algebraic variety

$H_c^* X = H_c^*(X; \mathbb{Z}_2)$

$$0 \subset W_0^* X \subset W_1^* X \subset \cdots \subset W_i^* X = H_c^* X$$

NATURALITY ✓

MANIFOLD ✓

NORMAL CROSSINGS ✓

+ other properties

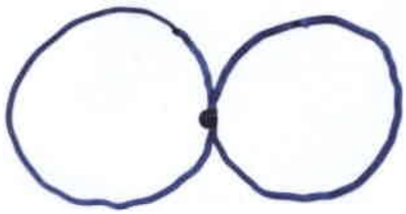
BUT some important properties of the complex weight filtration do <u>not</u> hold for the real weight filtration.

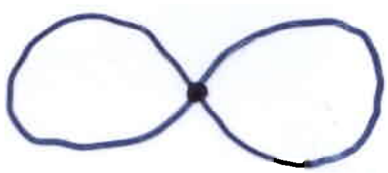
e.g. strict naturality

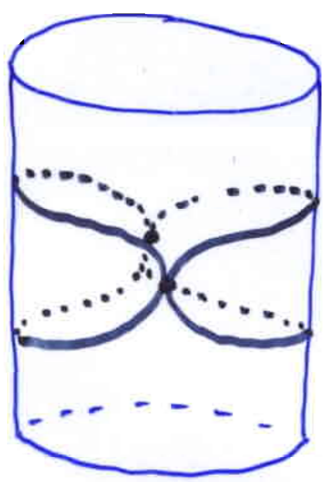
additivity

Examples: $\dim X = 1$

$$H_1 X = \hat{W}^0_1 X \supset \hat{W}^1_1 X \supset 0$$

X :  $H_1 X = \hat{W}^1_1 X$

X :  $H_1 X \neq \hat{W}^1_1 X$

X : 

$\dim H_1 X = 3$

$\dim \hat{W}^1_1 X = 2$

$f \big\downarrow$

$f_* H_1 X = H_1 Y$

$f_* \hat{W}^1_1 X = 0$

Y :  $H_1 Y = \hat{W}^1_1 Y$

Construction of weight filtration
- for complex or real varieties

Deligne's "hyperresolution" method greatly simplified by Guillén and Navarro Aznar.

Deligne's prototype:

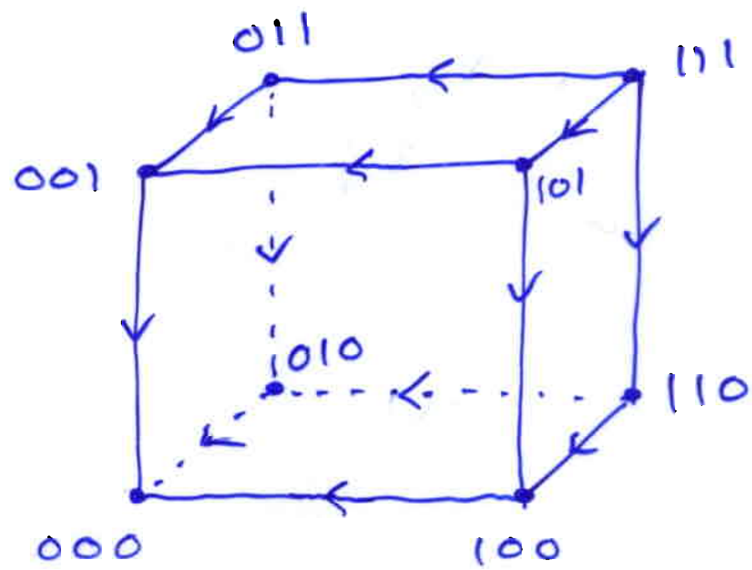
$$X = A \cup B \quad \rightsquigarrow \quad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

Navarro's prototype:

$$\begin{array}{l} f : \tilde{X} \to X \\ Y \subset X \\ \tilde{X} \setminus f^{-1}Y \xrightarrow{\approx} X \setminus Y \end{array} \quad \rightsquigarrow \quad \begin{tikzcd} f^{-1}Y \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r] & X \end{tikzcd}$$

CUBICAL HYPERRESOLUTIONS

<u>cube</u>: $\square^n = \{0,1\}^n$ poset :



A <u>cubical variety</u> is a functor

$$\square^n \longrightarrow (\text{varieties})$$

$$a = (a_1, \dots, a_n) \in \square^n$$

$$a \longmapsto X_a$$

$$X = X_{(0,\dots,0)}$$

$$\boxed{X_\bullet \longrightarrow X}$$

<u>cubical variety over X</u>

Topological realization $|X_\cdot| \to X$

$\{0,1\}^n \subset \mathbb{R}^n$

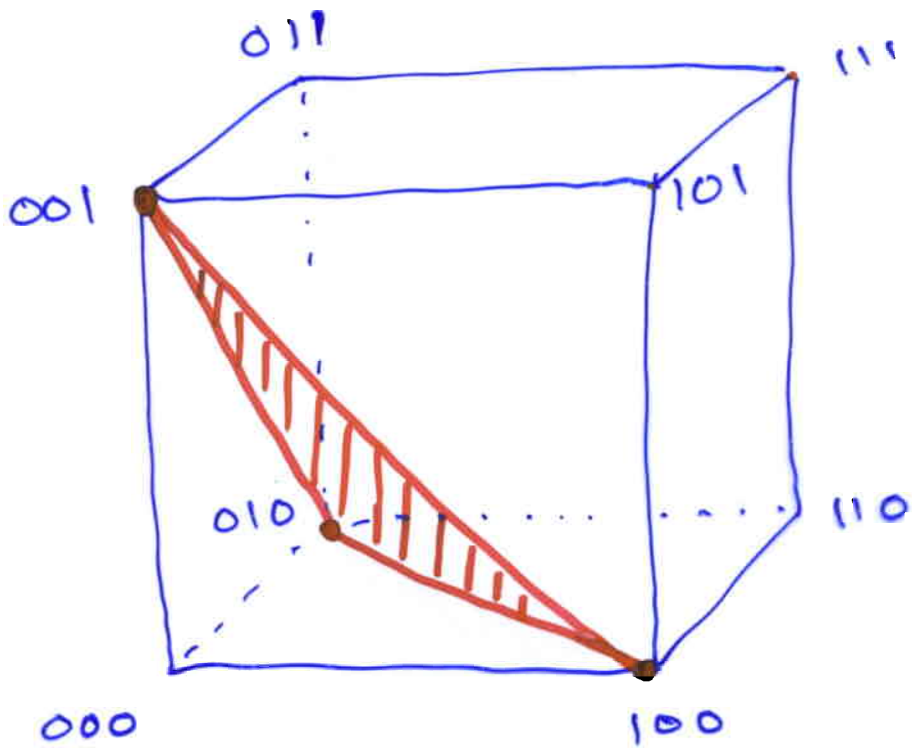
e^i on coordinate axes : $e^i_j = \delta_{ij}$

$a \in \{0,1\}^n$, $a = \sum a_i e^i$

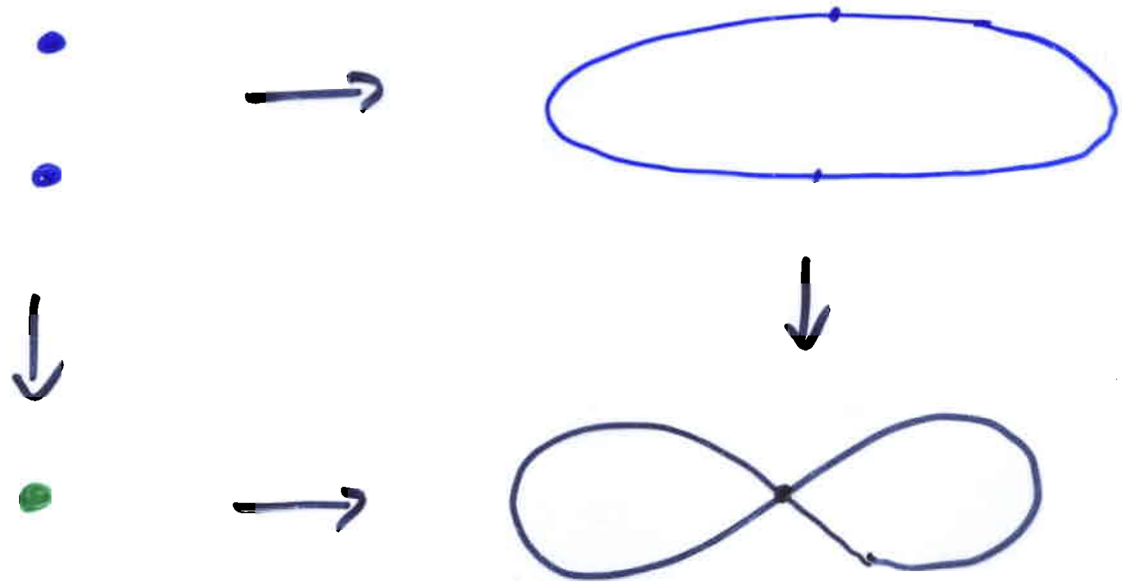
$a \neq 0$ $\quad \Delta_a \subset \mathbb{R}^n$

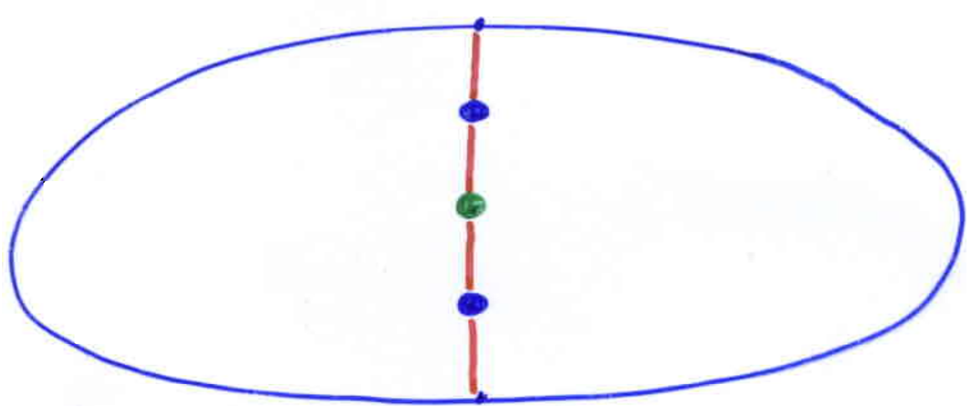
Δ_a simplex with vertices e^i s.t. $a_i \neq 0$.

$\dim \Delta_a = |a| - 1$, $|a| = \sum a_i$



$$\boxed{|X_\bullet| = \coprod X_a \times \Delta_a / \sim}$$





X compact

A <u>cubical hyperresolution</u> of X is a cubical variety over X

$$X_\bullet \longrightarrow X$$

such that

① X_a nonsingular $(a \neq 0)$

② the fibers of $|X_\bullet| \to X$ are contractible

- Condition ② is a strong version of the standard condition, which is homological.
- If X is not compact, a cubical hyperresolution of X is a cubical variety over $\bar{X} \setminus X \to \bar{X}$, $\bar{X}$ a compactification of X

<u>Theorem</u> (Guillen–Navarro Aznar)

(1) Every variety X has a cubical hyperresolution $X_\cdot \to X$, with $\dim X_a \le \dim X - |a| + 1$ for all a.

(2) The filtration of $|X_\cdot|$ by $p = |a|$ gives a spectral sequence

$$E_2^{pq} = H^q X^{(p)} \Rightarrow H^{p+q} X$$

and E_r^{pq} is independent of the hyperresolution for $r \ge 2$.

- similar result for X not compact – note that

$$H_c^{\cdot} X = H^{\cdot}(\bar{X} \setminus X \longrightarrow \bar{X})$$

$= H^{i}(\bar{X}, \bar{X} \setminus X)$

- for X complex, this is a simpler version of part of Deligne's mixed Hodge theory

The weight filtration is the filtration associated to this spectral sequence:

$$E_\infty^{pq} = W_p H_c^{p+q} X / W_{p-1} H_c^{p+q} X$$

The functors E_2^{*q} are additive:

$Y \subset X$ closed subvariety $\Rightarrow$

$$\cdots \to E_2^{pq}(X \setminus Y) \to E_2^{pq} X \to E_2^{pq} Y \to E_2^{p+1\,q}(X \setminus Y) \to \cdots$$

Thus we have additive euler characteristics

$$\beta_q(X) = \sum_p (-1)^p \dim E_2^{p,q}$$

These are the "virtual Betti numbers"

For complex varieties $E_2 = E_\infty$

$\rightsquigarrow$ mixed Hodge theory.

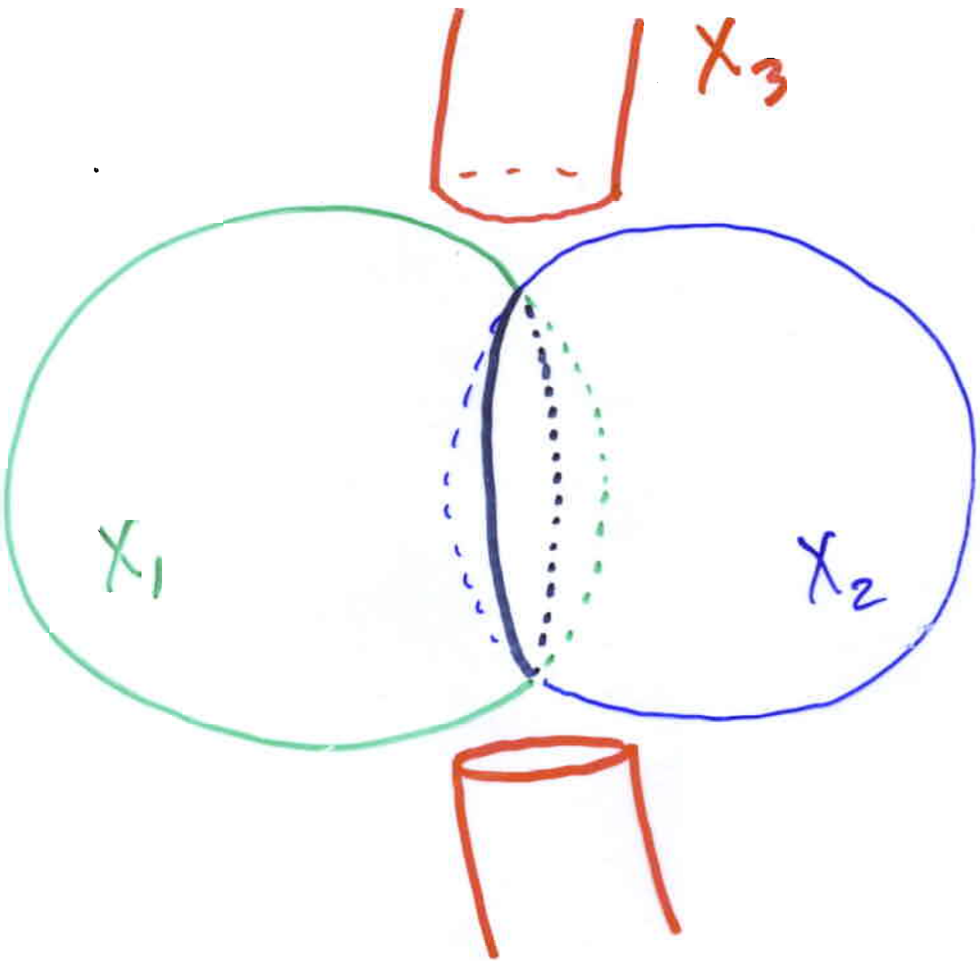
+ virtual Betti numbers are weighted Euler characteristics.

For real varieties $E_2 \neq E_\infty$ in general, and virtual Betti numbers are not associated to the weight filtration.

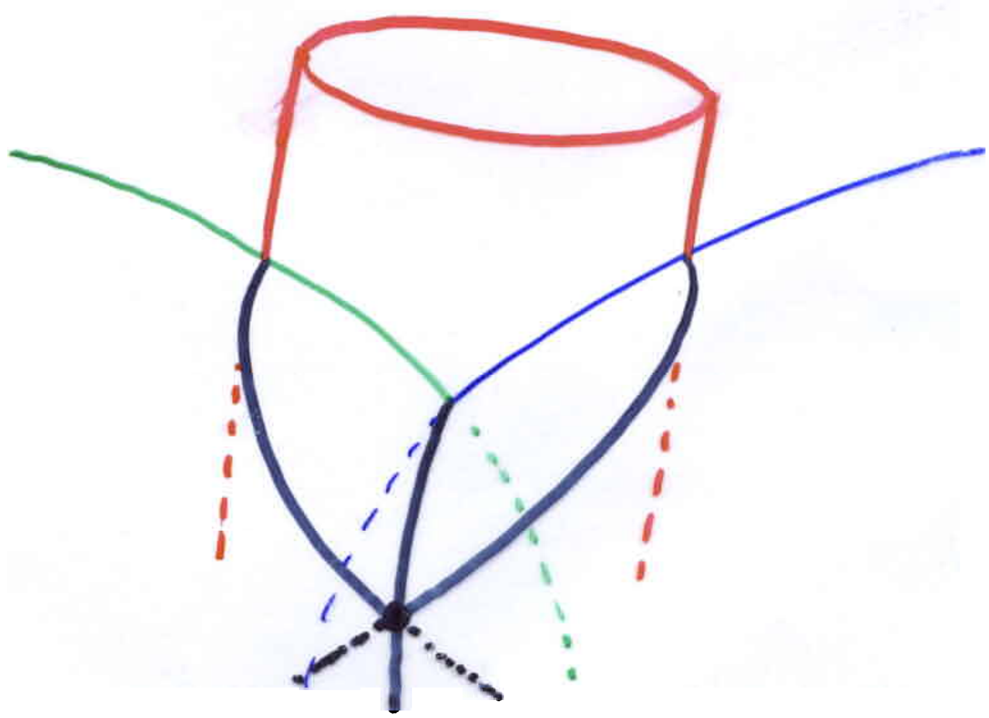
<u>EXAMPLE</u> (Parusiński)

X surface in $\mathbb{R}^3$, union of two spheres and a torus, intersecting transversely. So the weight spectral sequence is the Mayer-Vietoris spectral sequence.

$$d_2 \neq 0$$



X_3
X_1
X_2



$X = X_1 \cup X_2 \cup X_3$

$H_1 X = \mathbb{Z}_2$, generated by the "long" circle on the torus X_3

The "short" circle on X_3 represents

$\alpha \in H_1 X_3 \subset E_2^{01}$.

$\alpha \neq 0$ in E_2 because α does not bound in $X_1 \cup X_3$ or $X_2 \cup X_3$

$\alpha \in \mathrm{Im}\, d_2$

$\cancel{d_2 \alpha = 0}$ because α bounds in $X_1 \cup X_2 \cup X_3$.

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