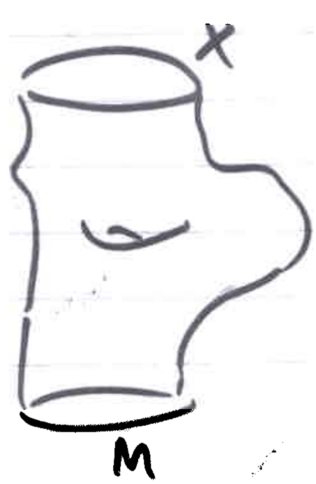
Isotoping Submanifolds to Subvarieties

- All results joint with S. Akbulut
- All varieties are affine unless stated otherwise

Thm. If V nonsingular r.a.v., M cpct manifold, $f : M \to V$. Then there is a nonsingular r.a.v. W and a polynomial $\varphi : W \to V$ approximating f iff. f is bordant to a polynomial map.

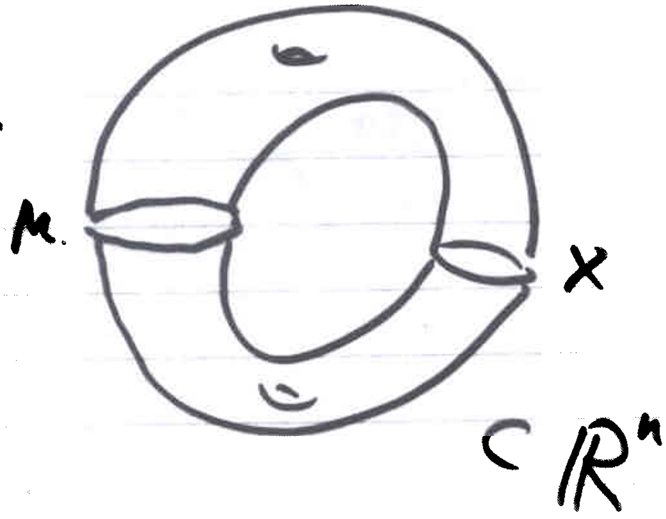


$\xrightarrow{F} V$

$F|_X$ poly

$F|_M = f$

Proof:



$\xrightarrow{F \times h} V \times E_{n,k}$

canon. bundle over Grassmann

approximate $F \times h$ by a polynomial p, take $p^{-1}(V \times G_{n,k})$. You get r.a.v.

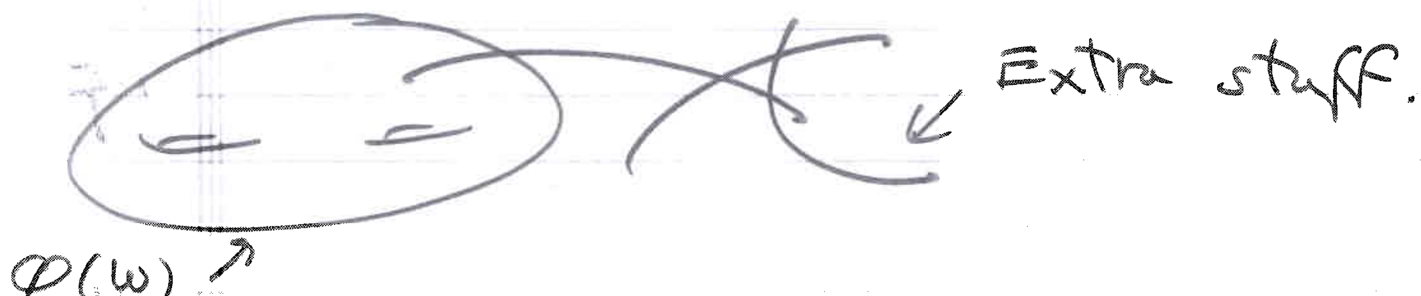
Now take a polynomial $g : \to \mathbb{R}$ with indicated signs (so $g|_X = 0$).

Let $W = g^{-1}(0) - X$. $\square$

Note: W is now in $\mathbb{R}^n \times \mathbb{R}^k$ for a <u>big</u> k.

Now, if $M \subset V$ let's try to make M isotopic to a r.a.v.

We have a polynomial $\varphi : W \to V$ so $\varphi(W) \approx M$, but the Zariski closure of $\varphi(W)$ may include lower dimensional garbage



This extra stuff comes from:

a) $\varphi_{\mathbb{C}}$(some points in $W_{\mathbb{C}} - W$)

b) If $\varphi_{\mathbb{C}}$ is not proper, limits of $\varphi_{\mathbb{C}}$(points near ∞ in $W_{\mathbb{C}}$)

GOAL – Modify $\varphi_{\mathbb{C}}$ on $W_{\mathbb{C}} - W$ to eliminate the extra stuff.

Thm. $\varphi : W_{\mathbb{C}} \to V_{\mathbb{C}}$ polynomial $/\mathbb{R}$

Suppose $T \subset \varphi^{-1}(V)$ is compact, $T = \overline{T}$, $\varphi|_T$ is finite-to-one.

Suppose $f : T \to \mathbb{C}$ continuous and $f(\bar{z}) = \overline{f(z)}$ all $z \in T$.

Then there is a polynomial $p : W_{\mathbb{C}} \to \mathbb{C}$ defined over $\mathbb{R}$ so

$$p|_T \approx f \qquad (C^0 \text{ approx})$$

Moreover, if $Q_{\mathbb{C}} \subset W_{\mathbb{C}}$ defined $/\mathbb{R}$ and $f|_u = 0$ u a nbhd of $Q_{\mathbb{C}} \cap T$

then we may also have:

$$p|_u \approx 0 \qquad (C^1 \text{ approx})$$

$$p|_{Q_{\mathbb{C}}} = 0$$

Proof: For each $x \in \varphi(T)$ it suffices to find $p_x \simeq f$ on $T \cap \varphi^{-1}(U_x)$

U_x a nbhd of x. $\quad p = \sum p_x \lambda_x \circ \varphi$

where λ_x a poly. approx. to a partition of unity for $\{U_x\}$.

If $T \cap \varphi^{-1}(x) = A \cup \bar{A}$ with $A \cap \bar{A} \subset W$

$$p_x = \sum_{y \in A} g_y h_y$$

g_y = poly. $/\mathbb{R}$, 0 on Q, $g_y(y) = f(y)$

$h_y \approx 1$ near $y, \bar{y}$

$h_y \approx 0$ near $T \cap \varphi^{-1}(x) - \{y, \bar{y}\}$

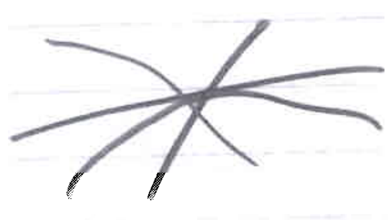
$\square$

Thm. An immersion $f: M \looparrowright \mathbb{R}^n$ is ε-isotopic to an immersion onto an almost nonsingular r.a.v. $Y \subset \mathbb{R}^n$ iff. f is immersion cobordant to an almost nonsingular r.a.v. of $\mathbb{R}^n$.

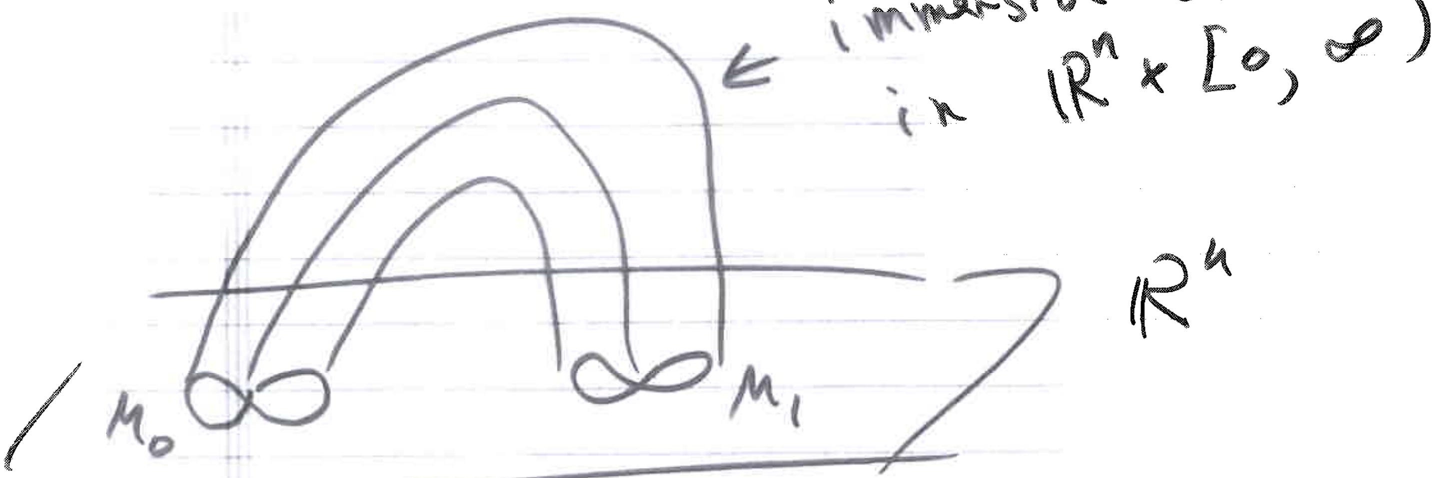
Corollary $M \subset \mathbb{R}^n$ is ε-isotopic to a nonsingular r.a.v. $Y \subset \mathbb{R}^n$ iff

$$[M] \in \pi_n^{alg} \Omega^\infty \Sigma^\infty MO(n-m)$$

Def. Y is almost nonsingular at x if locally Y = union of analytic manifolds Y_α and $Y_{\mathbb{C}}$ = union of the analytic complexifications of the Y_α.



Def. Immersion cobordism is the equiv. relation generated by:



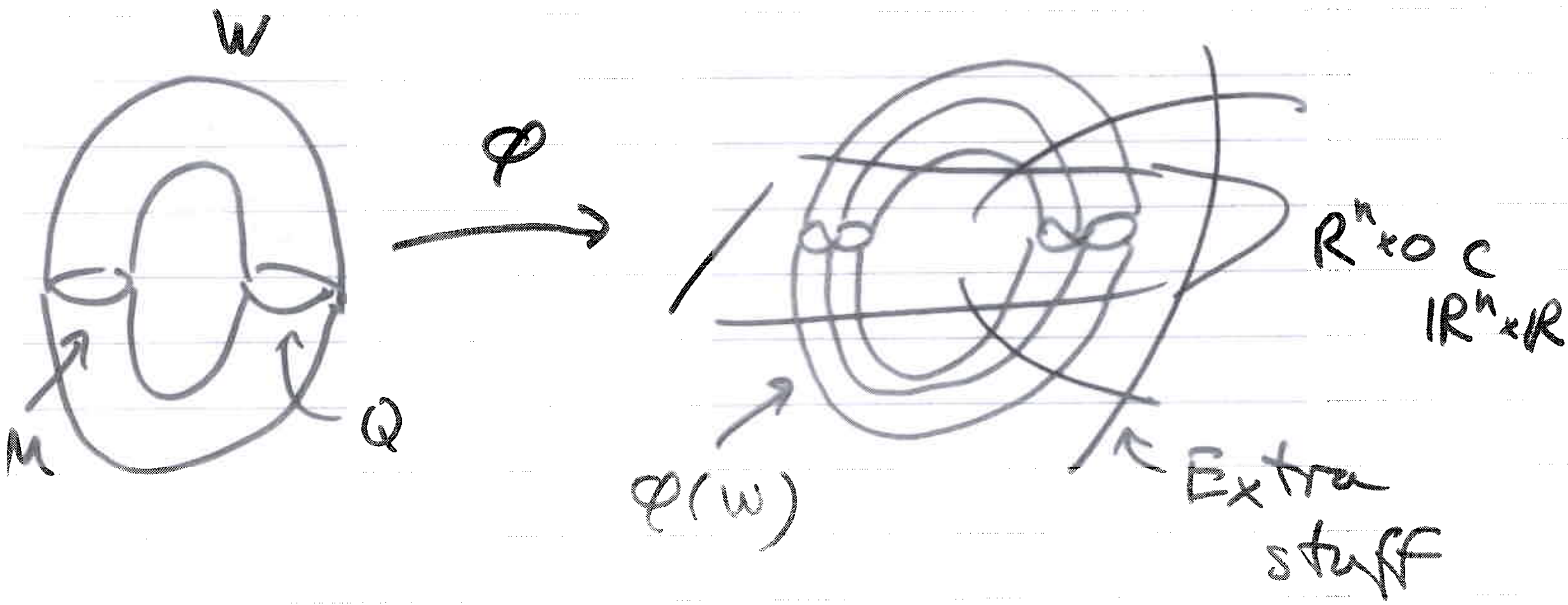
So $\partial N = M_0 \amalg M_1$
$\quad\;\, \| $
$\quad N \cap \mathbb{R}^n \times 0$

Related Results:

1) $M \subset \mathbb{R}^n \Rightarrow M \simeq \text{Nonsing } V$ for a r.a.v. $V \subset \mathbb{R}^n$

2) $M \subset \mathbb{R}^n \Rightarrow M \simeq V$ for a r.a.v. $V \subset \mathbb{R}^{n+1}$

Proof. Double the immersion of N, and represent it by a polynomial:



So $\varphi(Q)$ is an almost n.s. r.a.v.

For simplicity, assume extra stuff cpct.

$$\begin{tikzcd} & & \mathbb{C} \\ W_{\mathbb{C}} \arrow[r, "\varphi_{\mathbb{C}}"'] \arrow[urr, bend left] \arrow[drr, bend right, "\psi"'] & \mathbb{C}^n \times \mathbb{C} \arrow[ur] \arrow[dr] & \\ & & \mathbb{C}^n \end{tikzcd}$$

We may assume ψ is a finite map.

Let $T = \varphi^{-1}(\mathbb{R}^n)$

Pick $f : T \to \mathbb{R}$ so

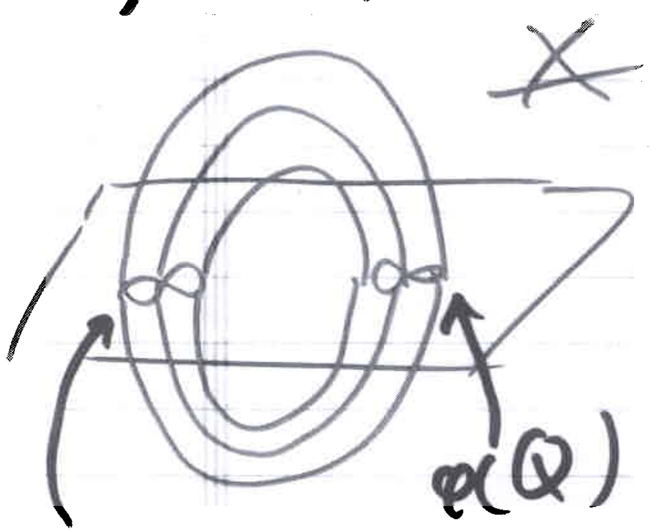
$f|_W = 0$, $f = 1$ outside

a small nbhd of W in T.

Approximate f by a polynomial p,

so $p|_Q = 0$. For large b,

$(\varphi, \sigma + bp)(W_\varepsilon) \cap \mathbb{R}^n \times \mathbb{R}$ is



Extra stuff moved up by b, no longer intersects $\mathbb{R}^n \times 0$.

immersion ε-isotopic to M, algebraic in $\mathbb{R}^n \times 0$. $\square$

Amusing Corollary – There is an $M \subset \mathbb{R}^n$ so that no matter how it is made nonsingular algebraic $V \subset \mathbb{R}^n$, then $V_{\mathbb{C}} \subset \mathbb{C}P^n$ must be singular.

$$M = \mathbb{R}P^m \times S^1 \subset \mathbb{R}^n$$

m even
$n = 2m-4$

$M = \partial(\mathbb{R}P^m \times D^2)$ so can be made algebraic

$$\begin{array}{ccc} H^2(\mathbb{C}P^n) & \xrightarrow{\approx} & H^2(V_{\mathbb{C}}) \\ \downarrow j^* & & \downarrow j^* \\ H^2(\mathbb{R}P^n) & & H^2(V) \\ & \searrow \quad H^2(\mathbb{R}^n) = 0 \quad \nearrow & \end{array}$$

$\mathbb{Z}/2\mathbb{Z}$ coefficients.

$\boxed{\text{Suppose } M \approx V \text{ with } V_{\mathbb{C}} \subset \mathbb{C}P^n \text{ nonsingular}}$

Thm.

$$j_*(H_A^{2k}(V_{\mathbb{C}})) = H_A^k(V)^2$$

Poincaré duals of homology of subvarieties defined over $\mathbb{R}$.

squares of Poincaré duals of homology of (real) subvarieties.

In our example, the LHS is 0. But the RHS contains $\omega_1^2 = \alpha^2 \times 1 \neq 0$