Polygon spaces

or

Spaces of clouds

Jean-Claude HAUSMANN

University of Geneva, Switzerland

Report on joint works with

Allen KNUTSON (1997–98) and Eugenio RODRIGUEZ (2002)

$$\mathcal{N}_d^m := \mathrm{Isom}^+(\mathbb{R}^d) \Big\backslash (\mathbb{R}^d)^m$$


Lever
Alkaïd
Petite Ourse
Polaire
Cassiopée
Chiens
Dubhé
Mérak
Grande Ourse
Girafe
Persée
Lynx
Capella
Cocher
Castor

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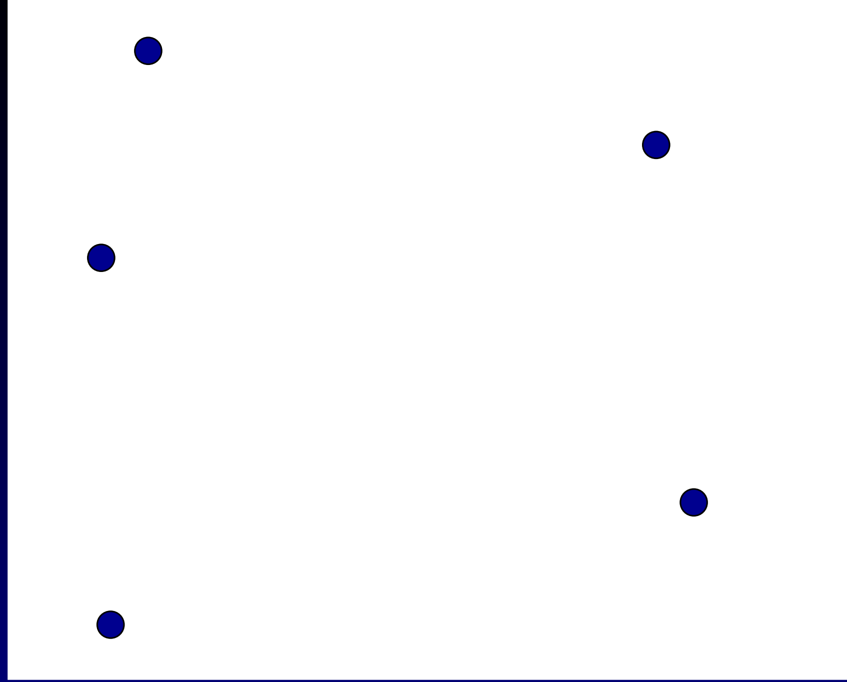

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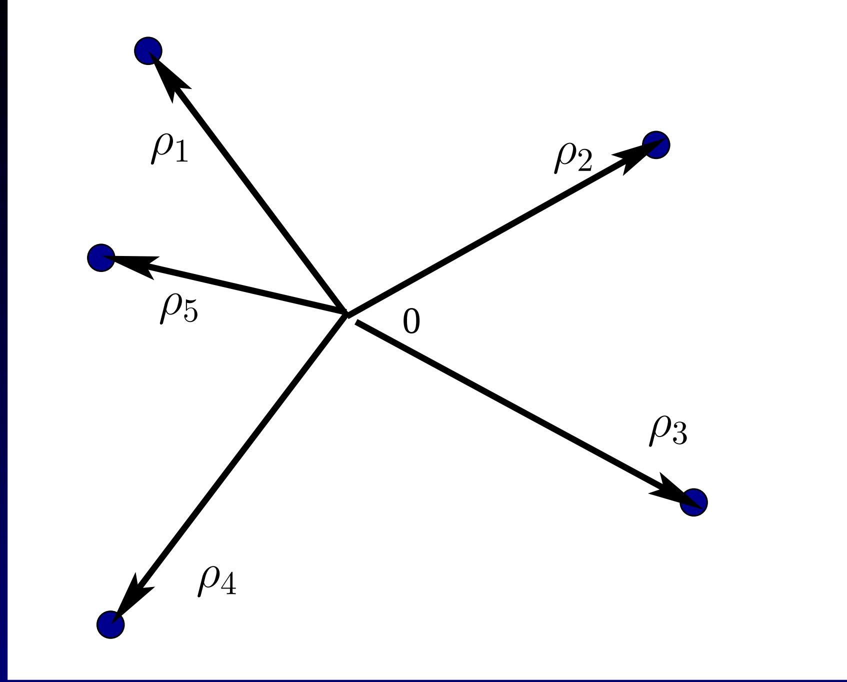

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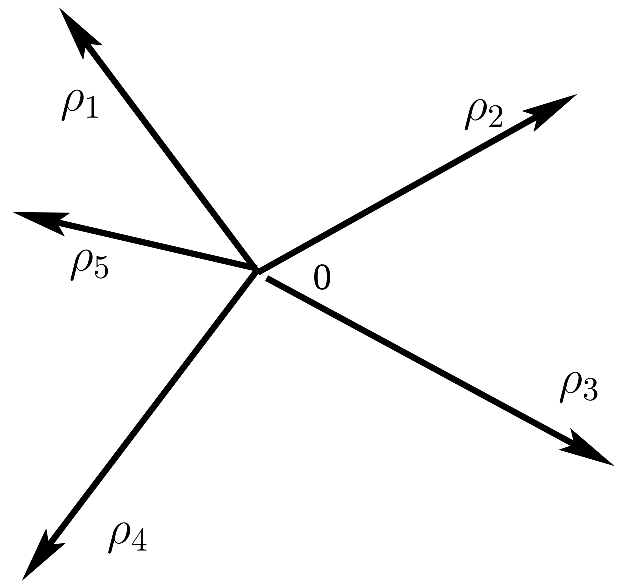

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where

$$\mathcal{V}_d^m := \left\{ (\rho_1, \ldots \rho_m) \in (\mathbb{R}^d)^m \,\middle|\, \sum \rho_i = 0 \right\}$$

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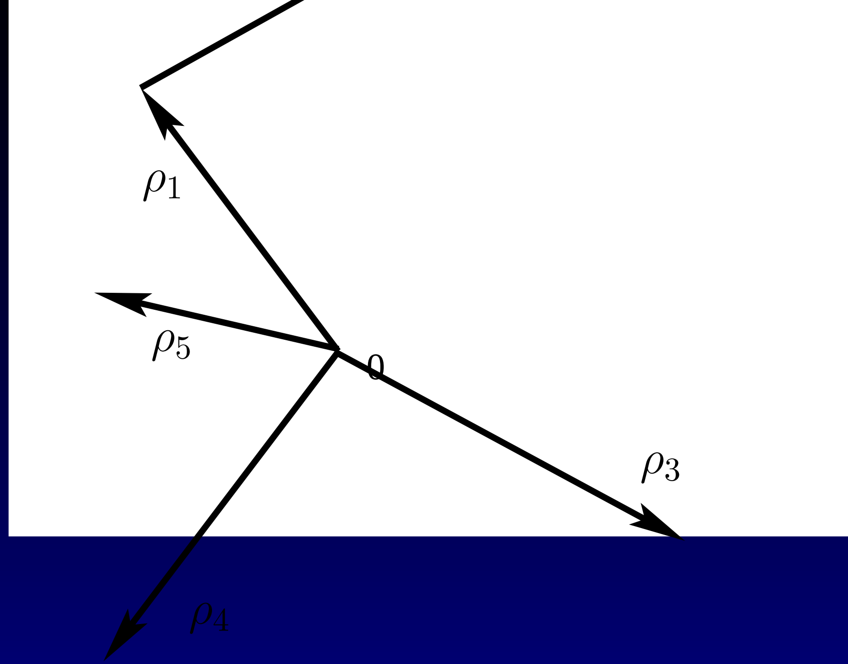

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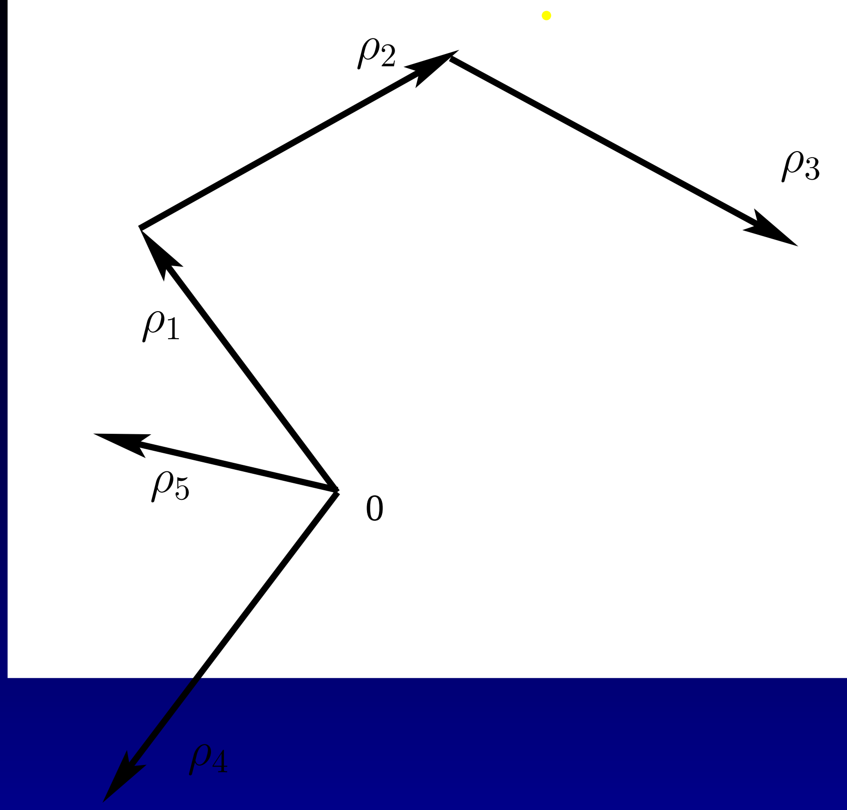

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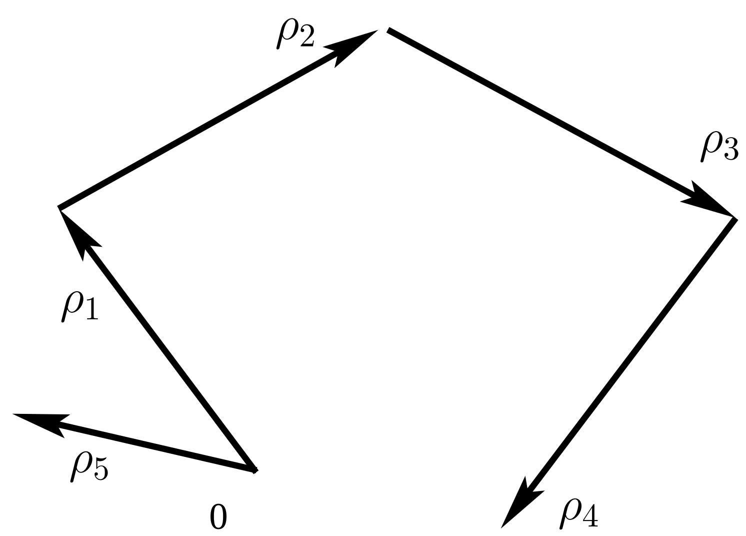

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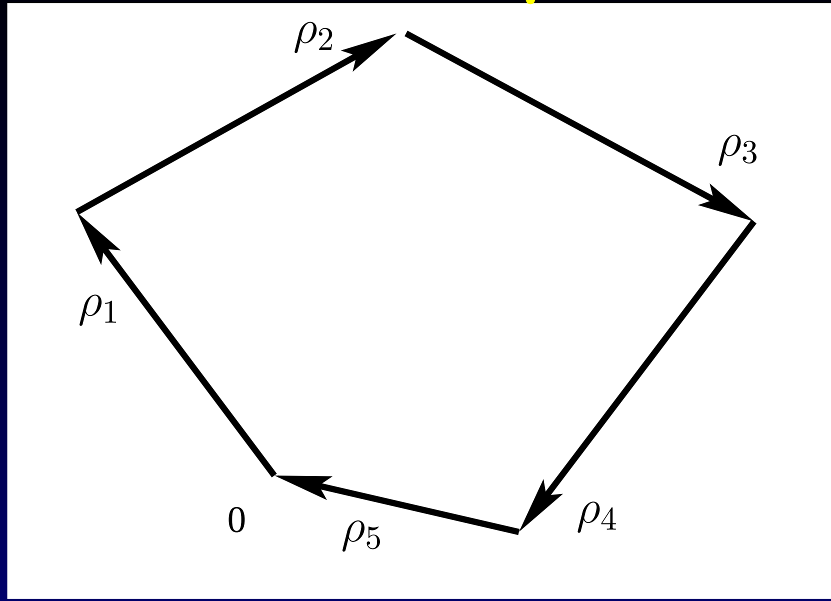

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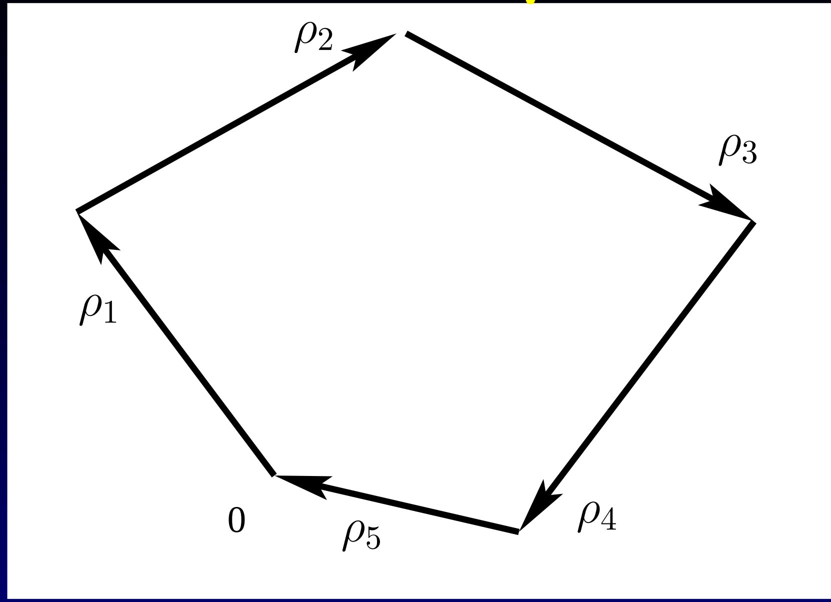

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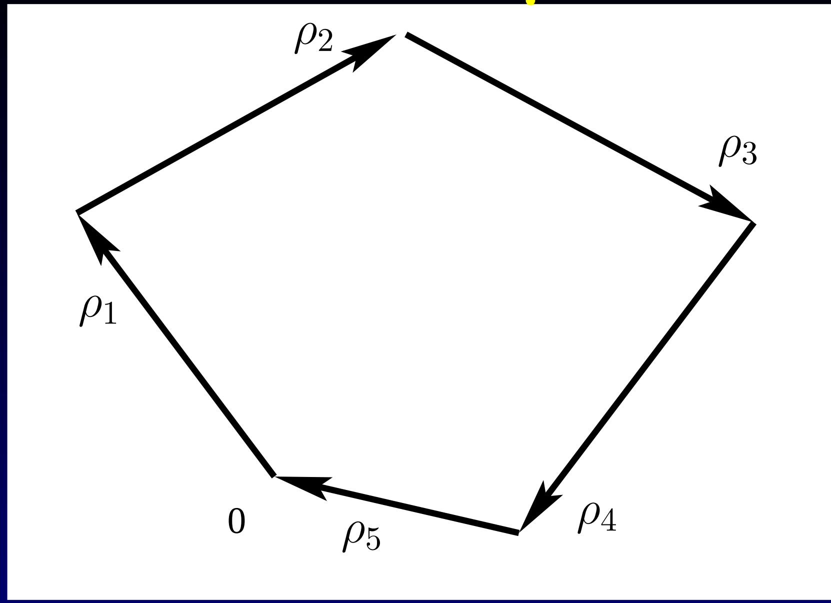

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Examples:

a	$\mathcal{N}_3^m(a)$	$\bar{\mathcal{N}}_2^m(a)$	$\mathcal{N}_2^m(a)$
$(1, \cdots, 1, m-2)$	$\mathbb{C}P^{m-3}$	$\mathbb{R}P^{m-3}$	S^{m-3}
$(\varepsilon, \cdots, \varepsilon, 1, 1, 1)$	$(S^2)^{m-3}$	T^{m-3}	$T^{m-3} \dot{\cup} T^{m-3}$

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Moreover, there is a ring isomorphism

$$H^{2*}(\mathcal{N}_3^m(a); \mathbb{Z}_2) \xrightarrow{\approx} H^*(\bar{\mathcal{N}}_2^m(a); \mathbb{Z}_2)$$

dividing the degrees in half. (A. Knutson – JCH, 1997–98)

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- $$\sum_{i \in I} a_i \neq \sum_{j \notin I} a_j$$

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PROPOSITION (A. Knutson – JCH, 1998)
Let a and a' be generic. Then

$$\mathcal{S}(a) \cong \mathcal{S}(a') \implies \mathcal{N}_d^m(a) \cong \mathcal{N}_d^m(a')$$

for all d

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There exists an algorithm giving all genetic codes (and then all polygon spaces). *(E. Rodriguez – JCH, 2002)*

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4	3	7
5	7	21

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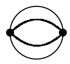
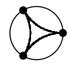
Generic 5-gons and all 4-gons

	genetic code	$a_{\min}$	$\bar{\mathcal{N}}_2^5(\alpha)$	$\mathcal{N}_2^5(\alpha)$	$a^-_{\min}$	$\mathcal{N}_2^4(a^-)$
1	$\langle\rangle$	$(0,0,0,0,1)$	$\emptyset$	$\emptyset$	$(0,0,0,1)$	$\emptyset$
2	$\langle 5\rangle$	$(1,1,1,1,3)$	$\mathbb{R}P^2$	S^2	$(1,1,1,3)$	1 point
3	$\langle 51\rangle$	$(0,1,1,1,2)$	K	T^2	$(1,1,1,2)$	S^1
4	$\langle 52\rangle$	$(1,1,2,2,3)$	Σ_2	Σ_2^{or}	$(1,2,2,3)$	$S^1 \vee S^1$
5	$\langle 521\rangle$	$(0,0,1,1,1)$	T^2	$T^2 \dot\cup T^2$	$(0,1,1,1)$	$S^1 \dot\cup S^1$
6	$\langle 53\rangle$	$(1,1,1,2,2)$	Σ_3	Σ_3^{or}	$(1,1,2,2)$	
7	$\langle 54\rangle$	$(1,1,1,1,1)$	Σ_4	Σ_4^{or}	$(1,1,1,1)$	

6-gon spaces

	α	$a_{\min}(\alpha)$	b_1	$\mathrm{r}_\cup$	s_2
1	$\langle\rangle$	$(0,0,0,0,0,1)$	0	0	0
2	$\langle 6\rangle$	$(1,1,1,1,1,4)$	1	1	1
3	$\langle 61\rangle$	$(0,1,1,1,1,3)$	2	2	2
4	$\langle 6321\rangle$	$(0,0,0,1,1,1)$	3	0	0
5	$\langle 621\rangle$	$(0,0,1,1,1,2)$	3	2	0
6	$\langle 62\rangle$	$(1,1,2,2,2,5)$	3	3	3
7	$\langle 632\rangle$	$(1,1,1,3,3,4)$	4	1	1
8	$\langle 631\rangle$	$(0,1,1,2,2,3)$	4	2	0
9	$\langle 621, 63\rangle$	$(1,1,2,3,3,5)$	4	3	1
10	$\langle 63\rangle$	$(1,1,1,2,2,4)$	4	4	4
11	$\langle 641\rangle$	$(0,1,1,1,2,2)$	5	2	0
12	$\langle 632, 64\rangle$	$(1,1,1,2,3,3)$	5	2	2
13	$\langle 631, 64\rangle$	$(1,2,2,3,4,5)$	5	3	1
14	$\langle 621, 64\rangle$	$(1,1,2,2,3,4)$	5	4	2
15	$\langle 64\rangle$	$(1,1,1,1,2,3)$	5	5	5
16	$\langle 651\rangle$	$(0,1,1,1,1,1)$	6	2	0
17	$\langle 641, 65\rangle$	$(1,2,2,2,3,3)$	6	3	1
18	$\langle 632, 65\rangle$	$(1,1,1,2,2,2)$	6	3	3
19	$\langle 631, 65\rangle$	$(1,2,2,3,3,4)$	6	4	2
20	$\langle 621, 65\rangle$	$(1,1,2,2,2,3)$	6	5	3
21	$\langle 65\rangle$	$(1,1,1,1,1,2)$	6	6	6

6-gon spaces

	α	$a_{\min}(\alpha)$	b	$\mathrm{r}_\cup$	s	$\bar{\mathcal{N}}_2^6(\alpha)$	$\mathcal{N}_2^6(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,0,1)$	0	0	0	$\emptyset$	$\emptyset$
2	$\langle 6\rangle$	$(1,1,1,1,1,4)$	1	1	1	$\mathbb{R}P^3$	S^3
3	$\langle 61\rangle$	$(0,1,1,1,1,3)$	2	2	2		
4	$\langle 6321\rangle$	$(0,0,0,1,1,1)$	3	0	0	T^3	$T^3 \amalg T^3$
5	$\langle 621\rangle$	$(0,0,1,1,1,2)$	3	2	0		
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10	$\langle 63\rangle$	$(1,1,1,2,2,4)$	4	4	4		
11	$\langle 641\rangle$	$(0,1,1,1,2,2)$	5	2	0		
12	$\langle 632,64\rangle$	$(1,1,1,2,3,3)$	5	2	2		
13	$\langle 631,64\rangle$	$(1,2,2,3,4,5)$	5	3	1		
14	$\langle 621,64\rangle$	$(1,1,2,2,3,4)$	5	4	2		
15	$\langle 64\rangle$	$(1,1,1,1,2,3)$	5	5	5		
16	$\langle 651\rangle$	$(0,1,1,1,1,1)$	6	2	0		
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19	$\langle 631,65\rangle$	$(1,2,2,3,3,4)$	6	4	2		
20	$\langle 621,65\rangle$	$(1,1,2,2,2,3)$	6	5	3		
21	$\langle 65\rangle$	$(1,1,1,1,1,2)$	6	6	6		

6-gon spaces

	α	$a_{\min}(\alpha)$	b	$\mathrm{r}_\cup$	s	$\bar{\mathcal{N}}_2^6(\alpha)$	$\mathcal{N}_2^6(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,0,1)$	0	0	0	$\emptyset$	$\emptyset$
2	$\langle 6\rangle$	$(1,1,1,1,1,4)$	1	1	1	$\mathbb{R}P^3$	S^3
3	$\langle 61\rangle$	$(0,1,1,1,1,3)$	2	2	2	$\mathbb{R}P^3 \sharp \overline{\mathbb{R}P}^3$	$S^2 \times S^1$
4	$\langle 6321\rangle$	$(0,0,0,1,1,1)$	3	0	0	T^3	$T^3 \amalg T^3$
5	$\langle 621\rangle$	$(0,0,1,1,1,2)$	3	2	0	$T^2 \times [0,1]/\sim$	$T^2 \times S^1$
6	$\langle 62\rangle$	$(1,1,2,2,2,5)$	3	3	3		
7	$\langle 632\rangle$	$(1,1,1,3,3,4)$	4	1	1		
8	$\langle 631\rangle$	$(0,1,1,2,2,3)$	4	2	0	$\Sigma_2^{or} \times [0,1]/\sim$	$\Sigma_2^{or} \times S^1$
9	$\langle 621, 63\rangle$	$(1,1,2,3,3,5)$	4	3	1		
10	$\langle 63\rangle$	$(1,1,1,2,2,4)$	4	4	4		
11	$\langle 641\rangle$	$(0,1,1,1,2,2)$	5	2	0	$\Sigma_3^{or} \times [0,1]/\sim$	$\Sigma_3^{or} \times S^1$
12	$\langle 632, 64\rangle$	$(1,1,1,2,3,3)$	5	2	2		
13	$\langle 631, 64\rangle$	$(1,2,2,3,4,5)$	5	3	1		
14	$\langle 621, 64\rangle$	$(1,1,2,2,3,4)$	5	4	2		
15	$\langle 64\rangle$	$(1,1,1,1,2,3)$	5	5	5		
16	$\langle 651\rangle$	$(0,1,1,1,1,1)$	6	2	0	$\Sigma_4^{or} \times [0,1]/\sim$	$\Sigma_4^{or} \times S^1$
17	$\langle 641, 65\rangle$	$(1,2,2,2,3,3)$	6	3	1		
18	$\langle 632, 65\rangle$	$(1,1,1,2,2,2)$	6	3	3		
19	$\langle 631, 65\rangle$	$(1,2,2,3,3,4)$	6	4	2		
20	$\langle 621, 65\rangle$	$(1,1,2,2,2,3)$	6	5	3		
21	$\langle 65\rangle$	$(1,1,1,1,1,2)$	6	6	6		

6-gon spaces

	α	$a_{\min}(\alpha)$	b	$\mathrm{r}_\cup$	s	$\bar{\mathcal{N}}_2^6(\alpha)$	$\mathcal{N}_2^6(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,0,1)$	0	0	0	$\emptyset$	$\emptyset$
2	$\langle 6\rangle$	$(1,1,1,1,1,4)$	1	1	1	$\mathbb{R}P^3$	S^3
3	$\langle 61\rangle$	$(0,1,1,1,1,3)$	2	2	2	$\mathbb{R}P^3 \sharp \overline{\mathbb{R}P}^3$	$S^2 \times S^1$
4	$\langle 6321\rangle$	$(0,0,0,1,1,1)$	3	0	0	T^3	$T^3 \amalg T^3$
5	$\langle 621\rangle$	$(0,0,1,1,1,2)$	3	2	0	$T^2 \times [0,1]/\sim$	$T^2 \times S^1$
6	$\langle 62\rangle$	$(1,1,2,2,2,5)$	3	3	3	$\mathbb{R}P^3 \sharp 2\,\overline{\mathbb{R}P}^3$	
7	$\langle 632\rangle$	$(1,1,1,3,3,4)$	4	1	1	$T^3 \sharp \overline{\mathbb{R}P}^3$	
8	$\langle 631\rangle$	$(0,1,1,2,2,3)$	4	2	0	$\Sigma_2^{or} \times [0,1]/\sim$	$\Sigma_2^{or} \times S^1$
9	$\langle 621, 63\rangle$	$(1,1,2,3,3,5)$	4	3	1	$T^2 \times [0,1]/\sim\ \sharp \overline{\mathbb{R}P}^3$	
10	$\langle 63\rangle$	$(1,1,1,2,2,4)$	4	4	4	$\mathbb{R}P^3 \sharp 3\,\overline{\mathbb{R}P}^3$	
11	$\langle 641\rangle$	$(0,1,1,1,2,2)$	5	2	0	$\Sigma_3^{or} \times [0,1]/\sim$	$\Sigma_3^{or} \times S^1$
12	$\langle 632, 64\rangle$	$(1,1,1,2,3,3)$	5	2	2	$T^3 \sharp 2\,\overline{\mathbb{R}P}^3$	
13	$\langle 631, 64\rangle$	$(1,2,2,3,4,5)$	5	3	1	$\Sigma_2^{or} \times [0,1]/\sim \sharp \overline{\mathbb{R}P}^3$	
14	$\langle 621, 64\rangle$	$(1,1,2,2,3,4)$	5	4	2	$T^2 \times [0,1]/\sim\ \sharp 2\,\overline{\mathbb{R}P}^3$	
15	$\langle 64\rangle$	$(1,1,1,1,2,3)$	5	5	5	$\mathbb{R}P^3 \sharp 4\,\overline{\mathbb{R}P}^3$	
16	$\langle 651\rangle$	$(0,1,1,1,1,1)$	6	2	0	$\Sigma_4^{or} \times [0,1]/\sim$	$\Sigma_4^{or} \times S^1$
17	$\langle 641, 65\rangle$	$(1,2,2,2,3,3)$	6	3	1	$\Sigma_3^{or} \times [0,1]/\sim \sharp \overline{\mathbb{R}P}^3$	
18	$\langle 632, 65\rangle$	$(1,1,1,2,2,2)$	6	3	3	$T^3 \sharp 3\,\overline{\mathbb{R}P}^3$	
19	$\langle 631, 65\rangle$	$(1,2,2,3,3,4)$	6	4	2	$\Sigma_2^{or} \times [0,1]/\sim \sharp 2\,\overline{\mathbb{R}P}^3$	
20	$\langle 621, 65\rangle$	$(1,1,2,2,2,3)$	6	5	3	$T^2 \times [0,1]/\sim\ \sharp 3\,\overline{\mathbb{R}P}^3$	
21	$\langle 65\rangle$	$(1,1,1,1,1,2)$	6	6	6	$\mathbb{R}P^3 \sharp 5\,\overline{\mathbb{R}P}^3$	