INSTANTON FLOER-HOMOLOGY WITH LAGRANGIAN BOUNDARY CONDITIONS

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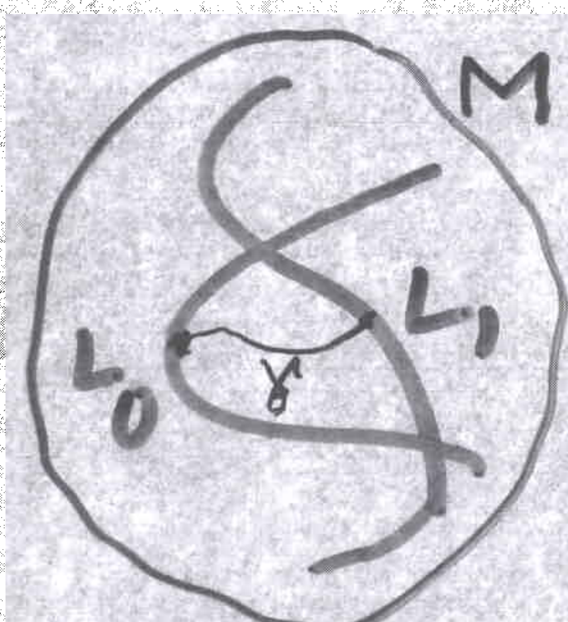
I BACKGROUND

1. SYMPLECTIC FLOER-HOMOLOGY

M, ω sympl. Mfld

$\pi_1 = 0$

$[\omega] = \lambda c_1$, $\lambda > 0$ "monotone"



$L_0, L_1 \subset M$

Lagr. Submfld

$\rightsquigarrow$

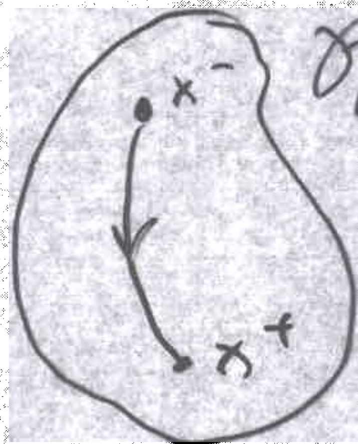
$HF^*_{symp}(L_0, L_1)$

intuitively $\quad \parallel$

"middle dim'l" $H^{\frac{1}{2}\infty}(\mathcal{P}_{L_0, L_1})$

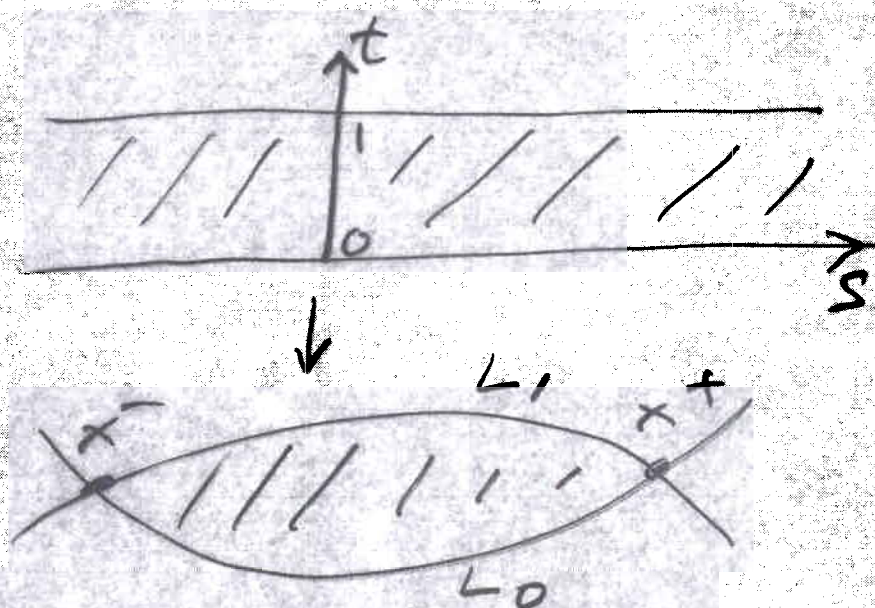
path space $\quad \mid \gamma(0) \in L_0$

$$J: TM \circlearrowleft \qquad J^2 = -\mathbb{1} \qquad \omega(\cdot, J\cdot) = \langle \cdot , \cdot \rangle$$



$\mathcal{P}_{L_0, L_1}$

$$u : [0,1] \times \mathbb{R} \longrightarrow M$$



$$\boxed{\begin{array}{l} \partial_s u + J_t(u)\, \partial_t(u) = 0 \\ u(s,0) \in L_0 \qquad u(s,1) \in L_1 \\ u(s,t) \longrightarrow x^{\pm}, \; s \to \pm\infty \end{array}}$$

$$CF^*(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{Z}\, x$$

$\circlearrowleft \partial$

δ "counts conn. orbits"

$$\delta \circ \delta = 0$$

$$\boxed{HF^*_{symp}(L_0, L_1) := \ker \delta / \operatorname{im} \delta}$$

EXPLES

$$\partial Y = \Sigma$$

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[r] \arrow[d] & SU(2) = G \\ \pi_1(Y) \arrow[ur] & \end{tikzcd}$$

$$\boxed{\begin{array}{ccl} \Sigma & \longrightarrow & M_\Sigma = \dfrac{\mathrm{Hom}(\pi_1(\Sigma), G)}{\mathrm{conj}} \\ & & \ \cup \\ Y & \longrightarrow & L_Y = \dfrac{\mathrm{Hom}(\pi_1(Y), G)}{\mathrm{conj}} \end{array}}$$

$$M_\Sigma = \frac{\{\text{flat conn'ns}\}}{\text{gauge equivalence}}$$

symplectic manifold

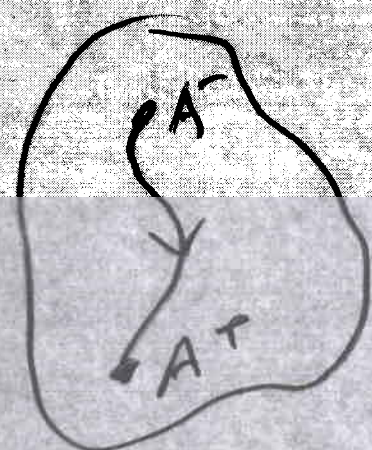
$\cup$ $\quad$ $\dim M_\Sigma = 6g - 6$

L_Y $\quad$ Lagrangian submfld

FLOER HOMOLOGY FOR 3-MFLDS

$$Y \longrightarrow \mathcal{B}_Y = \frac{\{SU(2)\text{-conn'ns}\}}{\text{gauge equival.}}$$

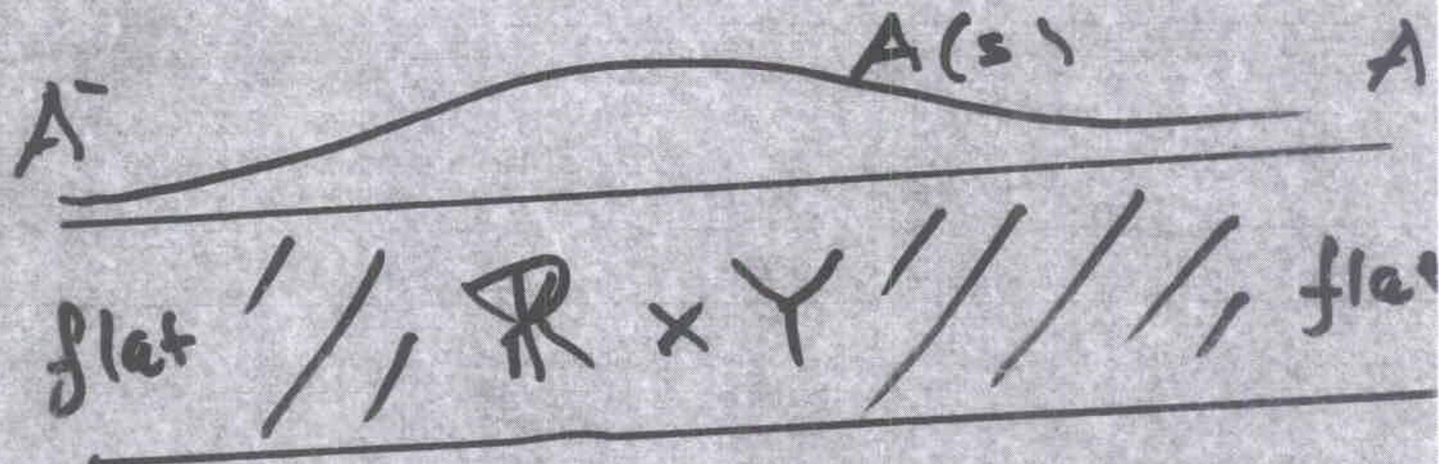
$\mathcal{B}_Y$



$$CS : \mathcal{B}_Y \longrightarrow \mathbb{R}/4\pi^2\mathbb{Z}$$

gradient lines

$$\boxed{\dot{A} + *F_A = 0}$$



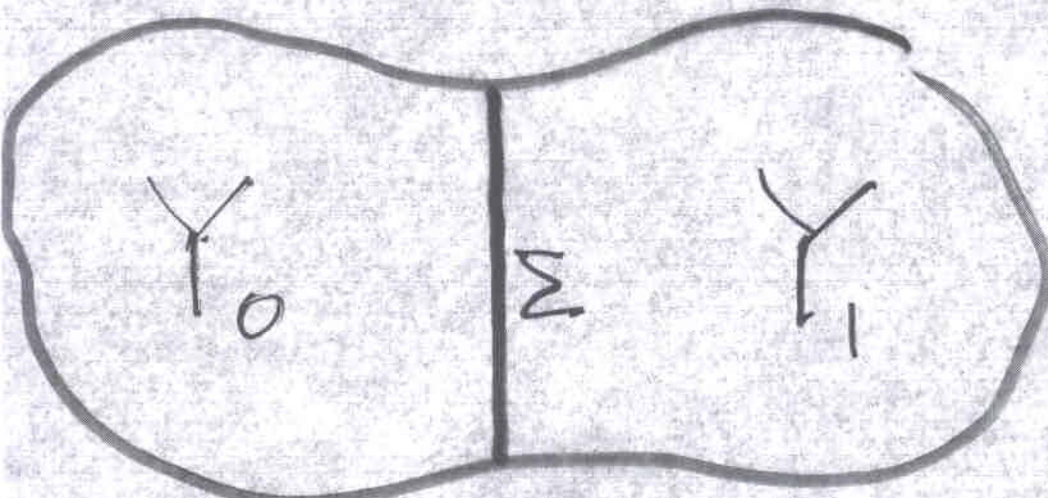
ASD instantons

$$\boxed{HF^{*}_{inst}(Y) = \text{``}H^{\frac{\infty}{2}}(\mathcal{B}_Y)\text{''}}$$

ATIYAH-FLOER CONJECTURE

$$Y = Y_0 \cup_\Sigma Y_1$$

HEEGARD SPLITTING OF HOMOLOGY 3-SPHERE



$$\downarrow$$

$$L_{Y_0} \subset M_\Sigma \supset L_{Y_1}$$

CONJECTURE: $\boxed{HF^*_{inst}(Y) \cong HF^*_{symp}(L_{Y_0}, L_{Y_1})}$

EVIDENCE:

1. $L_{Y_0} \cap L_{Y_1} = \mathcal{A}^{flat}(Y)/\mathcal{G}(Y)$

2. $\mathcal{P}_1 \qquad \subset \mathcal{B}$

PLAN OF PROOF

$$HF^*_{inst}(Y) \qquad HF^*_{symp}(L_{Y_0}, L_{Y_1}$$

$$\cong \qquad \cong \leftarrow \text{adiabatic limit}$$

$$HF^*_{inst}(\Sigma \times [0,1]; L_{Y_0} \times L_{Y_1}$$

TASK: Given 3-mfld Y
with $\partial Y = \Sigma$
and $L \subset M_\Sigma$

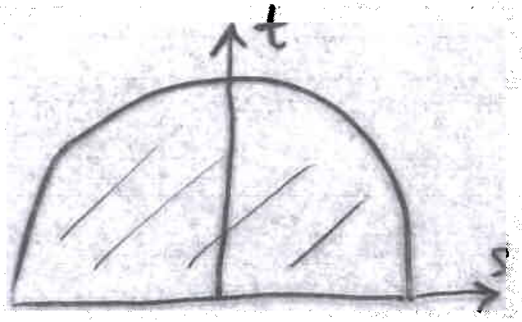
$$\boxed{\text{define } HF_{inst}(Y, L)}$$

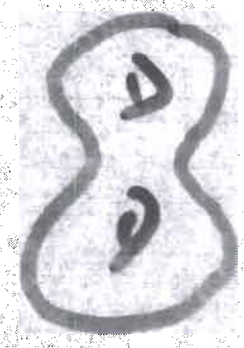
study: $s \longmapsto A(s) \in \Omega^1(Y, \mathfrak{g})$

(1) $\boxed{\mathring{A}(s) + \alpha F_{A(s)} = 0}$

II WORK OF KATRIN WEHRHEIM

$\mathbb{D} \times \Sigma$





Connection $\Xi \in \Omega^1(\mathbb{D}\times\Sigma, \mathfrak{g})$

$$\boxed{\Xi = A(s,t) + S(s,t)\,ds + T(s,t)\,dt}$$

$$A \in \Omega^1(\Sigma,\mathfrak{g}) \qquad S \in \Omega^0(\Sigma,\mathfrak{g}) \qquad T \in \Omega^0(\Sigma,\mathfrak{g})$$

$$\boxed{\begin{gathered}\partial_s A - d_A S + *(\partial_t A - d_A T) = 0\\ \partial_s T - \partial_t S + [S,T] + *F_A = 0\\ A(s,0) \in \mathcal{L}_Y\end{gathered}}$$

$$\rightsquigarrow \mathcal{L}_Y := \left\{ \begin{array}{l} A \in \Omega^1(\Sigma,\mathfrak{g}) \mid \\ \exists \tilde{A} \in \Omega^1(Y,\mathfrak{g}) \\ \tilde{A} \text{ flat}, \tilde{A}|_\Sigma = A \end{array} \right\}$$

handle-
body

Lagr. submfld

Thm A (COMPACTNESS)

$\Xi_1, \Xi_2, \Xi_3, \ldots$ sol'ns of (2).

$$\boxed{\sup_\nu \| F_{\Xi_\nu} \|_{L^p} < \infty}, \quad p > 2.$$

$\Rightarrow \exists g_\nu : \mathbb{D} \times \Sigma \to G$ s. that

$g_\nu^* \Xi_\nu$ has C^∞-conv't subseq.

Thm B (ENERGY QUANTIZATION)

$\exists \hbar > 0$ s. that the following holds:

Ξ_ν sol'ns of (2) $\boxed{|F_{\Xi_\nu}(s_\nu, t_\nu, z_\nu)| \to \infty}$

$s_\nu + it_\nu \to 0, \quad z_\nu \in \Sigma$

$$\Rightarrow \boxed{\lim_{\nu \to \infty} \| F_{\Xi_\nu} \|_{L^2(\mathbb{D}_\varepsilon \times \Sigma)} \geq \hbar} \quad \forall \varepsilon > 0$$

Thm C (REMOVABLE SINGULARITIES)

Ξ sol'n of (2) on $(\mathbb{D} \setminus 0) \times \Sigma$

$E(\Xi) = \frac{1}{2} \| F_\Xi \|_{L^2}^2 < \infty \Longrightarrow \exists g : (\mathbb{D} \setminus 0) \times \Sigma \to G$ s. that $g^* \Xi$

PROOF OF THM A

$$\boxed{\begin{gathered}\partial_s A + *\partial_t A = d_A S + *d_A T \\ A(s,0) \in \mathcal{L}_Y\end{gathered}}$$

analogous to

$$\boxed{\begin{gathered}\partial_s u + J(u)\partial_t u = f \\ u(s,0) \in L\end{gathered}}$$

regularity & estimates for S and T

+ CR equations in Banach spaces

$\Longrightarrow$ Thm A.

PROOF OF THEOREM B

WEHRHEIM TRICK (MODEL CASE: ASD-INST. ON $\mathbb{R}^4$

Step 1: $e := |F_\Xi|^2 : \mathbb{R}^4 \longrightarrow [0,\infty)$

$$\Longrightarrow \boxed{\Delta e \geqslant -a e^{3/2}} \qquad (3)$$

Step 2: $\exists\, C, \hbar$ such that

$$\boxed{(3) \ \& \ \int_{B_r} e < \hbar \Longrightarrow e(0) \leqslant \frac{C}{r^4}\int_{B_r} e}$$

Step 3: $|F_{\Xi_\nu}(x_\nu)| \to \infty, \quad x_\nu \longrightarrow 0$

$$\Longrightarrow \boxed{\int_{B_\varepsilon} |F_{\Xi_\nu}|^2 \geqslant \hbar} \qquad \forall \varepsilon > 0 \quad \forall \nu \text{ large}$$

Proof: $R_\nu^2 := |F_{\Xi_\nu}(x_\nu)| \to \infty, \quad \varepsilon_\nu R_\nu \to \infty$

$$\int_{B_\varepsilon} |F_{\Xi_\nu}|^2 < \hbar \overset{\text{Step 2}}{\Longrightarrow} R_\nu^4 = |F_{\Xi_\nu}(x_\nu)|^2 \leqslant \frac{C}{\varepsilon_\nu^4} \int_{B_{\varepsilon_\nu}(x_\nu)} |F_{\Xi_\nu}|^2$$

$$\Rightarrow \int |F_\Xi|^2 \geqslant \frac{R_\nu^4 \varepsilon_\nu^4}{C} \geqslant \hbar$$

ADAPTATION TO BVP

$$\boxed{e(s,t) := \|F_{\Xi}(s,t)\|^2_{L^2(\Sigma)}}$$

$$\Delta e = \text{pos} - \int_\Sigma \langle F_A \wedge *[\partial_s A \wedge \partial_s A]\rangle$$

$$\partial_t e(s,0) = -\int_Y \langle \partial_s \tilde{A} \wedge [\partial_s \tilde{A} \wedge \partial_s \tilde{A}]\rangle$$

$$\|F_{A(s,t)}\|_{L^\infty(\Sigma)} \leq c \|F_{\Xi}(s,t)\|^2_{L^2(\Sigma)}$$

$$\tilde{A}(s) \in \mathcal{A}_{flat}(Y)$$

$$\tilde{A}(s)\big|_\Sigma = A(s,0)$$

$$\|\partial_s \tilde{A}\|_{L^3(Y)} \leq c \|\partial_s A\|_{L^2(\Sigma)}$$

$$\boxed{\begin{aligned} \Delta e &\geq -a e^2 \\ \partial_t e(s,0) &\geq -a\, e(s,0)^{3/2} \end{aligned}}$$

$\Rightarrow$ mean value ineq. when $\int_{B_r} e \leq \hbar$

$\Rightarrow$ same argument works

DETAILS

STEP 1: $\sup_\nu \|F_{\Xi_\nu}\|_{L^\infty(B_\delta\times\Sigma)} = \infty \ \forall\delta$

Thm A $\Longrightarrow$ $\sup_\nu \|F_{\Xi_\nu}\|_{L^p(B_\delta\times\Sigma)} = \infty \ \forall\delta$, $2<p<3$

∇ $\overset{\Leftarrow}{\Rightarrow}$ $\boxed{\sup_\nu \sup_{B_\delta} e_\nu = \infty \quad \forall\delta}$

PROOF a priori in dim=4 & interpolation

STEP 2: HOFER TRICK: W.L.O.G.

$e_\nu(s_\nu,t_\nu)=R_\nu^2\to\infty$, $\sup_{B_{\varepsilon_\nu}(s_\nu,t_\nu)} e_\nu \le 2R_\nu^2$, $\varepsilon_\nu R_\nu\to\infty$

∇ $\overset{\Leftarrow}{\Rightarrow}$ $\boxed{\|F_{A_\nu(s,t)}\|_{L^\infty(\Sigma)} \le cR_\nu^2}$ $(s,t)\in B_{\varepsilon_\nu}(s_\nu,t_\nu)$

td-RESCALING

$\Longrightarrow \quad \Delta e_\nu \ge -cR_\nu^2 e_\nu$

STEP 3: $\partial_t e_\nu(s,0) \ge -c\, e_\nu(s,0)^{3/2}$

STEP 4: WEHRHEIM'S MEANVALUE INEQ.:

$$\left.\begin{array}{l}\text{Step 2 \& 3}\\ \displaystyle\int_{B_{\varepsilon_\nu}(s_\nu,t_\nu)} e \le \hbar\end{array}\right\} \Rightarrow R_\nu^2 = e_\nu(s_\nu,t_\nu) \le C\left(R_\nu^2+\frac{1}{\varepsilon_\nu^2}\right)\int_{B_{\varepsilon_\nu}} e_\nu$$

$C\hbar \le \frac{1}{2}$

$$\Longrightarrow R_\nu^2\varepsilon_\nu^2 \le 2C\hbar \le 1$$

INTERLUDE (THE ISOPERIMETRIC INEQUALITY)

KEY LEMMA (EXTENSION)

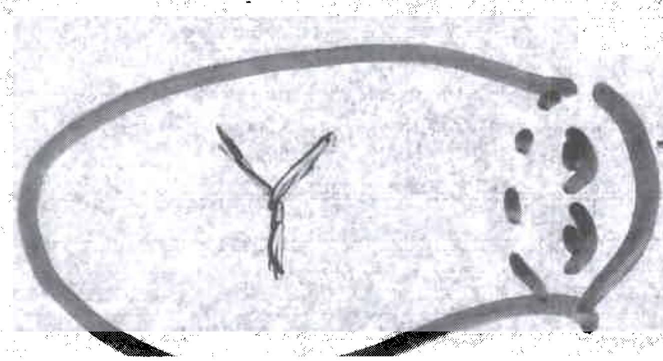
$\exists\ \delta, c > 0$ such that

$\forall\ A_0, A_1 \in \mathcal{L}_Y$, $\boxed{\|A_0 - A_1\|_{L^2(\Sigma)} < \delta}$

$\exists\ \tilde{A}_0, \tilde{A}_1 \in \mathcal{A}^{flat}(Y)$:

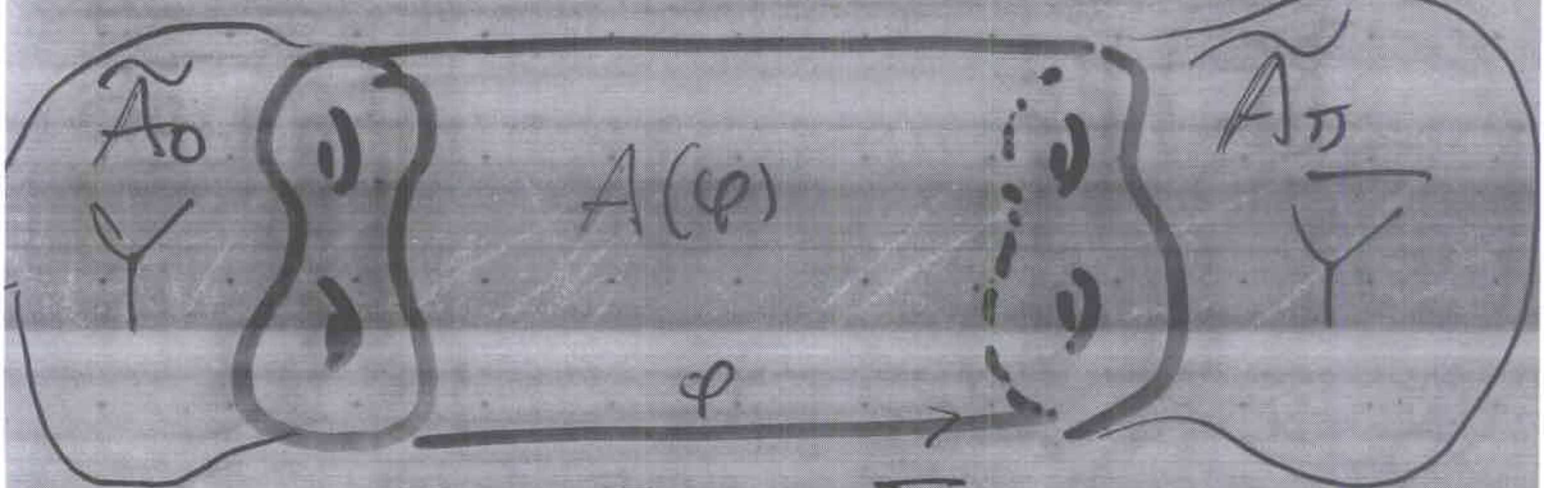
$$\tilde{A}_0\big|_{\Sigma} = A_0 \qquad \tilde{A}_1\big|_{\Sigma} = A_1$$

$$\|\tilde{A}_0 - \tilde{A}_1\|_{L^3(Y)} \leq c\,\|A_0 - A_1\|_{L^2(\Sigma)}$$



$\Sigma = \partial Y$

THE LOCAL CHERN-SIMONS FUNCTIONAL



$[0,\pi] \times \Sigma$

CONNECTION $\quad A(\varphi) \in \Omega^1(\Sigma, \mathfrak{g})$

$A(0) \in \mathcal{L}_Y \qquad A(\pi) \in \mathcal{L}_Y$

ASSUME $\boxed{\int_0^\pi \| \partial_\varphi A \|_{L^2(\Sigma)} \, d\varphi < \delta}$

$\Rightarrow \quad \| A(0) - A(\pi) \|_{L^2(\Sigma)} < \delta$

$\Rightarrow \quad \exists \, \tilde{A}_0, \tilde{A}_1 \in \mathcal{A}^{flat}(Y):$

$\tilde{A}_0|_\Sigma = A(0) \quad \tilde{A}_\pi|_\Sigma = A(\pi)$

$\| \tilde{A}_0 - \tilde{A}_\pi \|_{L^2(\ldots)} \leq C \, \| A(0) - A(\pi) \|_{L^2}$

$$\tilde{Y} = Y \cup \left([0,\pi] \times \Sigma \right) \cup \overline{Y}$$

$$\tilde{A} = \tilde{A}_0 \cup A(\varphi) \cup \tilde{A}_\pi$$

<u>LEMMA</u> (ISOPERIMETRIC INEQ)

$$\boxed{CS(\tilde{A}) \leq c \left(\int_0^\pi \|\partial_\varphi A\|_{L^2(\Sigma)} d\varphi \right)^2}$$

whenever $\int_0^\pi \|\partial_\varphi A\|_{L^2(\Sigma)} d\varphi < \delta$.

THE PROOF IS

MAGIC!

IDEA :

$$CS(\tilde{A}) = -\frac{1}{2}\int_0^{\pi}\int_{\Sigma}\langle A(\varphi)\wedge\dot{A}(\varphi)\rangle\, d\varphi$$

$$+\,CS(\tilde{A}_0) - CS(\tilde{A}_{\pi})$$

$$A(\varphi) = A(0) + \int_0^{\varphi}\dot{A}(\varphi')\,d\varphi'.$$

$\Longrightarrow$

$$\boxed{\begin{aligned} CS(\tilde{A}) = &-\frac{1}{2}\int_0^{\pi}\int_0^{\varphi}\int_{\Sigma}\langle\dot{A}(\varphi')\wedge\dot{A}(\varphi)\rangle\, d\varphi'\, d\varphi \\ &-\frac{1}{2}\int_{\Sigma}\langle A(0)\wedge A(\pi)\rangle \\ &+CS(\tilde{A}_0) - CS(\tilde{A}_{\pi}). \end{aligned}}$$

$$CS(\tilde{A}) = \frac{1}{2}\int_Y \langle \tilde{A} \wedge d\tilde{A}\rangle + \frac{1}{3}\langle \tilde{A} \wedge [\tilde{A} \wedge \tilde{A}]\rangle$$

$$\boxed{\frac{d}{dt} CS(\tilde{A}) = \int_Y \langle F_{\tilde{A}} \wedge \frac{d}{dt}\tilde{A}\rangle - \frac{1}{2}\int_{\partial Y} \langle \tilde{A} \wedge \frac{d}{dt}\tilde{A}\rangle}$$

$$\tilde{A}(t) := \tilde{A}_0 + t(\tilde{A}_\pi - \tilde{A}_0)$$

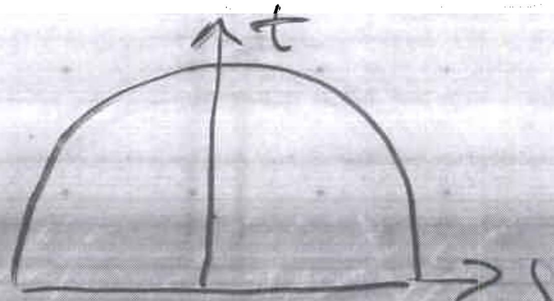
$\Longrightarrow$

$$\boxed{\begin{aligned} &CS(\tilde{A}_\pi) - CS(\tilde{A}_0) + \frac{1}{2}\int_\Sigma \langle A(0) \wedge A(\pi)\rangle \\ &= \int_0^\pi \int_Y \langle F_{\tilde{A}(t)} \wedge (\tilde{A}_\pi - \tilde{A}_0)\rangle \\ &= \frac{1}{12}\int_Y \langle (\tilde{A}_\pi - \tilde{A}_0) \wedge [(\tilde{A}_\pi - \tilde{A}_0) \wedge (\tilde{A}_\pi - \tilde{A}_0)]\rangle \end{aligned}}$$

$$\ldots \left(\int^\pi \| \ldots \| \ldots \right)^2 \ldots \|\tilde{A} - \tilde{A}_0\|^3$$

PROOF OF THM C

POLAR COORDINATES



$$s = r\cos\varphi$$
$$t = r\sin\varphi$$
$$0 \le \varphi \le \pi$$

$$\boxed{\Xi = A(r,\varphi) + \Phi(r,\varphi)\,d\varphi + R(r,\varphi)\,dr}$$

angular gauge $\Phi = 0$.

$$\boxed{\begin{aligned}\mathcal{E}(r) :&= \int_{B_r} \|F_{\Xi}\|^2_{L^2(\Sigma)} \\ &= \int_0^r \int_0^{\pi} \left(\rho \|F_{A(\rho,\varphi)}\|^2_{L^2(\Sigma)} + \frac{1}{\rho} \|\partial_\varphi A(\rho,\varphi)\|^2_{L^2(\Sigma)} \right) d\varphi\, d\rho\end{aligned}}$$

STEP 1 $r\|F_{A(r,\varphi)}\|_{L^2(\Sigma)} \to 0$

$\|\partial_\varphi A(r,\varphi)\|_{L^2(\Sigma)} \to 0$ $r \to 0$

Same analysis as in thm B.

STEP 2 ISOPERIMETRIC INEQ

$$\boxed{\frac{\mathcal{E}'(r)}{\mathcal{E}(r)} \geq \frac{\mu}{r}}$$

$\varphi \mapsto A(r,\varphi)$

$$\mathcal{E}(r) = CS(\tilde{A}_r)$$

$\hat{\tilde{A}}$ ✓

$$\leq \frac{1}{\mu}\int_0^\pi \|\partial_\varphi A(r,\varphi)\|^2_{L^2(\Sigma)}\, d\varphi$$

$$\leq \frac{r}{\mu}\int_0^\pi \left(r\|F_{A(r,\varphi)}\|^2_{L^2(\Sigma)} + \frac{1}{r}\|\partial_\varphi A(r,\varphi)\|^2_{L^2(\Sigma)}\right) d\varphi$$

$$= \frac{r}{\mu}\,\mathcal{E}'(r)$$

$$\mathcal{E}(r) \leq C\, r^\mu$$

STEP 3

$$r\,\|F_{A(r,\varphi)}\|_{L^2(\Sigma)} \leq C r^{\mu}$$

$$\|\partial_\varphi A(r,\varphi)\|_{L^2(\Sigma)} \leq C r^{\mu}$$

Like Step 1

STEP 4 $\exists\, g : \mathbb{D}\setminus 0 \times \Sigma \to G$

s.that

$$g^* \Xi \in W^{1,p}(\mathbb{D}\times\Sigma, \mathfrak{g})$$

$\underset{\Longrightarrow}{\text{THM A}}$ $g^*\Xi$ gauge equiv. to smooth conn'n.

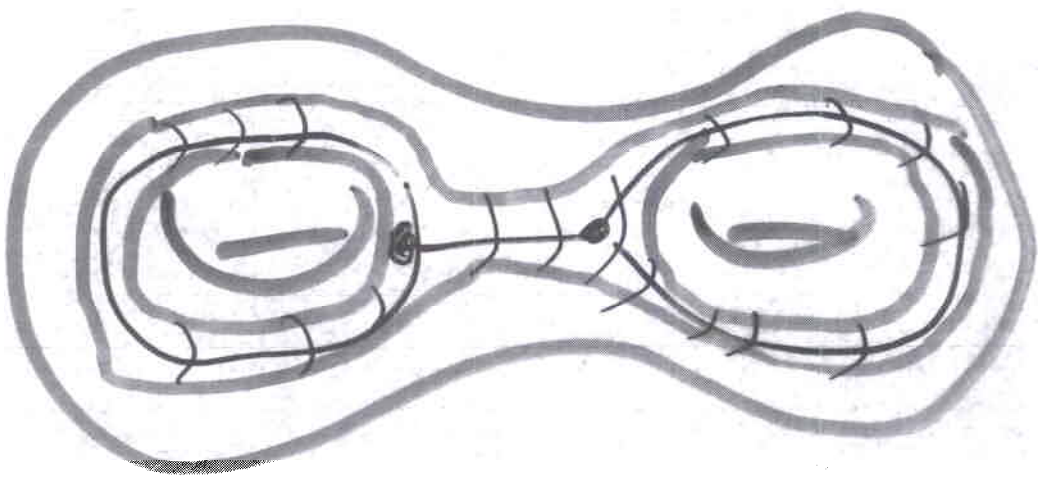
(a smart cutoff fct'n argument)

III OUTLOOK

1. DEFINE $HF^{*}_{inst}([0,1]\times\Sigma, \mathcal{L}_{Y_0}\times\mathcal{L}_{Y_1}$

2. PROVE

$$HF^{\alpha}_{inst}([0,1]\times\Sigma; \mathcal{L}_{Y_0}\times\mathcal{L}_{Y_1}) \cong HF^{\alpha}_{inst}(Y$$



Shrink Σ onto 1-skeleton

3. PROVE $HF^{\alpha}_{inst} \cong HF^{\alpha}_{symp}(L_{Y_0}, L_{Y_1})$

"ADIABATIC LIMIT"

$$\boxed{\begin{aligned} &\partial_s A - d_A S + *(\partial_t A - d_A T) = 0 \\ &\partial_s T - \partial_t S + [S, T] + \frac{1}{\varepsilon^2} * F_A = 0 \end{aligned}}$$