RECENT ADVANCE IN ORBIFOID THEOR

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Historical Prospective:

(I) ORBIFOID is an "OLD" subject dated back to 50's by Satake

(II) CLASSICAL ORBIFOID THEORY is "BORING" in the sense that it was viewed as a "generalized" smooth space

(III) Recent activities on orbifold explore "stringy" properties of orbifold and REVOLUTIONIZE our thinking of orbifold.

(IV) Current advance has deep connections to algebraic geometry, representation theory and algebraic topology which were difficult to imagine just a few years ago.

Examples:

(1) G - finite group acting smoothly on X. $Y = X/G$ has a natural orbifold structure with <u>one chart</u>

↑ global quotient

Ex: (i) $\cdot^G = \bullet/G \iff$ group theory

(ii) Symmetric product X^n/S_n

(iii) $T^4/\mathbb{Z}_2$ $\quad (z_1, z_2) \longleftrightarrow (-z_1, -z_2)$

$T^4 = \mathbb{C}^2/\Lambda$

(iv) $T^6/\mathbb{Z}_4$ $\quad (z_1, z_2, z_3) \to (iz_1, iz_2, -z_3)$

(v) $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad (z_1, z_2, z_3) \to (z_1, -z_2, -z_3)$

$(z_1, z_2, z_3) \to (-z_1, z_2, -z_3)$

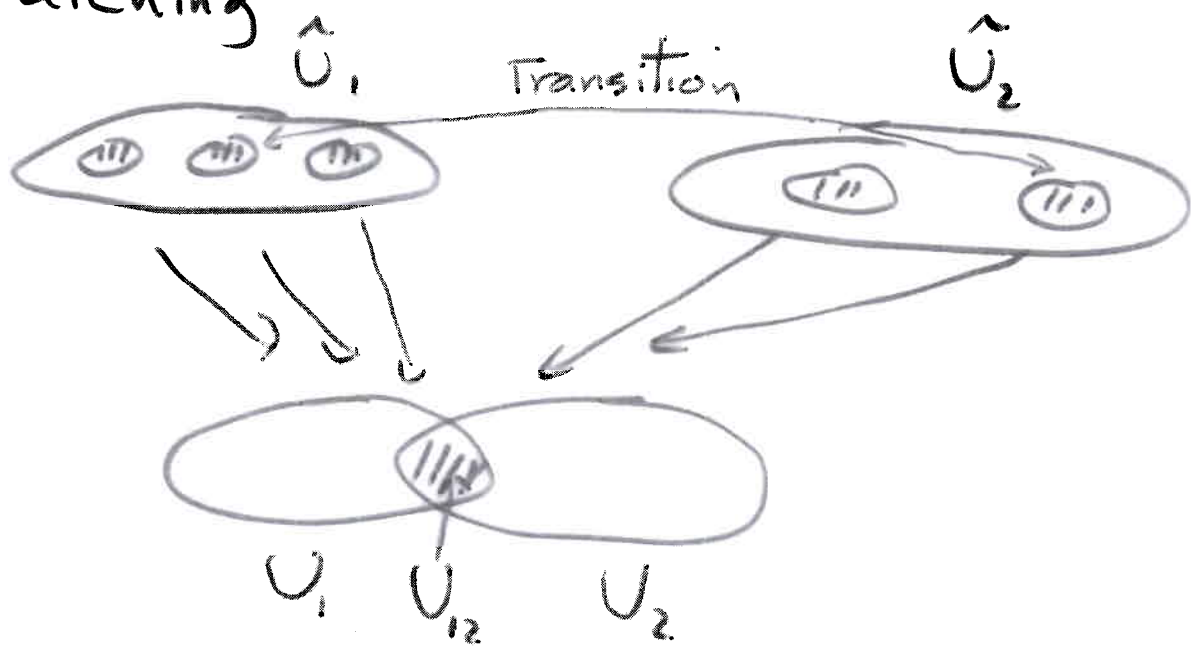
(vi) Mirror quintic

Orbifold structure:

(i) orbifold chart: Locally, $U = \hat{U}/G_{\hat{U}}$

$\hat{U}$ - smooth mfd, $G_{\hat{U}}$ - finite group acting smoothly on $\hat{U}$

(ii) Patching



(iii) Compatibility condition

open cover with previous cond is called ORBIFOLD ATLAS

Orbifold structure: Equivalence class of orbifold atlas under refinement

Quotient orbifold:

G - Lie group acting smoothly, properly on X with finite isotropy

$Y = X/G$ - has a natural orbifold str

Quotient orbifold

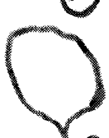
Ex(1): Weighted projective space

$$e^{i\theta}(z_0 \ldots z_n) = (e^{i\alpha_0\theta} z_0 \ldots e^{i\alpha_n\theta} z_n)$$

$$S^{2n+1}/S^1 = WP(\alpha_0, \ldots, \alpha_n)$$

$WP(1, \alpha_0)$ is not a global quotient unless $\alpha_0 = 1$

$\Updownarrow$

 — tear drop

$\mathbb{C}/\mathbb{Z}_{\alpha_0}$

Ex(2). Toric varieties

Classical Invariant of orbifold

- ordinary cohomology $- H^*(X, \mathbb{Q})$

  ↑ Invariant of underline space

- orbifold fundamental group $- \pi_1^{orb}(X)$

  ↑ Invariant of orbifold structure

- $(X, \mathcal{U}) \rightsquigarrow$ classifying space BX

(1) $\pi_1(BX) = \pi_1^{orb}(X)$

(2) classical orbifold cohomology

$$H^*_{orb}(X, \mathbb{Z}) = H^*(BX, \mathbb{Z})$$

Note: $H^*_{orb}(BX, \mathbb{Q}) = H^*(X, \mathbb{Q})$, but has interesting torsion classes

(3) orbifold homotopy group

$$\pi_k^{orb}(X) = \pi_k(BX)$$

- K-theory of orbifold vector bundles
- Derived category of orbifold sheaves

↑ Have their stringy aspect.

Algebraic geometry of Orbifold

$$Y \xrightarrow{\pi} X \text{ - crepant resolution}$$

(i) π is a resolution

(ii) $\pi^* K_X = K_Y$

Generalized McKay correspondence:

Using combinatoric str of X to describe topological (cohomology) - holomorphic inv of X

Representation theory of orbifold

X-smooth ⇝ vertex operator algebra

X-orbifold ⇝ orbifolding vertex operator algebra

Ex: T^{24}/G ⇝ Monster group

Beyond classical invariant of orbifold

Chen - Ruan orbifold cohomology

(I) Twisted sector

inertia orbifold (Kawasaki)

$$\widetilde{\Sigma_1 X} = \{(x, (g_1)_{G_x}),\ g \in G_x\}$$

Prop: $\widehat{\Sigma_1 X}$ has a natural orbifold str

components:

$$\widetilde{\Sigma_1 X} = \bigsqcup_{(g) \in T_1} X_{(g)}$$

Ex: $X = Y/G$ – global quotient

$$X_{(g)} = Y^g / C(g) \qquad C(g) \text{– centralizer}$$

$X_{(1)} = X$ — nontwisted sector

$X_{(g)}$ $g \neq 1$ — twisted sector

$\Downarrow$

geometric realization of twisted sector in orbifold conformal field theory

II. Degree shifting (X – almost complex)

$x \in X_{(g)}$ g acts on $T_x X$

write $g = \mathrm{diag}(e^{\frac{2\pi i m_1}{m}}, \dots, e^{\frac{2\pi i m_n}{m}})$ m = ord

Degree shifting number

$$\iota_{(g)} = \sum \frac{m_i}{m}$$

CR- orbifold cohomology group

$$H^*_{CR}(X, \mathbb{C}) = \bigoplus_{(g) \in T_1} H^*(X_{(g)}, \mathbb{C})[-2\iota_{(g)}]$$

III. Poincare Pairing

$$H^d(X_{(g)}, \mathbb{C})[-2\iota_{(g)}] \otimes H^{2n-d}(X_{(g^{-1})}, \mathbb{C})[-2\iota_{(g^{-1})}]$$

$$\downarrow$$

$$\mathbb{C}$$

$$\langle \alpha, \beta \rangle_{orb} = \int_{X_{(g)}} \alpha \wedge I^* \beta$$

$I: X_{(g)} \to X_{(g^{-1})}$

$(x, (g_1)_{G_x}) \to (x, (g_1^{-1})_{G_x})$

Cup Product:

A baby model: $X = \bullet G$

group algebra

$$\mathbb{C}G = \{ \lambda_1 \bar{g}_1 + \cdots + \lambda_k \bar{g}_k \}$$

$$(\lambda_1 \bar{g}_1)(\lambda_2 \bar{g}_2) = \lambda_1 \lambda_2 \overline{g_1 g_2}$$

center of group algebra

$Z[\mathbb{C}G]$ = generated by conjugacy class

$$\overline{(g)} = g_1 + \cdots + g_n$$

where $(g) = \{ g_1, \dots g_n \}$

Multiplication of conjugacy class

$$\overline{(g_1)} * \overline{(g_2)} = \sum_{\substack{h_1 \in (g_1) \\ h_2 \in (g_2) \\ (h_1, h_2)}} \frac{|C(h_1 h_2)|}{|C(h_1, h_2)|} \overline{(h_1 h_2)}$$

↑ conjugacy class of pair

$C(h)$ - centralizer of h

↑ Very important object in group theory

III. Cup product

$\alpha \in H^p(X_{(g_1)}, \mathbb{C})[-2i_{(g_1)}]$, $\beta \in H^p(X_{(g_2)}, \mathbb{C})[-2i_{(g_2)}]$

$$\alpha \cup \beta = \sum_{\substack{(h_1, h_2) \in T_2 \\ h_i \in (g_i)}} (\alpha \cup \beta)_{(h_1, h_2)}$$

$(\alpha \cup \beta)_{(h_1, h_2)} \in H^*(X_{(h_1 h_2)}, \mathbb{C})$ is defined by the relation

$$\langle (\alpha \cup \beta)_{(h_1, h_2)}, \gamma \rangle_{orb} = \int_{X_{(h_1, h_2)}} e_1^* \alpha \wedge e_2^* \beta \wedge e_3^* \gamma \wedge e_A(E_{(h_1, h_2)})$$

where

Ex: $X_{(h_1, h_2)} = Y_{h_1} \cap Y_{h_2} / C(h_1, h_2)$

$e_1 : X_{(h_1, h_2)} \to X_{(h_1)}$

$e_2 : X_{(h_1, h_2)} \to X_{(h_2)}$

$e_3 : X_{(h_1, h_2)} \to X_{((h_1 h_2)^{-1})}$

$E_{(h_1, h_2)} \to X_{(h_1, h_2)}$ is certain obstruction bundle.

Remarks:

- $H^*_{CR}(X, \mathbb{C})$ is combinatori and easy to compute at least in interesting examples

- $H^*_{CR}(X, \mathbb{C})$ is motivated by orbifold string theory model of Dixon-Harvey Vafa-Witten.

- $H^*_{CR}(X, \mathbb{C})$ ~~(its)~~ its ring structure in partic is new in physics.

APPLICATIONS

(I) Generalized Mckay correspondence

$\pi: Y \longrightarrow X$ – crepant resolution

TWO CONJECTURES: (Ruan)

Hyperkahler case: If π is a hyperkahler resolution, $H^*(Y, \mathbb{C})$ is ring isomorphic to $H^*_{CR}(X, \mathbb{C})$

$\Uparrow$

Hyperkahler Resolution conjecture

General case: $H^*(Y, \mathbb{C})$ is a ring deformation of $H^*_{CR}(X, \mathbb{C})$ by quantum correction coming from exceptional rational curves in an explicit form.

$\Rightarrow$

Cohomological Resolution Conjecture

Many, Many partial results!

Chen-Ruan cohomology of symmetric product X^n/S_n and its applications

Symmetry product: $S^n(X) = X^n/S_n$ ← symmetric group

Crepant resolution ($\dim_{\mathbb{C}} X = 2$)

$\pi: X^{[n]} \longrightarrow X^n/S_n$

↑ Hilbert scheme of points of length k

<u>Historical Remark</u>: The computation of cohomology of $X^{[n]}$ has been an active area of algebraic geometry for more than 15 years. A lot of machinery has been developed. However, it was not clear what kind of answer we should seek.

⇑

Chen-Ruan cohomology

$H^*_{CR}(X^n/S_n, \mathbb{C})$ As a vector space

Heisenberg Algebra:

$\alpha_n(\gamma)$ $n<0$ creation operators. $\gamma \in H^*(X$

$\alpha_n(\gamma)$ $n>0$ annihilation operators

Super commutator

$$\{\alpha_\ell(\gamma), \alpha_m(\gamma')\} = \alpha_\ell(\gamma)\alpha_m(\gamma') - (-1)^{\deg(\gamma)\cdot} \alpha_m(\gamma')\alpha_\ell(\gamma$$

Commutation Relation

$$\{\alpha_\ell(\gamma), \alpha_m(\gamma')\} = -\ell\,\delta_{\ell+m,0} \langle\gamma,\gamma'\rangle \, Id$$

<u>Vafa-Witten</u>: $\mathcal{H} = \bigoplus_{n=0}^{\infty} H^*_{CR}(X^n/S_n, \mathbb{C})$ is an irreducible representation of Heisenberg Algebra

$\Updownarrow$ isomorphic

Grojnowski-Nakajima: $\mathcal{H}' = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}, \mathbb{C})$ is an irreducible representation of Heisenberg Algebra.

Ring structure is much harder to compute and more interesting

Two Approaches towards ring structures

(I) Simple and direct approaches to compute obstruction bundle (Fantechi – Göttsche – Uribe + Lehn – Sorger

⇓

Solve Hyperkahler Resolution Conjecture for $X^{[n]} \longrightarrow X^n/S_n$ when $X = K3, T^4$

(II) Indirect and more sophisticated approach Li – Qin – Wang Axioms of C-R cohomology

(i) ∃ a set of combinatoric ring generators $\mathfrak{D}^k(\gamma)$

(ii) View $\mathfrak{D}^k(\gamma)$ as an operator by applying $\mathfrak{D}^k(\gamma) \cup \cdot$

(iii) Three axioms for commutation relations between $\mathfrak{D}^k(\gamma)$ and Heisenberg operators $\alpha_n(\gamma)$

Uniquely determine (det) Chen-Ruan cohomology ring

Solve Hyperkahler Resolution Conjecture for $X^{[n]} \longrightarrow X^n/S_n$ when $X = T^*\Sigma$, $\mathbb{C}^2/P$

<u>Historical Remark</u>: Above results employ deep connection between Chen–Ruan cohomology and such algebraic techniques as symmetric function, Heisenberg Algebra.

General Approach to Hyperkahler Resolution Conjecture

(I) On going by Caldararu, Ginzburg, Kaledin an others

(II) Use very different types of technologies such as derived category of complex of coherent sheaves, Fourier–Mukai transform Hochschild cohomology, ..

(III) Ginzburg–Kaledin's solution of Hyperkahler resolution conjecture for crepant resolution of symplectic singularities

Idea:

step I: D-McKay correspondence (Bridgeland, King, Reid)

$D^b(X)$ – Derived category of bounded complex of coherent sheaves

X is either smooth or orbifold.

integral transform

$$\begin{array}{ccc} & \mathcal{E} \downarrow & \\ & X \times Y & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ X & & Y \end{array}$$

$$\Phi : D^b(X) \longrightarrow D^b(Y)$$

$$\Phi(\mathcal{E}) = \pi_{2*}(\pi_1^* \mathcal{E} \otimes \xi)$$

step II. Hochschild cohomology of derived category

$$D^b(X) \rightsquigarrow HH^i(X),\ HH_i(X)$$

natural with respect to integral transf

step (III): In general, Hochschild cohomology is very different from Chen-Ruan cohom

However, in symplectic case

$$HH^{i}(X_{\mathbb{C}}) \overset{\text{expected}}{\approx} H^{*}_{CR}(X, \mathbb{C})$$

Cohomological Crepant Resolution Conjecture

① Involve quantum correction and much more difficult

② Very little is known.

- Li-Qin $X^{[2]} \longrightarrow X^n/S_2$ — already Highly Nontrivial

③ Approachable problem!

Application to Representation Theory

Historical Remarks:

(I) Vertex Operator Algebra on orbifold (orbifold conformal field theory) is a very important construction in representation theory and physics

(II) Orbifold vertex operator algebra contains deep information on even simple geometry such as $T^{24}/G \longleftrightarrow$ Monster moonshine

(III) Relation between algebraic-representation theoretic aspect and geometric-topological aspect of orbifold is still unraveling right now and will benefit both direction

A specific problem:

classically. smooth complex variety X

$\big\downarrow$ associate

sheaf of vertex operator algebra Ω^{ch}_X

↑ contain elliptic genus

orbifold X/G

$\big\downarrow$

$$\Omega^{ch}_{X/G} = \bigoplus_{(g)} \Omega^{ch,g}_X$$

↑ difficult to put product structure

Frenkel - Szczesny - Vaintrob: (on going)

Use construction of Chen-Ruan product to construct a product for $\Omega^{ch}_{X/G}$

⇓

marriage between Chen-Ruan cohomology and elliptic genus (elliptic cohomology?)

Application to Quantum cohomology of smooth manifold

- Dijkgraaf's speculation (2001)

Gromov – Witten Theory of algebraic surface X

$\Updownarrow$

cohomology ring of $X^{[n]}$ / Chen-Ruan cohomology of $S^n(X) = X^n / S_n$

In particular, $X = pt$

Intersection theory on $\overline{M}_{g,n}$

$\Updownarrow$

ring structure of $H^*_{CR}(\cdot G, \mathbb{C})$

- Costello: High genus GW-inv of arbitrary X (ge

related in a complicated ma

genus zero orbifold GW-inv of $S^{g+1}(X$

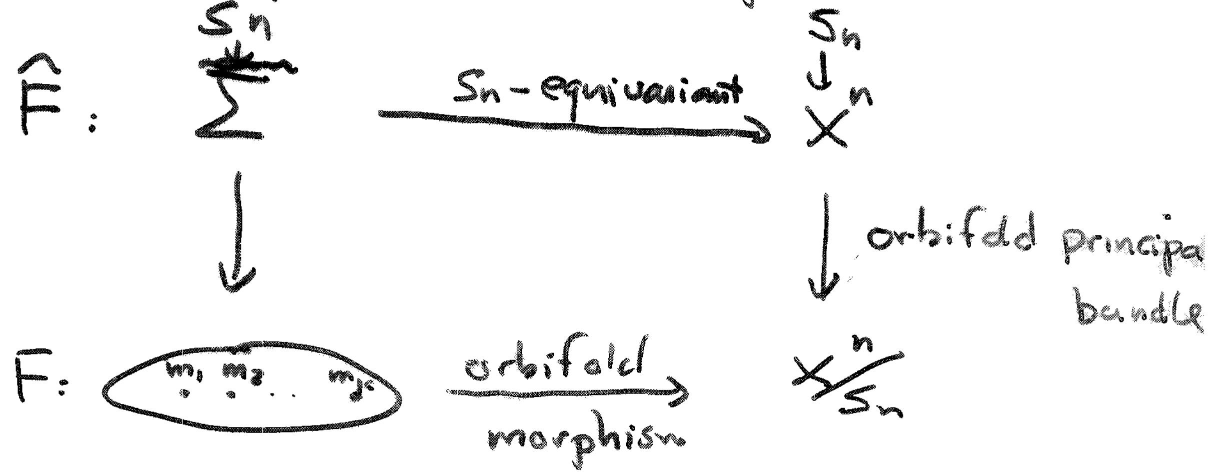
$\Uparrow$

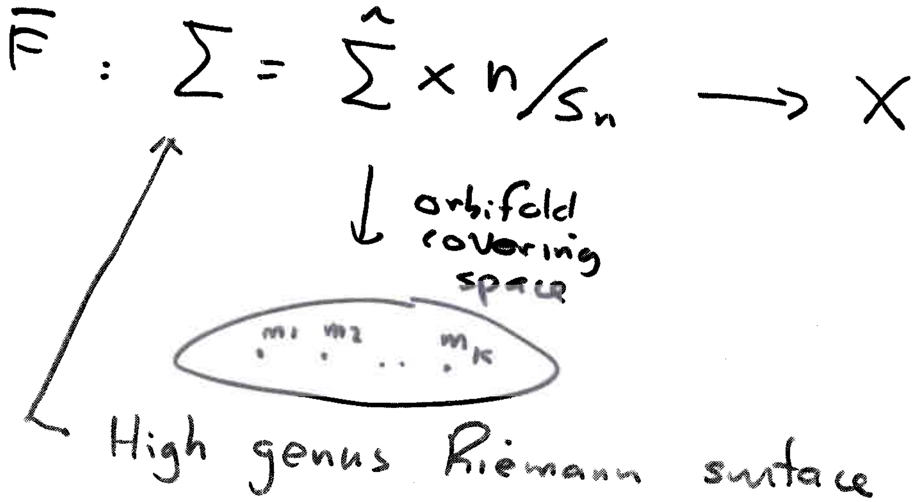
Chen – Ruan

$\Downarrow$

A new approach to Virasoro Conjecture

Key construction: orbifold morphism





$\bar{F}(x, i) = \hat{F}_i(x)$

$\hat{F}(x) = (\hat{F}_1(x) \ldots \hat{F}_n(x))$

Other Applications of Orbifold Theory

- Algebraic Topology and twisted K-theory

  Adem, Ruan, Lupercio, Uribe, Freed, Hopkins, Teleman

Idea: Many ideas from orbifold theory give new understanding and new results for twisted equivariant K-theory, twisted K-theory on stack, twisted algebraic K-theory and non-commutative geometry.

$\Downarrow$

Importance to understand more singular space such as Artin stack where local chart is U/G for compact Lie group G

- Deligne's l-adic version of Chen-Ruan cohomology

$\Downarrow$

Arithematic Application?