KAPUSTIN:
TOPOLOGICAL STRINGS & GENERALIZED COMPLEX GEOMETRY

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AK: hep-th/0310057
N. Hitchin : math 0209099) generalized cx structures
M Gualtieri

σ MODELS WITH $N=(2,2)$ SUPERSYMMETRY

$(\underset{\text{cpt mfd}}{X}, \underset{\text{cx str.}}{I}, \underset{\text{kähler form}}{\omega}, B)$

$B \in H^2(X,\mathbb{R})$ "flat B field"

TWO POSSIBLE TWISTED FIELD THEORIES

A MODEL $\xleftrightarrow[\text{SYMMETRY}]{\text{MIRROR}}$ B MODEL

underlying vs: $H^*(X)$ $\bigoplus_{p,q} H^q(\Lambda^p TX)$

mult = quantum product

CLASSICAL σ-MODELS:

FUNCTIONAL $S(\varphi, \psi_+, \psi_-)$

$\varphi : \Sigma \to X$ (Σ = worldsheet, X = target)

$\psi_\pm \in \Gamma(\varphi^* TX \otimes S^2_\pm)$

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ζ depends only on metric tensor g st $gI = \omega$, but other stuff depends on I, ω.

(X, g, I_+, I_-, B) s.t. (g, I_+) & (g, I_-) are both Kähler structures

$c_1(X, I_+) = c_1(X, I_-) = 0$
to preserve superconformal invariants on quantum level.

For HyperKähler manifolds, $\exists$ a 2-sphere of cx. structures compatible with ω.

SuperVirasoro algebra:

generators: L_n^+, Q_n^+, $\bar{Q}_n^+$, J_n^+
L_n^-, Q_n^-, $\bar{Q}_n^-$, J_n^-

Do topological twist on $N=2$ ~~topological~~ superconformal field theory.

Generalize: $C_2(X) \to U(1)$ s.t.

$$e^{2\pi i B}(\partial \alpha) = e^{2\pi i \int_\alpha H}$$
$$H \in \Omega^3_d(X)$$

Kähler condition $\nabla_{LC} I_\pm = 0$ and g is type $(1,1)$ (Hermitian)

is modified ~~to~~
$\nabla_\pm I_\pm = 0$, where $\nabla_\pm = \nabla_{L.C.} \pm g^{-1} H$

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Generalized Complex Structures (GC)

Recall CX structure is
$I : TX \circlearrowleft$ s.t. $I^2 = -\mathbb{1}_{TX}$ & $\ker(I - i)$ IS INTEGRABLE
($\Leftrightarrow$ CLOSED WRT LIE BRACKET)

GC: $\ell : TX \oplus T^*X \circlearrowleft$ s.t.

i) $\ell^2 = -\mathbb{1}$

ii) ~~q is the natural pairing on $TX \oplus T^*X$~~,
Define $q((v,\xi),(\tilde{v},\tilde{\xi})) = v(\tilde{\xi}) + \tilde{v}(\xi)$.
Then q is type $(1,1)$

iii) $\ker(\ell - i) := E$ is integrable,
i.e. closed wrt Courant bracket:
$[(v,\xi),(\tilde{v},\tilde{\xi})] := ([v,\tilde{v}]_{Lie}, \mathcal{L}_v \tilde{\xi} - i_{\tilde{v}} d\xi)$
This isn't skew, but satisfies a Jacobi identity (which ceases to hold if you skew symmetrize).
Restricting to an isotropic subbundle $\rightsquigarrow$ a Lie bracket.
Thus, E has a Lie bracket.

examples

i) complex structures, I, $I^2 = -\mathbb{1}$

$$\ell : \begin{pmatrix} v \\ \xi \end{pmatrix} \longrightarrow \begin{pmatrix} I & 0 \\ 0 & -\underbrace{I^\vee}_{\text{dual}} \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix}$$

ii) symplectic structures:

$$[\ell] = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

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$$[\ell] = \begin{pmatrix} \tilde{I} + \delta P \cdot B & -\delta P \\ \delta\omega + B\delta P \cdot B & -\tilde{I}^{\vee} - B\delta P \end{pmatrix}$$

$$[\mathcal{J}] = \left(\begin{array}{c|c} \delta I + \tilde{P} B & -\tilde{P} \\ \hline \tilde{\omega} + B\tilde{P}B + B\delta I + \delta I^{\vee} B & -\delta I^{\vee} - B\tilde{P} \end{array}\right)$$

compatible

i) $\ell\mathcal{J} = \mathcal{J}\ell$

ii) $q\,\ell\,\mathcal{J} > 0$

pairing

$$\tilde{I} = \tfrac{1}{2}(I_+ + I_-)$$

$$\tilde{\omega} = \tfrac{1}{2}(\omega_+ + \omega_-)$$

$$\tilde{P} = \tfrac{1}{2}(\omega_+^{-1} + \omega_-^{-1})$$

$$\delta I = \tfrac{1}{2}(I_+ - I_-)$$

$$\delta\omega = \tfrac{1}{2}(\omega_+ - \omega_-)$$

$$\delta P = \tfrac{1}{2}(\omega_+^{-1} - \omega_-^{-1})$$

Productsabove by considering 2-forms as maps $TX \to T^*X$ & bivectors as maps $T^*X \to TX$

This reduces in special cases to cx & symp.

test

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$C(X) \to E \to \dots \to \Lambda^p E^* \xrightarrow{d_E} \Lambda^{p+1} E^*$

from Lie algebroid structure

$H^*_{d_E}(\Lambda^* E^*)$ = space of states of a TFT

$\Gamma(\Lambda^* E)$ = fns. on ΠE

coordinates θ^a on fibres of E

~~differential~~

$$d_E : f(x,\theta) = \theta^a \Pi^i_a \frac{\partial f}{\partial x^i}$$

$$\Pi : E \to TX_{\mathbb{C}}$$

If ℓ is from a cx structure, i.e. $\ell = \begin{pmatrix} I & 0 \\ 0 & -I^\vee \end{pmatrix}$

$$H_{d_E}(\Lambda^* E^*) = \bigoplus_{p,q} H^p(\Lambda^q TX)$$

trace: $H_{d_E}(\Lambda^* E^*) \to \mathbb{C}$

(A model: integration of forms

B model: involves a triv of anticanonical l.b.)

$$c_1(I_+) = c_1(I_-) = 0$$

$$\langle \alpha, \beta, \gamma \rangle = \sum_{\substack{\text{holo maps } \Sigma \to X \\ \Sigma = \mathbb{P}^1,\ 3 \text{ marked pts} \to 3 \text{ cycles}}} e^{-S_{cw}}$$

holo maps are replaced by generalized holomorphic

$$\rho \circ d\varphi : T\Sigma \xrightarrow{d\varphi} TX \xleftarrow{\rho} TX \oplus T^*X$$

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Generalized holo map:

$(\rho \circ d\varphi) J_\Sigma = J (\rho \circ d\varphi)$

AK. Expects a "generalized Floer homology" interpolating between coherent sheaves and Fukaya category.