A NATURAL FLOER THEORY WITHOUT OBSTRUCTION

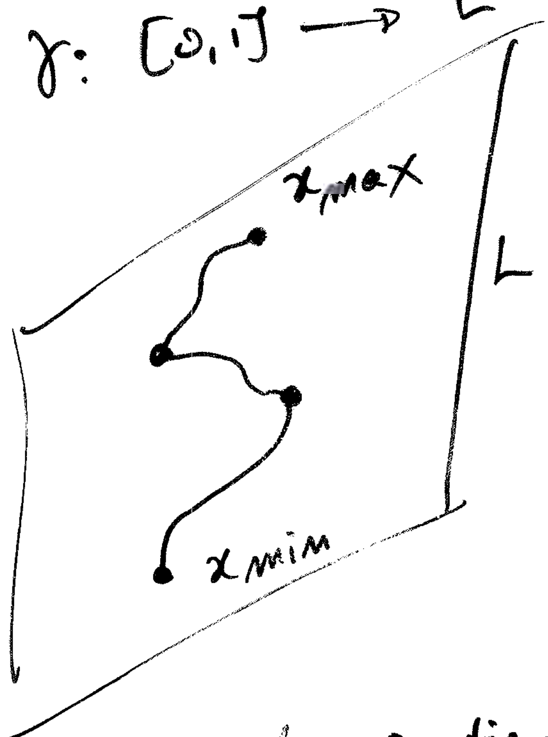
joint work with Octav CORNEA

$L \underset{\text{lagr.}}{\subset} (M,\omega)$

Fix:
- a generic a.c.s. J in $\mathcal{J}(\omega)$
- a Morse fct $f : L \to \mathbb{R}$ which is Smale generic

Join all critical points of f with a simple path

$\gamma : [0,1] \to L$



Denote by L_γ the quotient $L \to L/\gamma = L_\gamma$

Note that $L_\gamma \underset{h.e.}{\simeq} L$

Consider the following complex:

$$\mathbb{H}C(J,f,\gamma) \subset T(C_*(\Omega L_\gamma)) \otimes T(\mathbb{Q}(\mathrm{Crit}(f))) \otimes \Lambda$$

T stands for "Universal tensor algebra"

C_* is the complex of singular chains (cubic homology) with $\mathbb{Q}$-coefficients.

$$\Lambda = \left\{ \sum_{\lambda_i \in H_2^S(M,L)/\sim} q_i e^{\lambda_i} \;\middle|\; \text{for each } K>0,\ \exists \text{ only finitely non zero } q_i \in \mathbb{Q} \text{ with } \omega(\lambda_i) < K \right\}$$

ΩL_γ is the space of Moore loops:

$$\left\{ [a,b] \xrightarrow{\alpha} L_\gamma \;\middle|\; 0 \le a \le b \text{ and } \alpha(a) = \alpha(b) = * \right.$$

ΩL_γ has a neutral element $1_\Omega = \alpha_0 : [0,0] \to *$

$$\cancel{\Sigma}\ \underbrace{\Omega\Sigma(X)}$$

$$\|$$

$$\Omega L$$

$$\Omega\Sigma\Omega L$$

$$\mathbb{F}C(J,f,\gamma) = \Big[C_*(\Omega_{mc} L_\gamma) \otimes \mathbb{1}_f \bigoplus_{k \geq 1} C_*(\underbrace{\Omega L_\gamma \times \cdots \times \Omega L_\gamma}_{k+1}) \otimes Q(\mathrm{Crit}(f))^{\otimes k} \Big] \otimes \Lambda$$

$$= \Big\{ \sum_{\lambda} A(\lambda) \otimes \mathbb{1}_f \otimes e^{\lambda} + \sum_{\lambda, x_1, \ldots, x_k} A(\lambda, x_1, \ldots, x_k)\, x_1 \cdots x_k\, e^{\lambda} \Big\}$$

chain of loops in class λ

chain of $(k+1)$-tuples of loops in class λ

The differential of that complex :

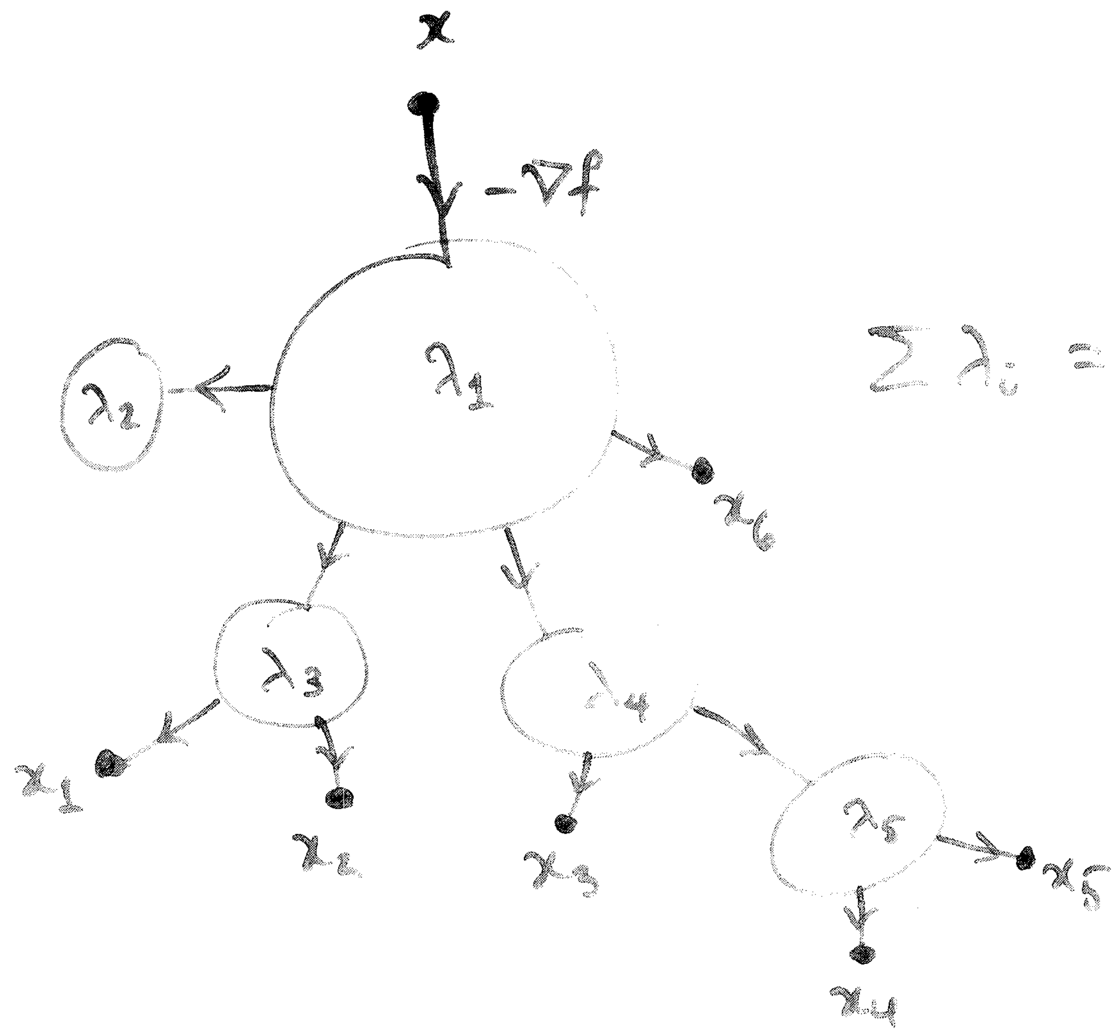
- d is Λ - linear and satisfies the Leibniz rule :

$$d(A \otimes x_1 \cdots x_k \otimes e^\lambda)$$
$$= d(A \otimes x_1 \cdots x_k) \otimes e^\lambda$$
$$= \Big(\partial A \, x_1 \cdots x_k + \sum_i A \, x_1 \cdots \overset{?}{(dx_i)} \cdots x_k\Big) \otimes e^\lambda$$

- $d(1\!\!1_\Omega) = d(1\!\!1_f) = 0$

- $$dx = \sum_{\lambda, x_1, x_2, \ldots, x_k} \overset{?}{m^x_{x_1 \cdots x_k}(\lambda)} \, x_1 \cdots x_k \, e^\lambda$$

- $m^x_{x_1 \cdots x_k}(\lambda)$ is a chain of $(k+1)$-tuples of loops associated with the "clustered moduli space" $\mathcal{M}^x_{x_1 \cdots x_k}(\lambda)$?



$\sum \lambda_i = \lambda$

$\mathcal{M}^{x}_{x_1 \dots x_n}(\lambda)$ is the space of all these configurations. It is obtained by gluing together a finite number of moduli spaces along all facets of codim 1 in the stratification of the boundary.

Thus we know what is:

$$dx = \sum_{\lambda, x_1 \dots x_k} m^{x}_{x_1 \dots x_k}(\lambda)\, x_1 \dots x_k\, e^{\lambda}$$

<u>Remarks.</u> (1) $k=0$ means $m^{x}(\lambda)$

(2) $\lambda = 0$ is permitted only for $\begin{cases} k=1 & \mathbb{F}C \\ k\geq 1 & \check{\mathbb{F}}C \end{cases}$

<u>PRODUCTS.</u>

$$\boxed{x_i \text{ exist } \forall\ 1 \leq i \leq k}$$

$$(\underset{\substack{\cap \\ C_*(\Omega L_\gamma^{k+1})}}{A}\, x_1 \dots x_k) \times_i (\underset{\substack{\cap \\ C_*(\Omega L_\gamma^{l+1})}}{B}\, y_1 \dots y_l)$$

$$\underset{\text{def.}}{=} \quad A \circ_i B \quad x_1 \dots x_{i-1}\, y_1 \dots y_l\, x_{i+1} \dots x_k$$

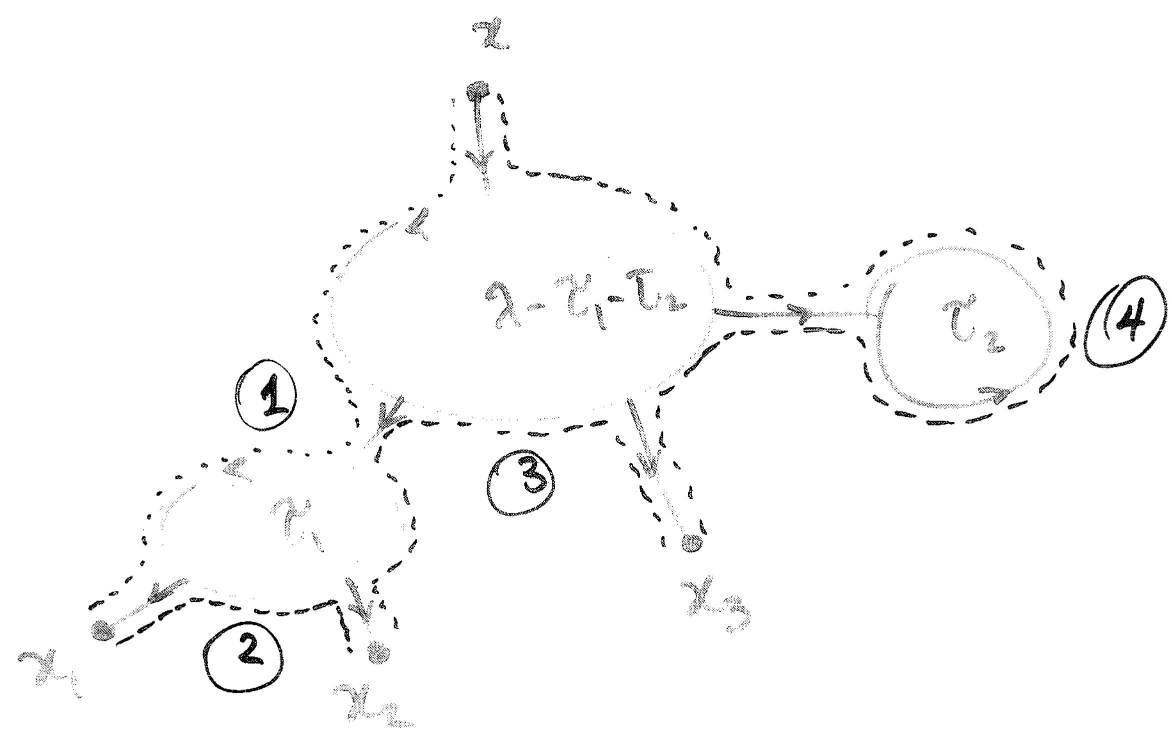
$$\Delta_p \xrightarrow{f} (\Omega L_\gamma)^{k+1}$$

$$\Delta_q \xrightarrow{g} (\Omega L_\gamma)^{l+1}$$

Consequence: $\partial \mathcal{M}^{x}_{x_1 \ldots x_k}(\lambda)$ is made of configurations with one broken flow line and nothing else.

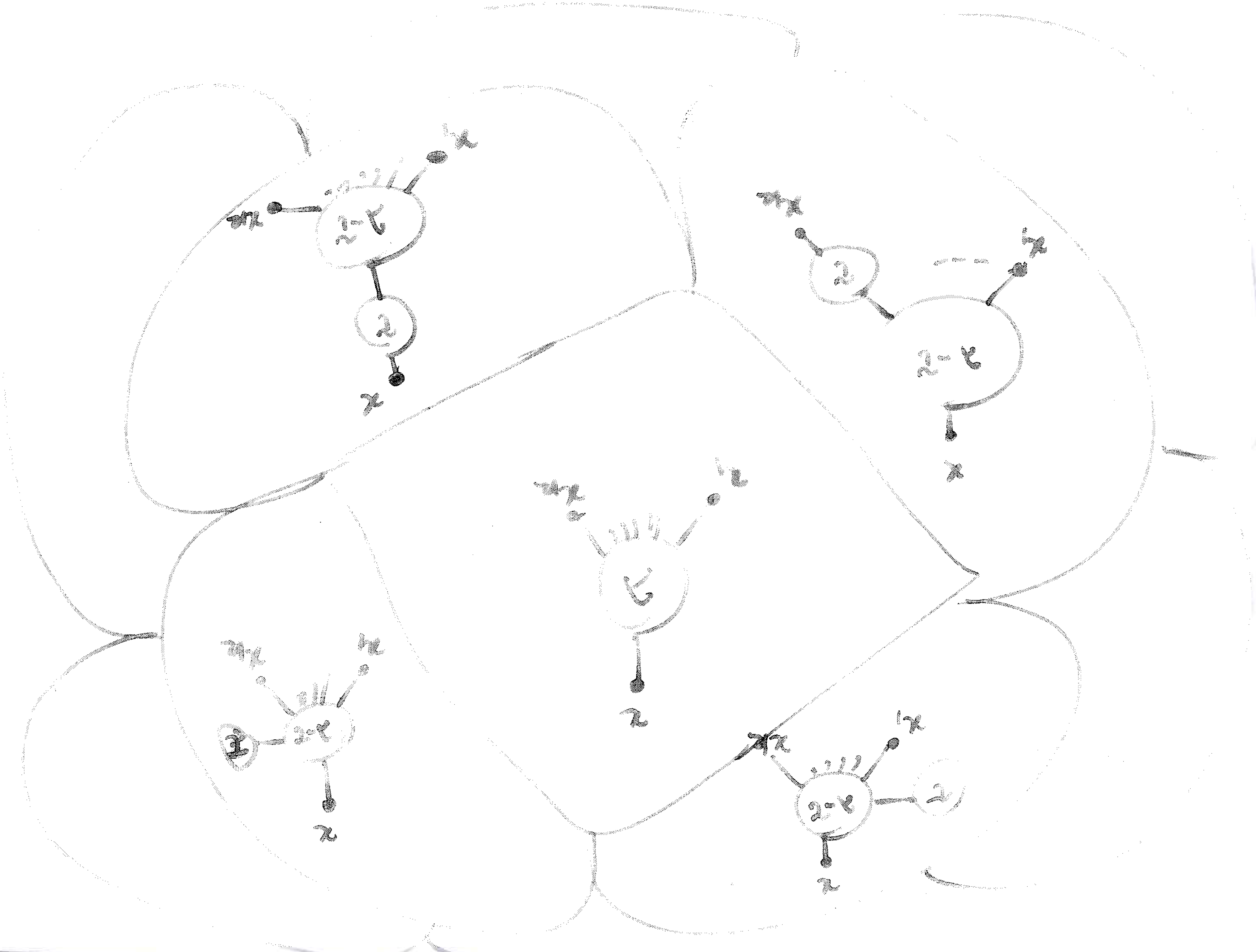
(9)

$$\mathcal{M}^{x}_{x_1 \cdots x_k}(\lambda) \rightsquigarrow \mathfrak{M}^{x}_{x_1 \cdots x_k}(\lambda) \; ?$$



$$\mathcal{M}^{x}_{x_1 \cdots x_k}(\lambda) \longrightarrow \underbrace{\Omega L_\gamma \times \cdots \times \Omega L_\gamma}_{k+1 \text{ times}}$$

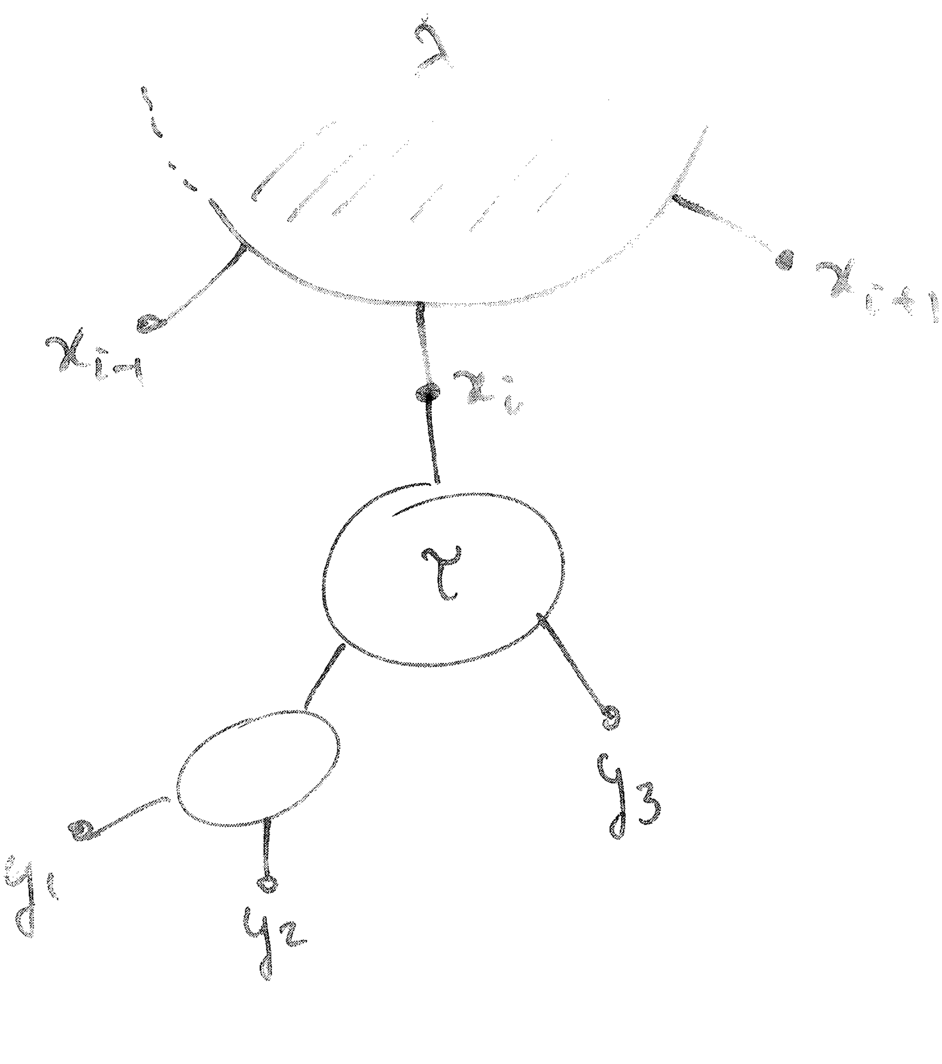
$$\mathfrak{M}^{x}_{x_1 \cdots x_k}(\lambda) \in C_{\dim \mathcal{M}}(\Omega L_\gamma \times \cdots \times \Omega L_\gamma)$$



This is what we mean by the Leibniz rule:

$$d(m\, x_1 \cdots x_k e^{\lambda}) =$$

$$\Big(\partial m\, x_1 \cdots x_k + \sum_i m\, x_1 \cdots x_{i-1} \underset{\displaystyle \sum m^{x_i}_{y_1 \cdots y_k}(\tau)\, y_1 \cdots y_k e^{\tau}}{\underset{\|}{(dx_i)}} x_{i+1} \cdots x_k\Big) e^{\lambda}$$




x
λ
x₁
x_k
λ-τ
τ
ℓ
∂-component bubbling
∂ as ℓ→0
∂ as ℓ→∞

$$\Delta_p \times \Delta_q \longrightarrow (\Omega L_\gamma)^{k+l+1}$$

$$(a,b) \longmapsto \big(f_1(a), \dots, f_{i-1}(a), f_i(a)\circ g_1(b), g_2(b), \dots$$

$$\dots g_l(b), g_{l+1}(b)\circ f_{i+1}(a), f_{i+2}(a), \dots, f_{k+1}(a)\big)$$

IMPLICITELY: $A\, x_1 \dots x_{i-1} (B\, y_1 \dots y_l)\, x_{i+1} \dots x_k$ is defined as the x_i-product between $A\, x_1 \dots x_k$ and $B\, y_1 \dots y_l$.

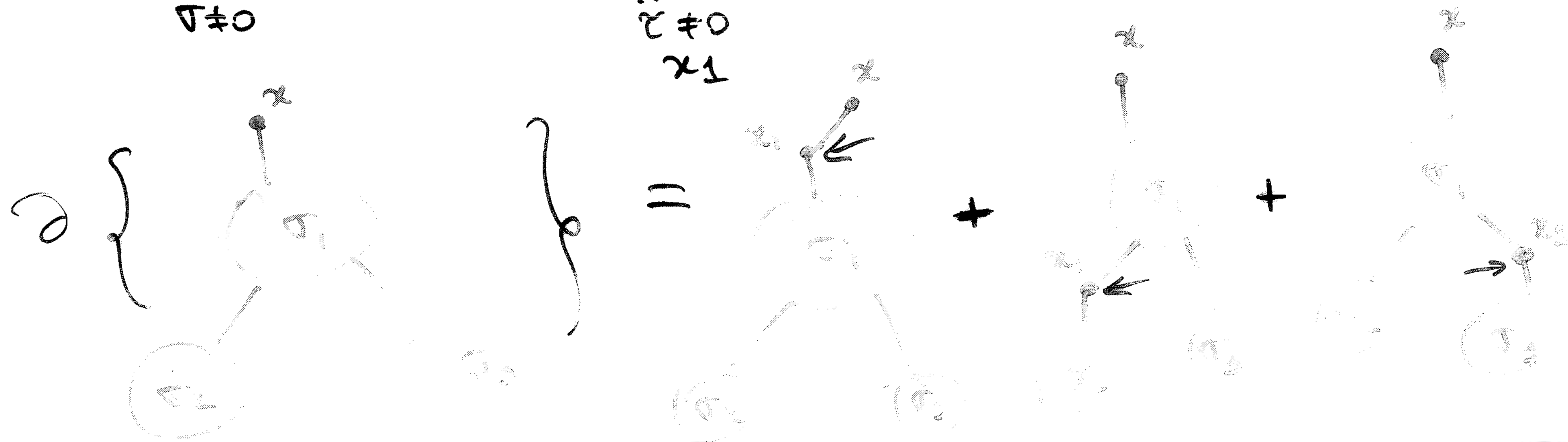
Lemma. $d^2 = 0$.

Proof. $$dx = \sum_{\lambda\neq 0} m^x(\lambda)e^\lambda + \sum_{\lambda, x_1} m^x_{x_1}(\lambda)\, x_1 e^\lambda + \sum_{\lambda, x_1, x_2} m^x_{x_1 x_2}(\lambda)\, x_1 x_2 e^\lambda + \dots$$

$$d^2x = \sum_{\lambda\neq 0} \partial m^x(\lambda)e^\lambda + \sum_{\lambda, x_1} \partial m^x_{x_1}(\lambda)e^\lambda + \sum_{\lambda, \tau, x_1} m^x_{x_1}(\lambda)\left[m^{x_1}(\tau)\mathbb{1}\right]e^{\lambda+\tau} + \dots$$

$$\langle d^2x, \mathbb{1}e^0\rangle = \sum_{\substack{\tau=-\lambda\neq 0\\ x_1}} m^x_y(\lambda)m^y(\tau)e^{\lambda+\tau=0} \quad \text{empty set} \quad = 0$$

$$\langle d^2x, \mathbb{1}e^\sigma\rangle_{\sigma\neq 0} = \partial m^x(\sigma) + \sum_{\substack{\lambda+\tau=\sigma\\ \tau\neq 0\\ x_1}} m^x_{x_1}(\lambda)m^{x_1}(\tau) \quad = 0$$



THM. $\mathbb{F}H_*(J, f, \gamma)$ is invariant under changes of J's, f's and γ's

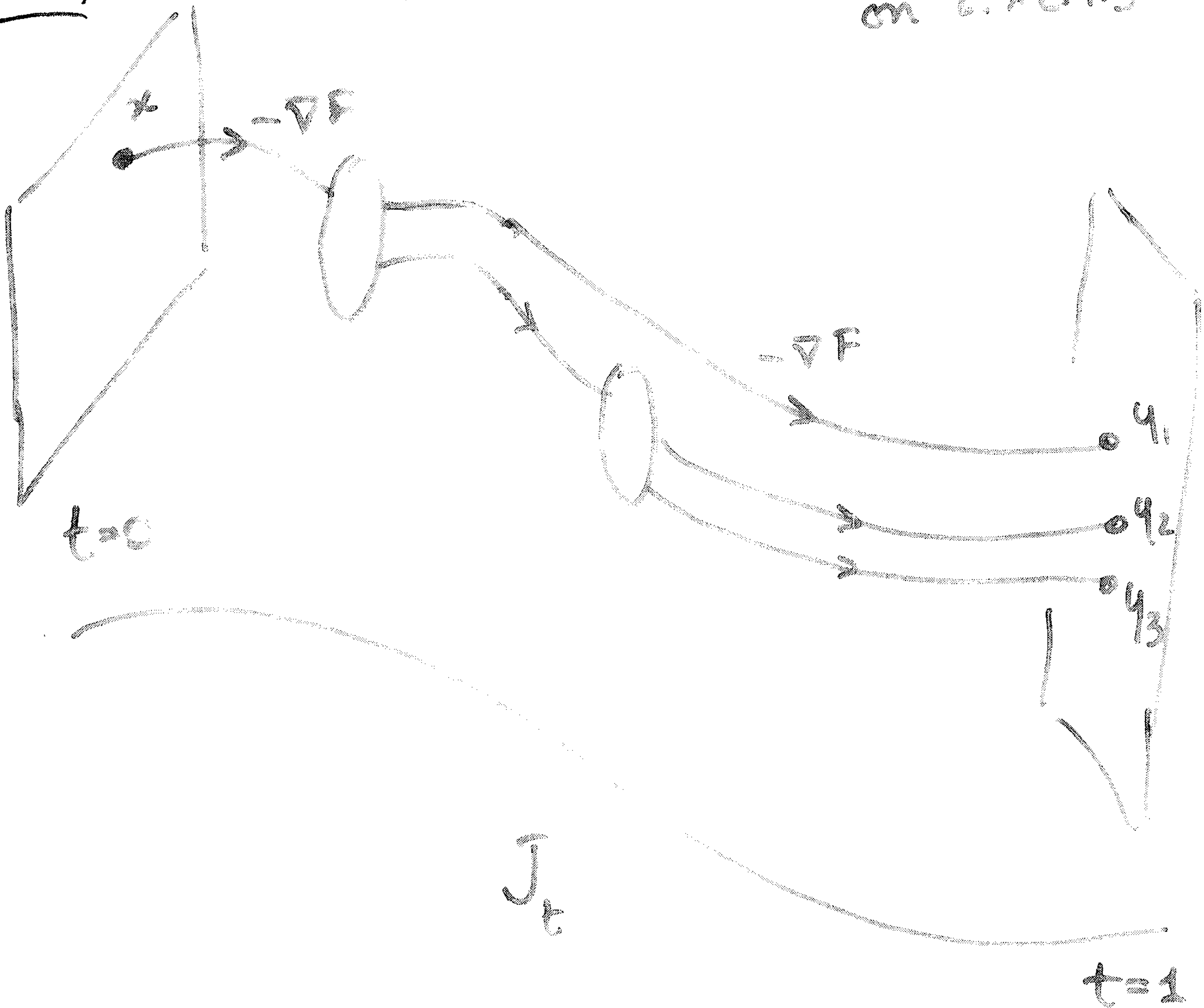
$$\Downarrow$$

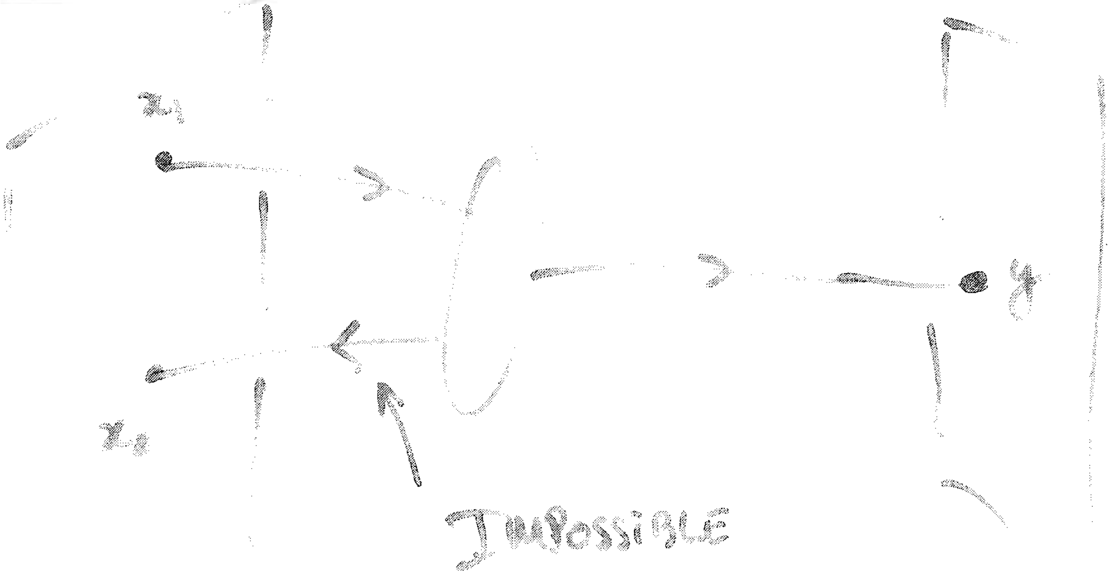
$\mathbb{F}H_*(L)$ (and $\check{\mathbb{F}}H_*(L)$)

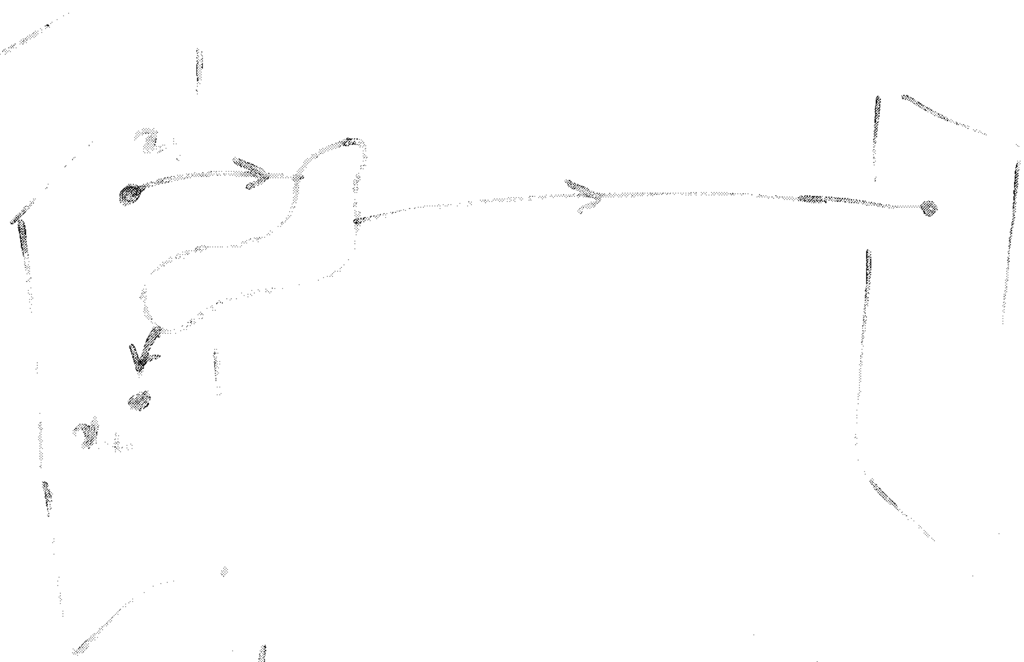
Proof.

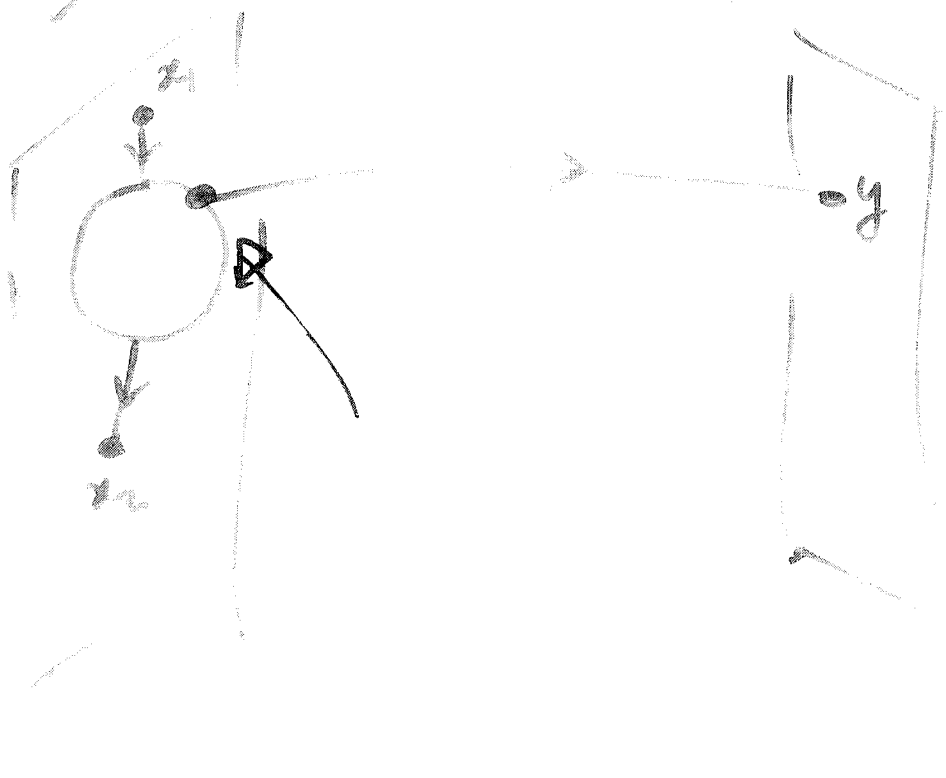
$$F_t = \left(\frac{\cos \pi t + 1}{2}\right)(f_0 + \varepsilon) + \left(\frac{1 - \cos \pi t}{2}\right)(f_1)$$

on $L \times [0,1]$




x1
x2
y
IMPOSSIBLE


x1
x2


x1
x2
y

Reductions:

$$L \xrightarrow{h} X$$

$$\Omega L \xrightarrow{h} \Omega X$$

$$\mathbb{F}C(L) \longrightarrow \mathbb{F}_X C(L)$$

Examples:

1) $X = S^n$

$L \xrightarrow[\text{Thom map}]{} S^n$ } localization of holomorphic curves

2) $X = pt$

$L \longrightarrow pt$

$\Omega L \longrightarrow \Omega pt = pt$

$\mathbb{F}_{pt} C(J, f, \gamma) \simeq T(\mathbb{Q}(\mathrm{Crit}(f))) \otimes \Lambda$

with differential given by counting.

Filtrations: Each filtration (according to action, Maslov index, powers of tensor products, ...)

$\rightsquigarrow$ Spectral sequence $\longrightarrow E^\infty = \mathbb{F}_X H(L)$.

FONDAMENTAL PROBLEM

COMPUTE $\mathbb{F}H_*(L)$.

<u>THM 1.</u> Suppose L can be disjoined from itself by some Hamiltonian isotopy. Then

$$\check{\mathbb{F}H}(L) \simeq Q\langle \mathbb{1} \rangle$$

<u>THM 2.</u> Suppose that $\nexists$ non-constant hol. disc with boundary in L. Then:

$$\mathbb{F}H(L) \simeq H_*(\Omega L_{nc}) \bigoplus_{k \geq 1} \rho^{-k}(H_*(\Omega L))$$

<u>Cor. of THM 1.</u> (in progress). Suppose that L can be disjoined from itself by a Hamiltonian isotopy. Then $\exists$ a non-constant hol. disc passing through each point of L.

* Do not need to assume that L is relatively spin

Proof of Corollary.

Suppose not, by contr., let $p \in L$ be such a point. Let f has its unique min $x_{min} \in Crit(f)$ at p. Then

$$d(x_{min}) = 0 \cdot \mathbb{1} + 0 = 0.$$

Thus $\exists\, e \in \check{\mathcal{F}}C(L)$ s.t. $de = x_{min}$. Thus e must contain a term of the form

- $\mathcal{M}\, x$ with $\dim \mathcal{M} = 0$ and $\mathcal{M}^{x}_{x_{min}}(\lambda \neq 0) \neq \phi$

OR

- $\mathcal{M}\, x\, x_{min}$ or $\mathcal{M}\, x_{min}\, x$ with $\dim \mathcal{M} = 0$ and $\mathcal{M}^{x}(\lambda \neq 0) \neq \phi$

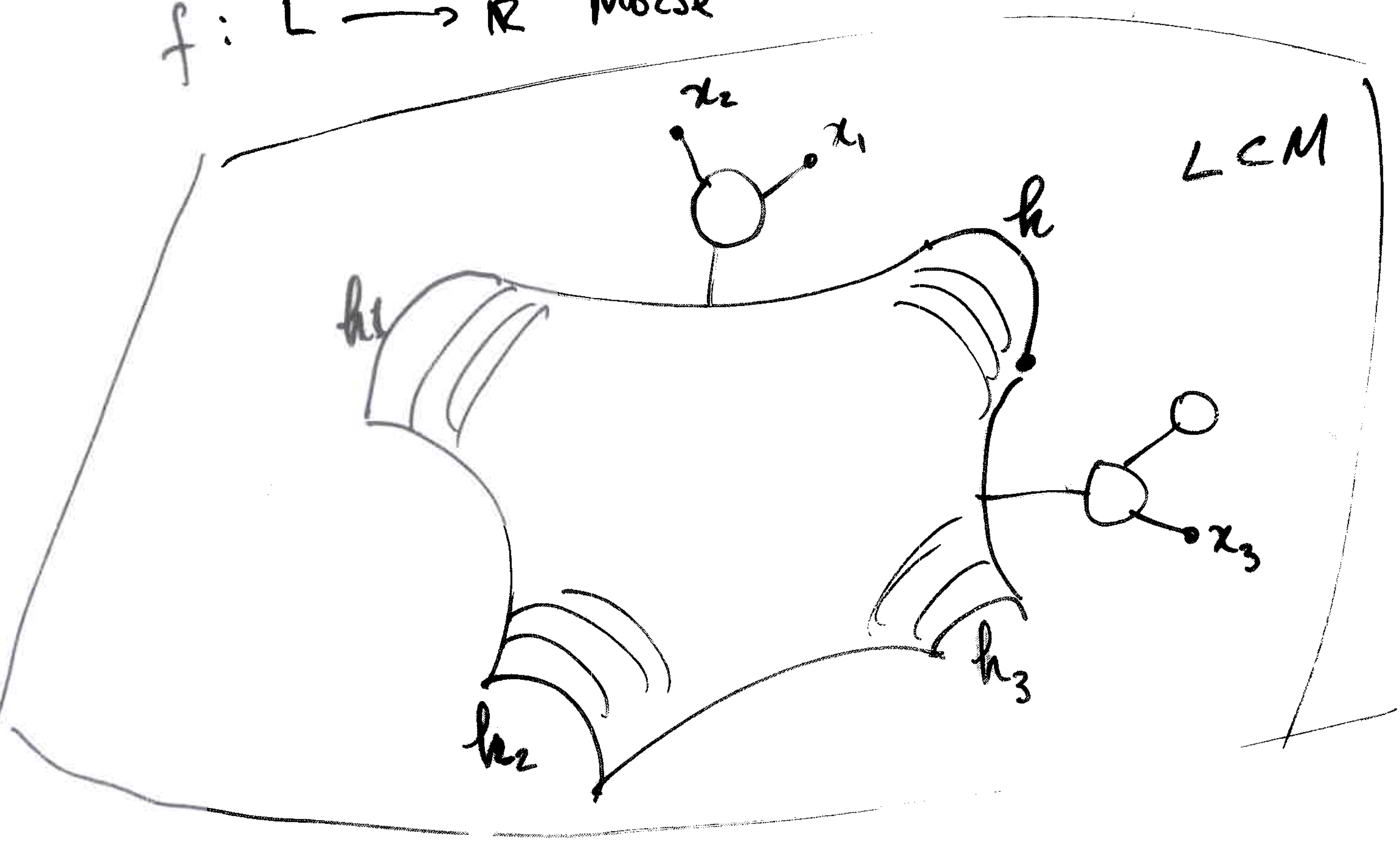
In both cases, the dim counting implies $\mu(\alpha) \leq 1$.

This proves Cor. in case $\mu(\alpha) > 1$.

Proof of THM 1.

$H_t \rightsquigarrow \varphi_{t \in [0,1]}$ on M

$f : L \longrightarrow \mathbb{R}$ Morse



The h, h_i's are elements of the finite set

$$C(H) = \left\{ h : [0,1] \longrightarrow M \;\middle|\; \begin{array}{l} h(0) = h(1) \in L \\ \dot{h}(t) = X_{H_t} \end{array} \right\}$$

$$FFC(H, f^\infty) = \left\{ \underbrace{m_1^f \, f_1 \cdots f_{k_1}}, \underbrace{m_1^H h_1}\, \underbrace{m_2^f \, f_{k_1+1} \cdots f_{k_2}}, \cdots \;\middle|\; \right.$$

$f_i \in \mathrm{Crit}(f)$ $\quad$ $m_i^f \in C_*(\Omega L^{k_i+1})$

$h_i \in C(H)$ $\quad$ $m_i^H \in C_*(\Omega L_{MC})$

The differential is given by the above picture.
Note:

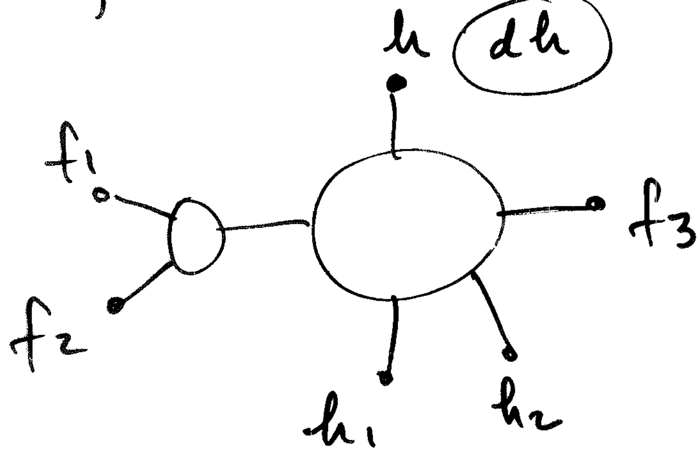
- the differential on f_i's are internal
- the differential on a h_i involves both the h_j's and the f_j's.

THM. $\mathbb{F}C(H, f^\infty)$ is a ∞-sided $\mathbb{F}C(f)$-module, homology invariant under Hamiltonian isotopy of H_t.

IF H_t disjoins L from itself $\Longrightarrow \mathbb{F}H(H, f^\infty) \simeq \mathbb{F}H(f^\infty) \overset{\text{def}}{=} T(\mathbb{F}H(f))$

IF $H_t \longrightarrow H : \mathcal{U}(T_*L) \longrightarrow L \underset{H}{\longrightarrow} \mathbb{R}$ autonomous

Then we may take $\mathrm{Crit}(H) = \mathrm{Crit}(f)$ and set



$$\begin{tikzcd} \mathbb{FC}(f^\infty) \arrow[r, hook, "\alpha"'] & \mathbb{FC}(H, f^\infty) \arrow[d, "\bar\beta"] \\ & \mathbb{FC}(H) = \mathbb{FC}(f) \end{tikzcd}$$

$\bar\beta$ = forgetful map
= All $f_i \mapsto h_i$

Let $z \in \mathbb{FZ}(H)$, $z \neq \mathbb{1}_H$. Then $z \in \mathbb{FZ}(H, f^\infty)$

Since α is an iso in homology:

$\exists$ $e \in \mathbb{FC}(H, f^\infty)$ and $z_f \in \mathbb{FC}(f^\infty)$ s.t.

$$de = z' - z_f$$
$$\bar\beta(de) = \bar\beta(z') - \bar\beta(z_f)$$
$$d(\bar\beta e) = z - 0 = z$$

QED.

PROOF OF THM 2. There are 2 parts:

1) $H_*(C_*(\Omega L) \otimes \mathbb{Q}\langle \mathrm{Crit}(f)\rangle, d) \simeq \mathbb{Q}$

(due to Barraud-Cornea

2) $\mathrm{FH}_*(f) \simeq \left(\bigoplus_{k \geq 0} p^{-k} H_*(\Omega L)\right) \otimes \Lambda$

The idea of (1):

Filtration $\mathcal{F}$

$$\mathcal{F}_k(C_*(\Omega L) \otimes \underbrace{\mathbb{Q}\langle \mathrm{Crit}(f)\rangle}_{C(f)}) = C_*(\Omega L) \otimes C_{* \leq k}(f)$$

Want to prove that this can be identified with the spectral sequence of

$$\begin{array}{ccc} \Omega L & \hookrightarrow & PL \\ & & \downarrow \\ & & L \end{array}$$

starting at term E^2.

2

$$C_*(\Omega L) \otimes C(f) \xrightarrow{\varphi} C_*(PL)$$

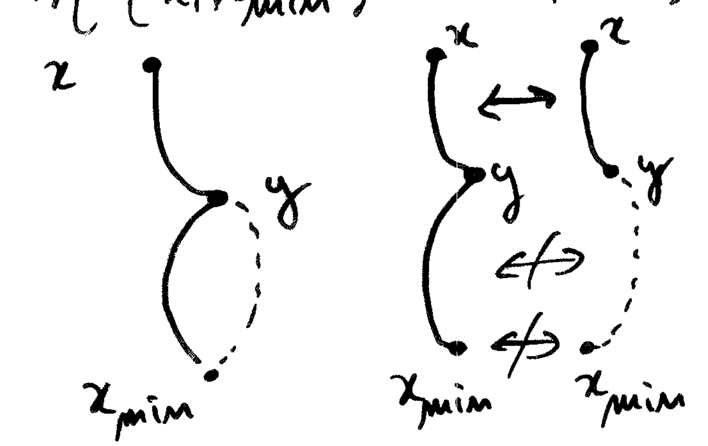
$$\uparrow$$

$$C_*(P_k L)$$

$$P_k L = \pi^{-1}(L_k \subset L)$$
$$L_k = \bigcup_{|x| \le k} W^u(x)$$

$$\varphi(x) = [\hat{\mathcal{M}}(x, x_{min})]$$

$$\bar{\mathcal{M}}(x, x_{min}) \times [0, f(x)]/\sim \; = \hat{\mathcal{M}} \text{ contractible}$$



$(\alpha, t) \sim (\beta, t)$ if
$\alpha(\tau) = \beta(\tau) \ \forall \tau \geq t$

$\hat{\mathcal{M}}$ is a kind of blow-up of trajectories.

$$\partial \hat{\mathcal{M}}(x, x_{min}) = \bigcup_y \bar{\mathcal{M}}(x, x_{min}) \times \hat{\mathcal{M}}(y, x_{min})$$

$$\begin{array}{ccc} x & \overset{\varphi}{\longmapsto} & \hat{\mathcal{M}}(x, x_{min}) \\ \downarrow & & \downarrow d \\ \sum \bar{\mathcal{M}}(x,y)\, y & \overset{\varphi}{\longmapsto} & \sum \bar{\mathcal{M}}(x,y)\, \hat{\mathcal{M}}(y, x_{min}) \end{array}$$

Relations with:

- Floer homology
- Operads
- Hoschild cohomology
- A^∞- structure and obstructions in Lagrangian Floer theory
- SFT

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