Symplectic geometry of the adjoint quotient I

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partly inspired by ideas of Mike Khovanov

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The adjoint quotient $\chi : \mathfrak{sl}_m \to \mathbb{C}^{m-1}$ is the map taking a matrix A to the coefficients of the characteristic polynomial $det(\lambda I - A)$. Away from the diagonals of $\mathbb{C}^{m-1}$, the fibre $\chi^{-1}(t)$ is smooth, and these form a fibre bundle over $\mathbb{C}^{m-1} \setminus \Delta \cong Conf_m^0(\mathbb{C})$. We will be interested in $m = 2n$ and the restriction of χ to a certain affine subspace $\mathcal{S}_n \subset \mathfrak{sl}_m$, which explicitly comprises the matrices

$$A = \begin{pmatrix} A_1 & I & & & \\ A_2 & & I & & \\ \vdots & & & & \\ A_{n-1} & & & & I \\ A_n & & \cdots & & 0 \end{pmatrix}$$

where the A_k are 2×2 matrices, and with $tr(A_1) = 0$. The characteristic polynomial is $det(\lambda I - A) = det(\lambda^n - A_1 \lambda^{n-1} - \cdots - A_n)$. The smooth fibres $Y_{n,t} = \chi^{-1}(t) \cap \mathcal{S}_n$ are smooth complex affine varieties of dimension $2n$. For $n = 1$, $\mathcal{S}_n = \mathfrak{sl}_2$ and the smooth fibres are affine quadrics (T^*S^2 to you); the fibre over 0 is a quadric with a node ($x^2 + yz = 0$).

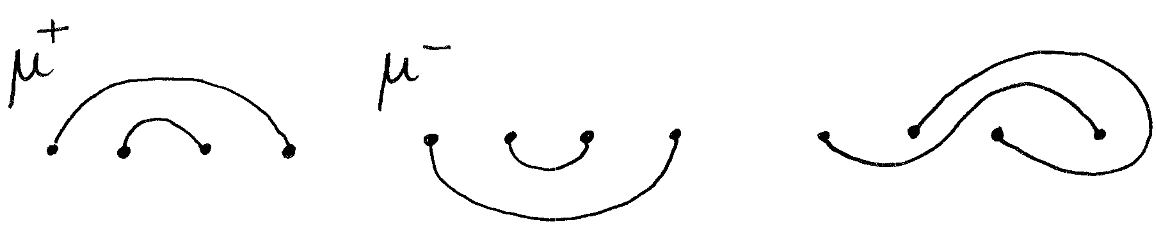
Over $Conf^0_{2n}(\mathbb{C})$, $\chi|\mathcal{S}_n$ is again a fibre bundle. If we equip $\mathcal{S}_n$ with the restriction of the constant Kähler form, we get a symplectic fibre bundle. The pullback of this by the quotient map $\mathbb{C}^{2n-1} \to \mathbb{C}^{2n-1}/S_{2n}$ admits a simultaneous resolution (Grothendieck). Hence the monodromy fits into

$$\begin{array}{ccc} Br_{2n} = \pi_1(Conf^0_{2n}(\mathbb{C})) & \longrightarrow & \pi_0(\mathsf{Symp}(Y_{n,t})) \\ \downarrow & & \downarrow \\ S_{2n} & \longrightarrow & \pi_0(\mathsf{Diff}(Y_{n,t})) \end{array}$$

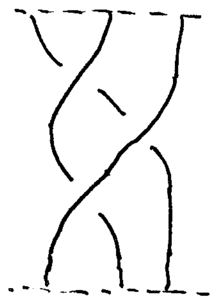
For $n = 1$, the generator of $Br_2 \cong \mathbb{Z}$ maps to the Dehn twist on T^*S^2.

The $\mathfrak{sl}_2$ case is a model for what happens when we degenerate to a fibre of χ where 2 eigenvalues coincide. Fix a fibre $Y_{n,\tau}$ corresponding to eigenvalues $(\lambda_1 = \lambda_2, \lambda_3, \dots, \lambda_{2n})$. The singular set is then a submanifold of complex codimension 2, and an open neighbourhood looks like its product with $\{x^2 + yz = 0\}$.

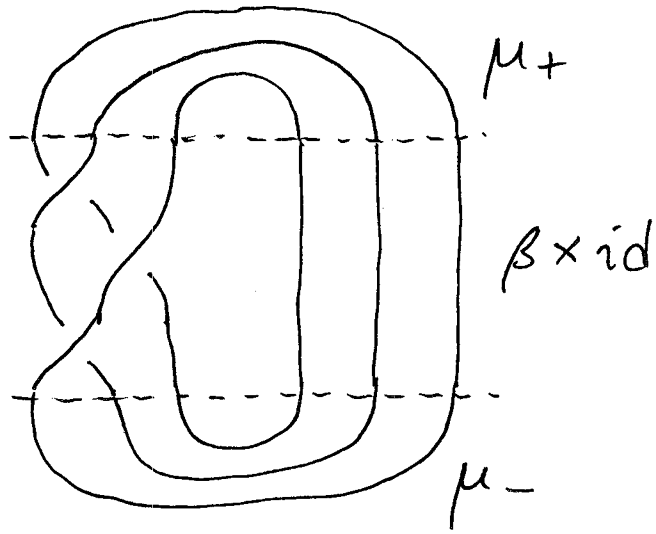
If the double eigenvalue is $\lambda_1 = \lambda_2 = 0$, the singular set consists of matrices with $A_n = 0$, hence is isomorphic to the (smooth) fibre $Y_{n-1,\tau'}$ of $\chi|\mathcal{S}_{n-1}$ over $\tau' = (\lambda_3, \ldots, \lambda_{2n})$. Given a Lagrangian submanifold $L_{n-1} \subset Y_{n-1,\tau'}$, we get an induced Lagrangian submanifold L_n in a nearby smooth fibre $Y_{n,t}$ $(t \approx \tau)$. Topologically $L_n \cong L_{n-1} \times S^2$. Iteratively bringing the $2n$ eigenvalues together in n distinct pairs, we get a Lagrangian submanifold $(S^2)^n \cong L_\mu \subset Y_{n,t}$ for every crossingless matching μ.



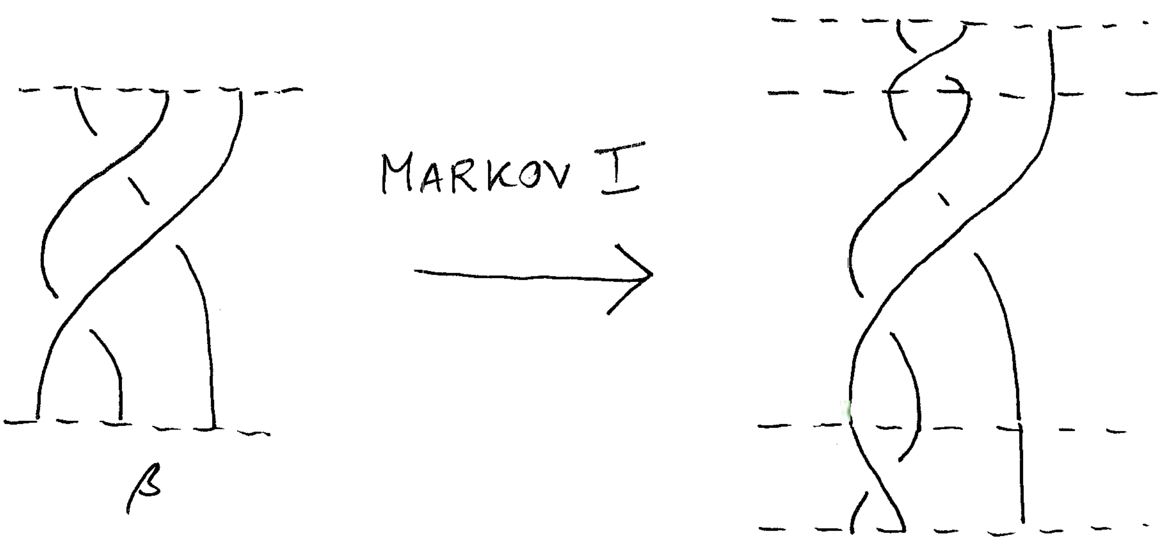
An oriented link $K \subset S^3$ can be presented as the closure of a braid $\beta \in Br_n$, $K = K_\beta$.



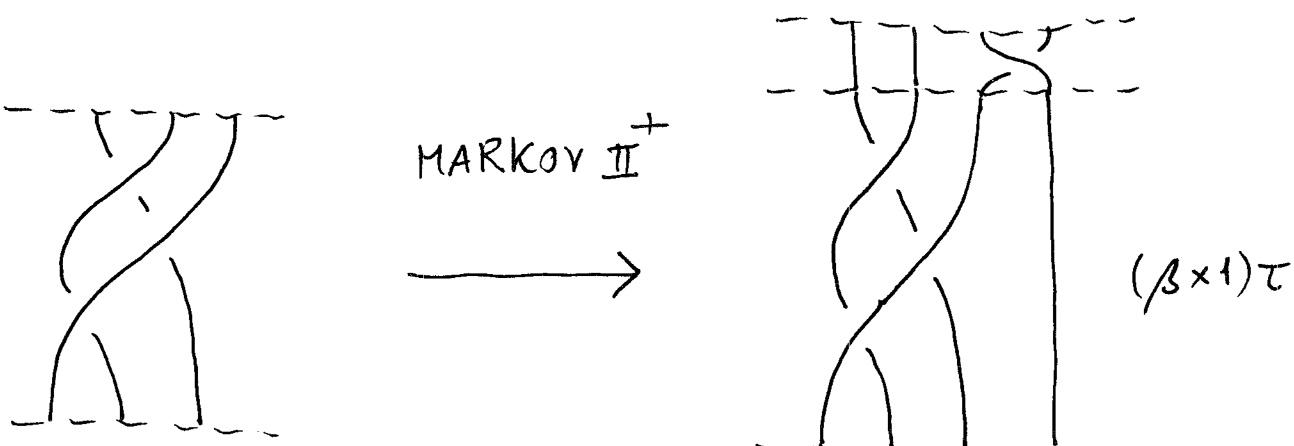
β



K_β



β $\qquad$ $u\beta u^{-1}$

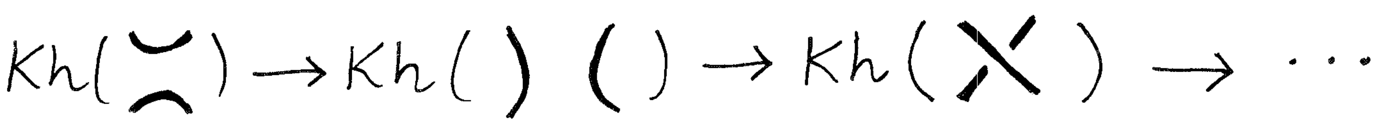


β

Conjecture: the invariant defined above is isomorphic to Khovanov's combinatorially defined invariant $Kh^{*,*}(K)$ (with the bigrading collapsed). Basic properties of Khovanov's theory include:

(1) For a d component unlink, $Kh^{*,*}(K) \cong H^*((S^2)^d)[d]$.

(2) There is a skein long exact sequence



(3) The graded Euler characteristic, defined as $\sum_{i,j}(-1)^i q^j \operatorname{rank} Kh^{i,j}(K)$, is the Jones polynomial.

(1) holds in our theory because $L \cong (S^2)^n$ and $HF^*(L, L) \cong H^*(L)$. (2) should hold by a generalization of the long exact sequence in Floer cohomology. (3) relies essentially on the bigrading, which we do not yet know how to recover geometrically.

Taking a configuration of points $t \in Conf^0_{2n}(\mathbb{C})$ which lie on the real line, we get two crossingless matchings $\mu_\pm$, hence two Lagrangian submanifolds of $Y_{n,t}$ which in fact are equal, $L_{\mu_+} = L_{\mu_-}$. For $\beta \times id \in Br_n \times Br_n \subset Br_{2n}$ we have a monodromy map on $Y_{n,t}$, also denoted by $\beta \times id$.

Theorem: *$HF^*(L_{\mu_\pm}, (\beta \times id)L_{\mu_\pm})$ is an invariant of the oriented link $K = K_\beta$.*

The theorem is proved by verifying invariance under the two Markov moves: the first one is fairly obvious from the general structure of the definition, and the second relies on the local geometry of the $\chi|\mathcal{S}_n$.

$Y_{n,t}$, $t = (\lambda_1, \ldots, \lambda_{2n})$, is also a moduli space of solutions to Nahm's equations. These can be viewed as equations for instantons on $\mathbb{R}^4$ with symmetry, which shows that $Y_{n,t}$ is naturally hyperkähler. More precisely, we want

$$T : (0; +\infty) \to \mathfrak{sl}_{2n} \otimes \mathbb{R}^3,$$
$$dT_i/dr + [T_j, T_k] = 0$$

with the following boundary conditions. As $r \to +\infty$, $T_1 \to 0$ and $T_2 + iT_3 \to$(matrix with eigenvalues λ_k). As $r \to 0$, the T_i have simple poles, whose residues define a homomorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{sl}_{2n}$ mapping $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to the nilpotent $A_1 = \cdots = A_n = 0$ in $\mathcal{S}_n$.

Because the centralizer of ρ in $\mathfrak{sl}_{2n}$ is $\mathfrak{sl}_2$, we have a (fixed point free) $SU(2)$-action on the space of solutions. On $Y_{n,t}$, this is given by conjugation of each A_k. There is an $SU(2)$-equivariant version of the link invariants.

There is also an involution, which is not part of a continuous symmetry group and is more special to our boundary conditions. The requirement that $T_1 \to 0$ breaks the $SO(3)$-symmetry of Nahm's equations, writing $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$. It follows that there is a holomorphic involution ι on the space of solutions, which is $(T_1, T_2, T_3) \mapsto (T_1^t, T_3^t, T_2^t)$ followed by a conjugation. In our original description of $Y_{n,t}$, this is simply $A_k \to A_k^t$, $k = 1, \ldots, n$ (this preserves eigenvalues).

Notation: Σ is the double branched cover of $\mathbb{C}P^1$ with branch points $(\lambda_1, \ldots, \lambda_{2n})$. It has genus $g = n - 1$. We denote by $\pi : \Sigma \to \mathbb{C}P^1$ the hyperelliptic quotient, and by $\pm\infty \in \Sigma$ the two preimages of $\infty \in \mathbb{C}P^1$.

Proposition: *The fixed point set $(Y_{n,t})^\iota$ is canonically isomorphic to $Sym^n(\Sigma) \setminus (\Delta_1 \cup \Delta_2)$, where Δ_1 consists of configurations including a fibre of π, and Δ_2 are configurations containing $+\infty$ or $-\infty$.*

Equivalently, $(Y_{n,t})^\iota$ is a (trivial) $\mathbb{C}^*$-bundle over $Pic^{g-1}(\Sigma)\backslash\Theta$. This description of hyperelliptic Jacobians goes back to Mumford.

The Lagrangian submanifolds L_μ associated to crossingless matchings are ι-invariant, with fixed point sets $(S^1)^n \subset (S^2)^n$. There are two more oriented link invariants, defined by taking Floer cohomology in the fixed point set, or $\mathbb{Z}/2$-equivariant Floer cohomology in $Y_{n,t}$.

To investigate the relation between all these variant theories (and others in the literature), there are two basic tools, one well-understood and the other currently being developed.

Adding a divisor: Let X be Kähler, $D \subset X$ a divisor with nonnegative normal bundle. Given two Lagrangians L_0, L_1 in $X\backslash D$ (and if suitable technical conditions are satisfied), there are groups $HF^*(L_0, L_1; X\backslash D)$ and $HF^*(L_0, L_1; X)$.

The chain groups coincide and the differential can be written as

$$d_X = d_{X\backslash D} + qd^{(1)} + q^2 d^{(2)} + \cdots$$

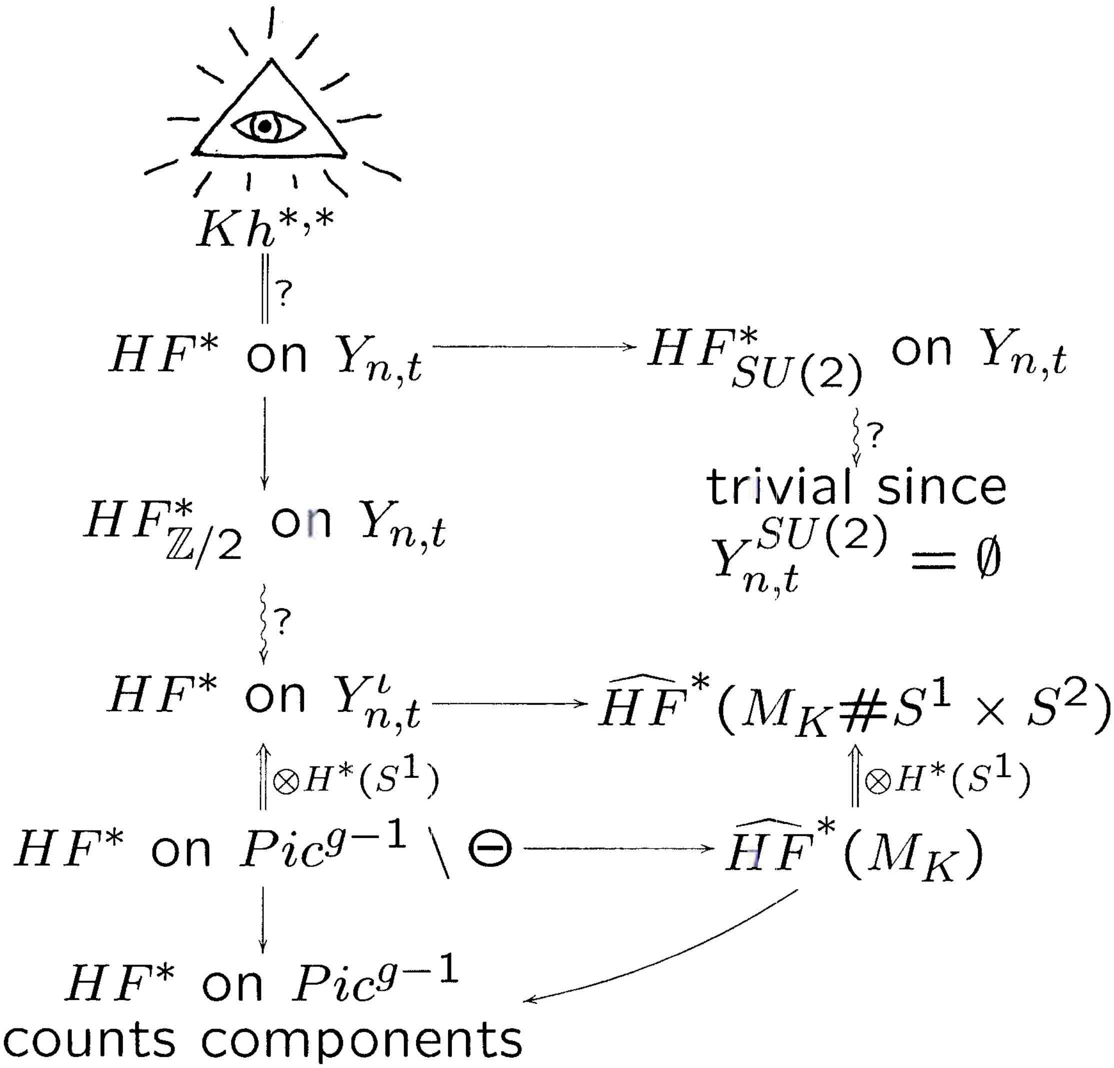
where $d^{(k)}$ counts discs having intersection number k with D. Consequently, we have a spectral sequence $HF^*(L_0, L_1; X\backslash D) \Rightarrow HF^*(L_0, L_1; X)$, in particular the rank of the former group is $\geq$ that of the latter.

Localization: Let G act on X preserving two Lagrangian submanifolds L_0, L_1. Under suitable technical assumptions, we have equivariant Floer cohomology $HF^*_G(L_0, L_1)$ and a spectral sequence converging to $HF^*_G(L_0, L_1)$, whose E_1 term is $C^*(BG) \otimes HF^*(L_0, L_1)$, and E_2 term $H^*(BG; HF^*(L_0, L_1))$. In particular, if $G = \mathbb{Z}/2$ and we are working with $\mathbb{Z}/2$-coefficients, then

$$rank_{H^*_G(pt)} HF^*_{\mathbb{Z}/2}(L_0, L_1) \leq rank_{\mathbb{Z}/2} HF^*(L_0, L_1).$$

The localization theorem for ordinary cohomology (of finite-dimensional G-manifolds M) says that the map $H^*_G(M) \to H^*_G(M^G) = H^*_G(pt) \otimes H^*(M^G)$ becomes an isomorphism after tensoring with the quotient field of $H^*_G(pt)$. A consequence is the (P.) Smith theorem, which for $G = \mathbb{Z}/2$ says $rank_{\mathbb{Z}/2} H^*(M^G) \leq rank_{\mathbb{Z}/2} H^*(M)$.

One can hope to mimic this in Floer cohomology, at least in the case where the normal bundle to the fixed locus has vanishing c_1, which holds in our case. If this works out, we will get the following crazy diagram of relations:



$=$ is isomorphism, $\rightsquigarrow$ is localization, $\rightarrow$ denotes a spectral sequence. The spectral sequence going to Pic^{g-1} is analogous to the Lee-Rasmussen spectral sequence in Khovanov cohomology, and indeed a similar picture appears in Ozsvath-Szabo's work.

One can speculate that the spectral sequence associated to the $SU(2)$-equivariant theory may also give nontrivial invariants. Finally, an important question is whether one can use this additional information to equip the symplectic theory with a bigrading.