Abelian categories and stability conditions

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1. The basic idea

Let $\mathcal{A}$ be an abelian category. We will define *configurations* (σ, ι, π) in $\mathcal{A}$, collections of objects and morphisms in $\mathcal{A}$ attached to a *finite poset* $(I, \preceq)$, satisfying axioms. They are a new tool for describing *how an object in $\mathcal{A}$ breaks up into subobjects*. They are useful for studying *stability conditions* on $\mathcal{A}$.

Let $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$ be a *slope function* with *phase* θ. Under conditions on $\mathcal{A}, Z$ we can define *moduli spaces* $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{st}}(I, \preceq, \kappa, \theta)$ of $(I, \preceq)$-configurations (σ, ι, π) with $\sigma(\{i\})$ θ-(*semi*)*stable*, $i \in I$. Let $I_{\mathrm{ss}}, I_{\mathrm{st}}(I, \preceq, \kappa, \theta)$ be their *Euler characteristics*. They are a system of invariants of $\mathcal{A}, Z$. We prove *identities* for them, and *transformation laws* to change from Z to $\tilde{Z}$.

2. Abelian categories

A category $\mathcal{A}$ has *objects* X, Y in $\mathcal{A}$ or $\mathrm{Obj}(\mathcal{A})$, *morphisms* f in $\mathrm{Mor}(\mathcal{A})$, or $f : X \to Y$. Write $\mathrm{Hom}(X,Y) = \{f : X \to Y\}$.

$\mathcal{A}$ is an *abelian category* if

- $\mathrm{Hom}(X, Y)$ is an *abelian group* for $X, Y \in \mathcal{A}$, and composition is *biadditive*.
- there is a *zero object* $0 \in \mathcal{A}$.
- *direct sums* $X \oplus Y$ exist.
- *kernels* and *cokernels* exist.

Exact sequences make sense.

For $\mathbb{K}$ a field, $\mathcal{A}$ is $\mathbb{K}$-*linear* if Hom(X,Y) is a $\mathbb{K}$-vector space, and composition is bilinear.

Examples

- category of *abelian groups*
- category of $\mathbb{K}$-*vector spaces*
- coh(P), the category of *coherent sheaves* on a *projective variety* P over $\mathbb{K}$.
- mod-A, the category of representations of a *finite-dimensional algebra* A over $\mathbb{K}$.

Subobjects

Let $X \in \mathcal{A}$ and $i : S \to X$, $i' : S' \to X$ be injective. We call i, i' *equivalent* if there is an isomorphism $h : S \to S'$ with $i = i' \circ h$. A *subobject* $S \subset X$ is an equivalence class of $i : S \to X$. Examples:

- subgroups of abelian groups.
- subspaces of a vector space.
- vector subbundles (subsheaves) of a vector bundle (coherent sheaf).

Call $\mathcal{A}$ *noetherian* (*artinian*) if ascending (descending) chains of subobjects must stabilize. Call $\mathcal{A}$ *finite length* if it is noetherian and artinian. Call $0 \neq X \in \mathcal{A}$ *simple* if the only subobjects $S \subset X$ are $0, X$.

Jordan-Hölder Theorem. *For $\mathcal{A}$ of finite length and X in $\mathcal{A}$, there exist subobjects $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ with $S_k = A_k/A_{k-1}$ simple, and n, S_k unique up to order, iso.*

Then we call $S_1, \ldots, S_n$ the *simple factors* of X, and $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ a *composition series* for X.

Let $S_1, \ldots, S_n$ be *pairwise non-isomorphic*. Write $\{S_1, \ldots, S_n\} = \{S^i : i \in I\}$, for I a finite *indexing set*, $|I| = n$. Then for each composition series $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$ with $T_k = B_k / B_{k-1}$, there is a unique *bijection* $\phi : I \to \{1, \ldots, n\}$ with $S^i \cong T_{\phi(i)}$, all $i \in I$.

Define a *partial order* $\preceq$ on I by $i\preceq j$ if $\phi(i) \leqslant \phi(j)$ for all ϕ from composition series as above.

Call $J \subseteq I$ an *s-set* if $i \in I$, $j \in J$ and $i\preceq j \Rightarrow i \in J$.

Call $J \subseteq I$ an *f-set* if $i \in I$, $h, j \in J$ and $h\preceq i\preceq j \Rightarrow i \in J$.

The *finite poset* $(I, \preceq)$ encodes all information on *subobjects* $S \subset X$, and their *inclusions* $S \subset T \subset X$, when X has non-isomorphic simple factors.

There are unique *1-1 correspondences*:

- *subobjects* $S \subset X \leftrightarrow$ *s-sets* $J \subseteq I$, where S has simple factors S^j, $j \in J$. If $S, T \leftrightarrow J, K$ then $S \subset T \Leftrightarrow J \subseteq K$.
- *factors* $F = T/S$ for $S \subset T \subset X \leftrightarrow$ *f-sets* $J \subseteq I$, where F has simple factors S^j, $j \in J$.
- *composition series* $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$ $\leftrightarrow$ *bijections* $\phi : I \to \{1, \ldots, n\}$ with $i \preceq j \Rightarrow \phi(i) \leqslant \phi(j)$.

Definition. Let $(I,\preceq)$ be a finite poset. Write $\mathcal{F}_{(I,\preceq)}$ for the set of f-sets of I. Define $\mathcal{G}_{(I,\preceq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preceq)}\times\mathcal{F}_{(I,\preceq)}$ such that $J\subseteq K$, and if $j\in J$ and $k\in K$ with $k\preceq j$, then $k\in J$. Define $\mathcal{H}_{(I,\preceq)}=\{(K,K\setminus J):(J,K)\in\mathcal{G}_{(I,\preceq)}\}$.

Define an $(I,\preceq)$-*configuration* (σ,ι,π) in an abelian category $\mathcal{A}$ to be maps $\sigma:\mathcal{F}_{(I,\preceq)}\rightarrow\mathrm{Obj}(\mathcal{A})$, $\iota:\mathcal{G}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, and $\pi:\mathcal{H}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, where $\iota(J,K),\pi(J,K)$ are morphisms $\sigma(J)\rightarrow\sigma(K)$.

These should satisfy the conditions:

(A) Let $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and set $L=K\setminus J$. Then the following is exact in $\mathcal{A}$:

$$0\longrightarrow\sigma(J)\xrightarrow{\iota(J,K)}\sigma(K)\xrightarrow{\pi(K,L)}\sigma(L)\longrightarrow 0.$$

(B) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{G}_{(I,\preceq)}$ then $\iota(J,L)=\iota(K,L)\circ\iota(J,K)$.

(C) If $(J,K)\in\mathcal{H}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$ then $\pi(J,L)=\pi(K,L)\circ\pi(J,K)$.

(D) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$ then

$$\pi(K,L)\circ\iota(J,K)=\iota(J\cap L,L)\circ\pi(J,J\cap L).$$

This encodes the properties of the set of subobjects $S \subset X$ when X has nonisomorphic simple factors.

Theorem 1. *Let $\mathcal{A}$ have finite length, $X \in \mathcal{A}$ have nonisomorphic simple factors $\{S^i : i \in I\}$, and $\preceq$ be as before. Then there exists an $(I, \preceq)$-configuration (σ, ι, π) with $\sigma(I) = X$, unique up to isomorphism, such that if a subobject $S \subset X$ has simple factors $\{S^j : j \in J\}$, then S is represented by $\iota(J, I) : \sigma(J) \to X$.*

I derived the idea of configuration for $\mathcal{A}$ of *finite length* and X with *nonisomorphic simple factors*. But it is useful much more generally, as a tool for describing how objects decompose into subobjects.

For example, a *short exact sequence* $0 \to X \to Y \to Z \to 0$ is the same as a $(\{1,2\},\leqslant)$-configuration (σ,ι,π) with $\sigma(\{1\}) = X$, $\sigma(\{1,2\}) = Y$ and $\sigma(\{2\}) = Z$. Essentially this says that Y has a subobject X.

Quotient configurations

Let $(I,\preceq)$, $(K,\trianglelefteq)$ be finite posets, and $\phi : I \to K$ surjective with $i \preceq j$ implies $\phi(i) \trianglelefteq \phi(j)$. Let (σ,ι,π) be an $(I,\preceq)$-configuration. Define a $(K,\trianglelefteq)$-configuration $(\tilde\sigma,\tilde\iota,\tilde\pi)$ to be $(\sigma\circ\phi^*,\iota\circ\phi^*,\pi\circ\phi^*)$, where $\phi^* : \mathcal{F}_{(K,\trianglelefteq)}, \mathcal{G}_{(K,\trianglelefteq)}, \mathcal{H}_{(K,\trianglelefteq)} \to \mathcal{F}_{(I,\preceq)}, \mathcal{G}_{(I,\preceq)}, \mathcal{H}_{(I,\preceq)}$ pulls back subsets of K to subsets of I.

We call $(\tilde\sigma,\tilde\iota,\tilde\pi)$ the *quotient* $(K,\trianglelefteq)$*-configuration* of (σ,ι,π). We call (σ,ι,π) a *refinement* of $(\tilde\sigma,\tilde\iota,\tilde\pi)$. If $I=K$ and $\phi=\mathrm{id}_I$ we call (σ,ι,π) an *improvement* of $(\tilde\sigma,\tilde\iota,\tilde\pi)$. Improvements split short exact sequences. We call $(\tilde\sigma,\tilde\iota,\tilde\pi)$ *best* if it admits no strict improvements.

Slope stability

Let $K(\mathcal{A})$ be the Grothendieck group of $\mathcal{A}$. A *slope function* on $\mathcal{A}$ is a homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ with $Z([X]) \in \{r\mathrm{e}^{i\pi\theta} : r > 0,\ \theta \in (0,1]\}$ for all $0 \ncong X \in \mathcal{A}$.

Define the *phase* $\theta([X]) \in (0,1]$ by $Z([X]) = r\mathrm{e}^{i\pi\theta([X])}$. Define X to be

(i) θ-*stable* if $\theta([S]) < \theta([X])$ for all $S \subset X$ with $S \neq 0, X$.

(ii) θ-*semistable* if $\theta([S]) \leqslant \theta([X])$ for all $0 \neq S \subset X$.

(iii) θ-*unstable* otherwise.

A research programme

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $\mathcal{A}$ be some interesting abelian category over $\mathbb{K}$. Let $(I, \preceq)$ be a finite poset and $\kappa : I \to K(\mathcal{A})$ a map. Define a *moduli space* $\mathcal{M}_{\mathsf{all}}(I, \preceq, \kappa)$ to be the set of isomorphism classes of $(I, \preceq)$-configurations (σ, ι, π) with $[\sigma(\{i\})] = \kappa(i)$ in $K(\mathcal{A})$ for all $i \in I$. Let Z be a slope function with phase θ. Define *subspaces* $\mathcal{M}_{\mathsf{st}}, \mathcal{M}_{\mathsf{ss}}(I, \preceq, \kappa, \theta)$ of $[(\sigma, \iota, \pi)]$ in $\mathcal{M}_{\mathsf{all}}(I, \preceq, \kappa)$ with $\sigma(\{i\})$ θ-(semi)stable for all $i \in I$, and $\mathcal{M}^{\mathsf{b}}_{\mathsf{all}}$, $\mathcal{M}^{\mathsf{b}}_{\mathsf{st}}, \mathcal{M}^{\mathsf{b}}_{\mathsf{ss}}(\ldots)$ with (σ, ι, π) *best*.

We call $(I,\preceq,\kappa)$ $\mathcal{A}$-*data*. For the examples I am interested in, $\mathcal{M}_{\mathrm{all}}(I,\preceq,\kappa)$ is an *Artin stack*, and $\mathcal{M}_{\mathrm{st}},\mathcal{M}_{\mathrm{ss}},\mathcal{M}_{\mathrm{all}}^{\mathrm{b}},\mathcal{M}_{\mathrm{st}}^{\mathrm{b}},\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(\ldots)$ are *constructible subsets* (finite unions of substacks of finite type over $\mathbb{K}$), with well-defined *Euler characteristics*. Quotient configurations induce *morphisms* $\mathcal{M}_{\mathrm{all}}(I,\preceq,\kappa)\rightarrow\mathcal{M}_{\mathrm{all}}(K,\trianglelefteq,\mu)$. Define $I_{\mathrm{st}},I_{\mathrm{ss}},I_{\mathrm{st}}^{\mathrm{b}},I_{\mathrm{ss}}^{\mathrm{b}}(I,\preceq,\kappa,\theta)$ to be the *Euler characteristics* of $\mathcal{M}_{\mathrm{st}},\mathcal{M}_{\mathrm{ss}},\mathcal{M}_{\mathrm{st}}^{\mathrm{b}},\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(I,\preceq,\kappa,\theta)$.

Universal identities between the invariants

Let $\trianglelefteq, \preceq$ be partial orders on I. We say that $\trianglelefteq$ *dominates* $\preceq$ if $i \preceq j$ implies $i \trianglelefteq j$. Here is how to transform between *best* and *non-best* invariants.

Theorem 2. *Let $\mathcal{A}$ satisfy some assumptions, and Z be a permissible slope function on $\mathcal{A}$ with phase θ. Then for all $\mathcal{A}$-data $(I, \trianglelefteq, \kappa)$ we have*

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} I^{\mathrm{b}}_{\mathrm{st}}(I, \preceq, \kappa, \theta) = I_{\mathrm{st}}(I, \trianglelefteq, \kappa, \theta),$$

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} I^{\mathrm{b}}_{\mathrm{ss}}(I, \preceq, \kappa, \theta) = I_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \theta),$$

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} n(I, \preceq, \trianglelefteq) I_{\mathrm{st}}(I, \preceq, \kappa, \theta) = I^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \theta),$$

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} n(I, \preceq, \trianglelefteq) I_{\mathrm{ss}}(I, \preceq, \kappa, \theta) = I^{\mathrm{b}}_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \theta).$$

Here $n(I, \preceq, \trianglelefteq)$ are explicitly defined constants. The proof of these uses $\chi(\mathbb{K}^l) = 1$.

Theorem 3. *Let $\mathcal{A}$ satisfy some assumptions, and Z be a permissible slope function on $\mathcal{A}$ with phase θ. Then for all $\mathcal{A}$-data $(K,\trianglelefteq,\mu)$ we have*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \phi:I\to K\text{ is surjective},\\ i\preceq j\text{ implies }\phi(i)\trianglelefteq\phi(j),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K,\\ \theta\circ\mu\circ\phi\equiv\theta\circ\kappa:I\to(0,1]}} I^{\mathrm{b}}_{\mathrm{st}}(I,\preceq,\kappa,\theta) = I_{\mathrm{ss}}(K,\trianglelefteq,\mu,\theta),$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ (I,\preceq,K,\phi)\text{ is allowable},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K,\\ \theta\circ\mu\circ\phi\equiv\theta\circ\kappa:I\to(0,1]}} I^{\mathrm{b}}_{\mathrm{st}}(I,\preceq,\kappa,\theta) = I^{\mathrm{b}}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\theta),$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ (I,\preceq,K,\phi)\text{ is allowable},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K,\\ \theta\circ\mu\circ\phi\equiv\theta\circ\kappa:I\to(0,1]}} N(I,\preceq,K,\phi)\cdot I^{\mathrm{b}}_{\mathrm{ss}}(I,\preceq,\kappa,\theta) = I^{\mathrm{b}}_{\mathrm{st}}(K,\trianglelefteq,\mu,\theta).$$

Here $N(I,\preceq,K,\phi)$ are explicitly defined constants. Theorems 2 and 3 mean that each of the four families $I_{\mathrm{st}},I_{\mathrm{ss}},I^{\mathrm{b}}_{\mathrm{st}},I^{\mathrm{b}}_{\mathrm{ss}}(\dots)$ determines the other three.

Here are the *transformation laws* between the invariants for two different slope functions.

Theorem 4. *Let $\mathcal{A}$ satisfy some assumptions, and $Z, \tilde{Z}$ be permissible slope functions on $\mathcal{A}$ with phases $\theta, \tilde{\theta}$. Then for all $\mathcal{A}$-data $(K, \trianglelefteq, \mu)$ we have*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preceq,K,\phi) \text{ is allowable},\\ \trianglelefteq = \mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K}} T^{\mathrm{b}}_{\mathrm{st}}(I,\preceq,\kappa,K,\phi,\theta,\tilde{\theta})\cdot I^{\mathrm{b}}_{\mathrm{st}}(I,\preceq,\kappa,\theta) = I^{\mathrm{b}}_{\mathrm{st}}(K,\trianglelefteq,\mu,\tilde{\theta}),$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preceq,K,\phi) \text{ is allowable},\\ \trianglelefteq = \mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K}} T^{\mathrm{b}}_{\mathrm{ss}}(I,\preceq,\kappa,K,\phi,\theta,\tilde{\theta})\cdot I^{\mathrm{b}}_{\mathrm{ss}}(I,\preceq,\kappa,\theta) = I^{\mathrm{b}}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tilde{\theta}).$$

There are only finitely many terms in each sum with $T^{\mathrm{b}}_{\mathrm{st}}, T^{\mathrm{b}}_{\mathrm{ss}}(\ldots)$ and $I^{\mathrm{b}}_{\mathrm{st}}, I^{\mathrm{b}}_{\mathrm{ss}}(\ldots)$ both nonzero.

Here $T^{\mathrm{b}}_{\mathrm{st}}, T^{\mathrm{b}}_{\mathrm{ss}}(\ldots)$ are explicitly defined constants. So, if we know the invariants for one slope function Z, we can compute them for all slope functions $\tilde{Z}$.

Conclusions

Moduli spaces of configurations are the right tools to use to understand how moduli spaces of θ-(semi)stable sheaves, etc., change as we vary the stability condition Z, θ.

For instance, we can compute how the Euler characteristics of moduli spaces of θ-(semi)stable sheaves change using Theorem 4.

There is also an extension of all this to *triangulated categories*, using Bridgeland's notion of stability. This extension should be important in *Homological Mirror Symmetry* of Calabi–Yau m-folds, and Π-*stability* and *branes* in String Theory.

Applied to the derived category of coherent sheaves, the invariants $I_{\mathrm{st}}, \ldots$ are an extension of the Gromov–Witten invariants, I think.

Applied to the derived category of the Fukaya category, the invariants count configurations of *special Lagrangian m-folds*. This is how I started working on all this.