Kähler manifold

(X, g, ω)

Lagrangian submfd

$L \subset X$, $\omega|_L = 0$

Kähler-Einstein mfd.

$Ric_g = c\,g$

minimal Lagrangian submanifold

Calabi-Yau mfd.

$(c=0)$, $Ric_g = 0$

special Lagrangian submanifold

Q: Where to find natural minimal Lag submfds in Kähler-Einstein mfds?

A: Represent vanishing cycles of degeneration by minimal Lag submfds.

General framework (math):

(Alg geo): Degeneration family of cpx mfds.

$$\begin{array}{ccc} X_t & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \{t\} & \hookrightarrow & D \end{array}$$

X_t — smooth $(t \neq 0)$, X_0 — Singular.

topological change —— vanishing cycles.

<u>Polarization</u> !

(Diff geo): Degen of Kähler mfds. (Gromov-Hausdorff conv.)

A family of Kähler mfds "Gromov-Hausdorff limit"

$$(X_t, g_t) \xrightarrow{GH} \text{"}(X_0, g_0)\text{"}$$

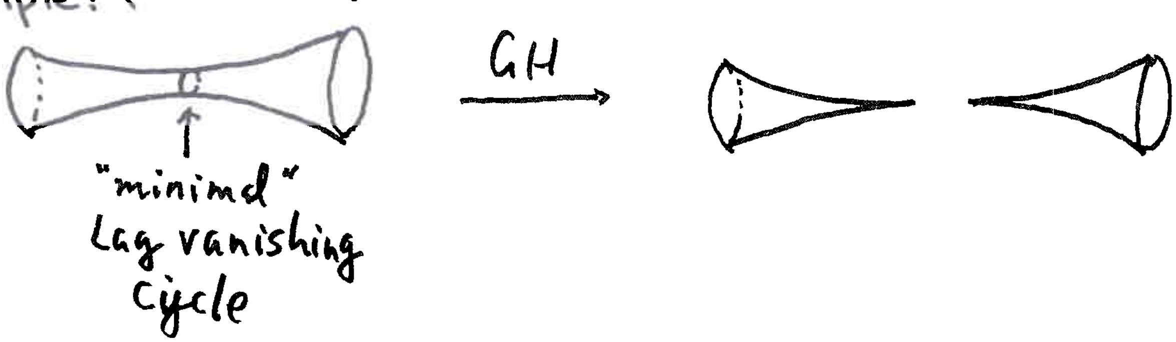
(Sym geo): Symplectic morphism

$$\varphi_t : (X_t, \omega_{g_t}) \longrightarrow (X_0, \omega_{g_0})$$

Vanishing cycles rep by "minimal" Lag submfds.

(more general: coisotropic structures)

Example: (nodal degen).



Example 2: (non-alg degen)

CY: $X_t \xrightarrow{\quad} $ "Large cpx limit" $\cong S^3$

$\uparrow$

Convergence (Cheeger-Gromov)

SYZ Conj: Vanishing cycle rep by special Lag torus that form fibration over S^3.

Related works:

R - Lag fibration, Symplectic SYZ.

M. Gross: Topological mirror.

R. - (math. DG/0309450)

"Generalized special Lag torus fibration for CY hypersurface in toric variety."

(special Lag fibration for almost CY).

In rest of the talk, we will concentrate on:

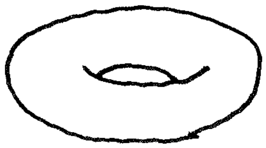
Degen of general type variety

$c_1(X) < 0$, K_X - ample.

math. DG/0311098

Riemann surface ($g=\frac{(d-1)(d-2)}{2}$) Alg curve in $\mathbb{P}^2$ (d.)

Conic curve



Cubic curve

quartic curve

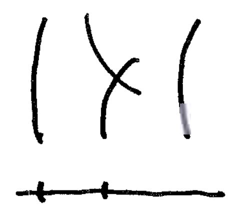
Q: How to see topology from alg structure?

Deligne – Mumford stable degeneration.

$\mathcal{M}_g \subset \overline{\mathcal{M}_g}$

smooth curves of genus g — stable curves of genus g (nodal singularities)

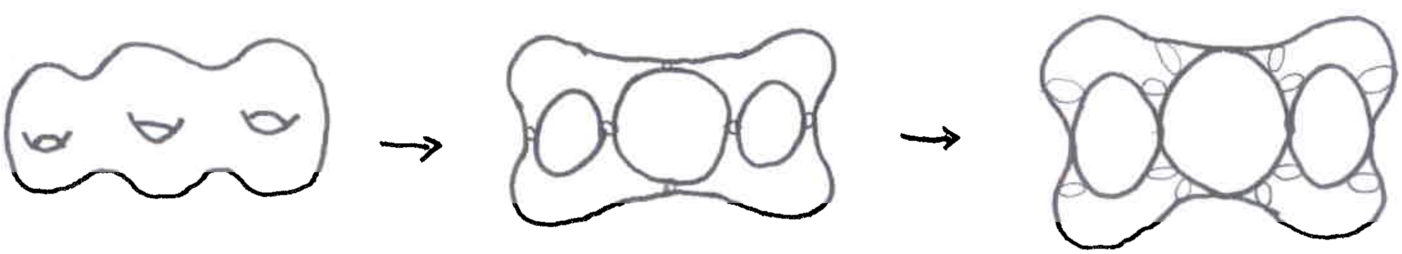
$C_t \to C_0$



$t \to 0$

Max degen:

$C_t \longrightarrow C_0 =$ union of rational curves $(\simeq (\mathbb{CP}^1, 0, 1, \infty))$ (pair of pants)

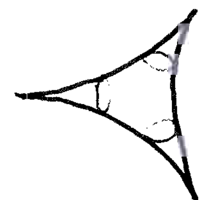


D–M also consider $\mathcal{M}_{g,n}$: $(C_g, x_1, \cdots x_n)$

Example: "the pair of pants" $(\mathbb{CP}^1, 0, 1, \infty)$

Diff geo pt of view: (metric, curvature, geodesic etc.)

$(C_t, g_t) \longrightarrow (C_0 \setminus Sing(C_0), g_0)$



Poincare metric w/ curvature -1.

union of hyperbolic pair of pants

Local model of degen:

hyperbolic cylinder:

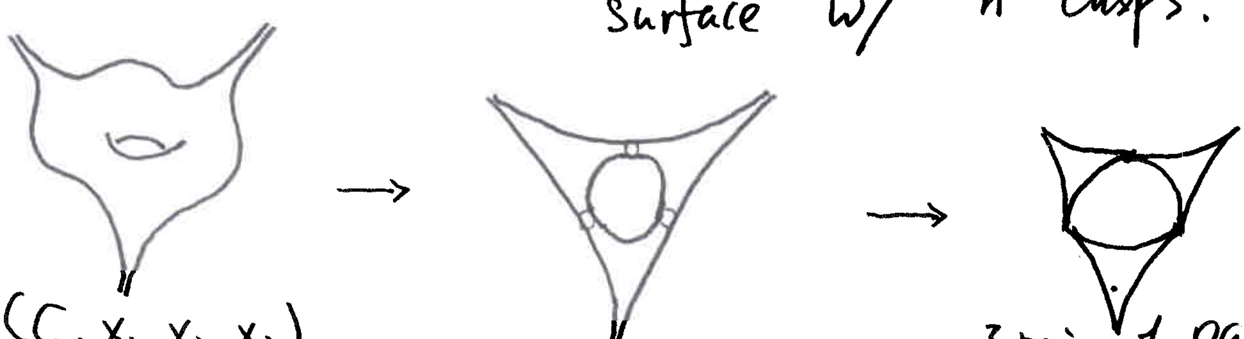
minimal geodesic.

cusp.

$\hookrightarrow$ enable canonical pair of pants decomposition for C_t, $t \neq 0$.

In general:

$(C, x_1 \cdots x_n; g_t)$: Complete hyperbolic Riemann surface w/ n cusps.



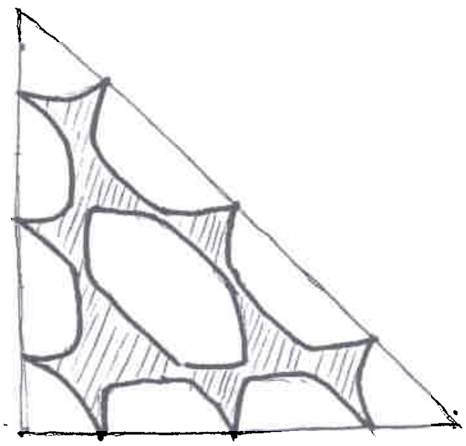
(C, x_1, x_2, x_3)

3 pair of pants

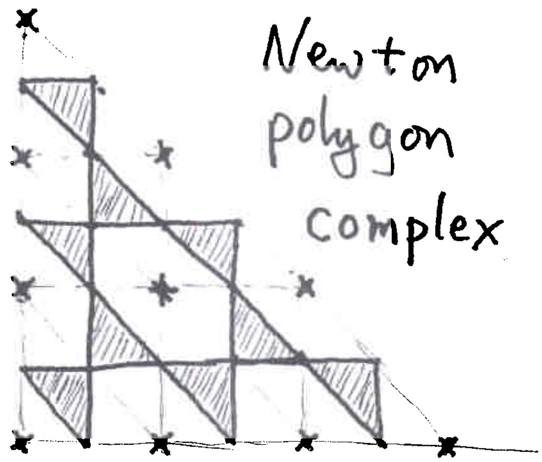
Symplectic (toric) geo pt of view through.
moment map, Amoeba (Viro).

$F_{FS}: \mathbb{CP}^2 \longrightarrow \mathbb{R}^2$

moment map (Fubini-Study): $(z_1, z_2) \longrightarrow \dfrac{(|z_1|^2, |z_2|^2)}{1+|z_1|^2+|z_2|^2}$



$\longrightarrow$



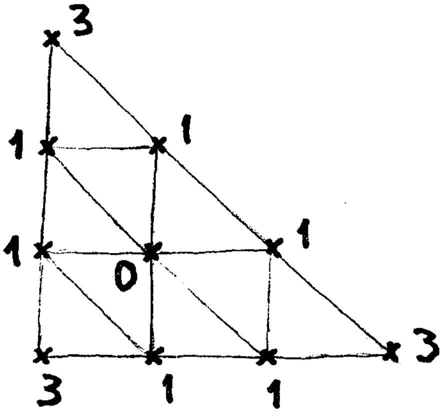
Newton polygon complex

For cubic curve near "large cpx structure limit"

① See torus structure of cubic curve.

② $E - \{9 \text{ pts}\} \longrightarrow$ union of 9 "pair of pants"

E.g.

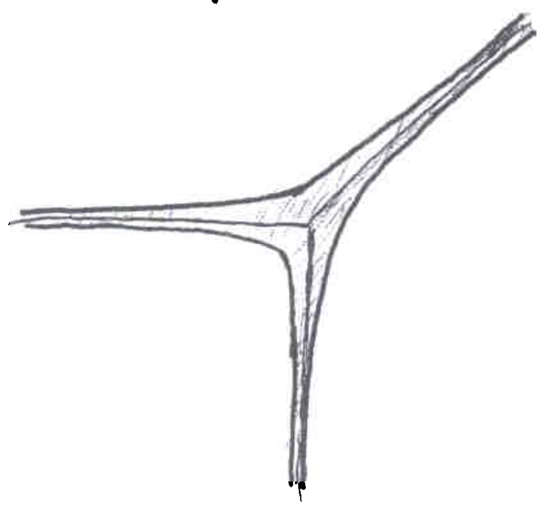


$$P_t(z) = \sum_{m \in \Delta} t^{w_m} z^m$$

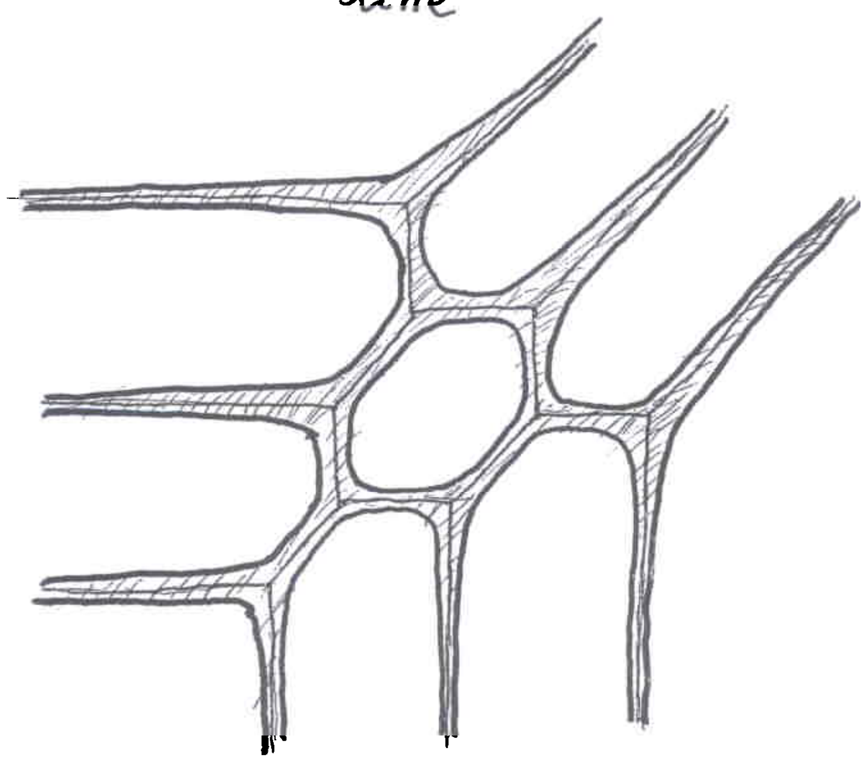
Forsberg - Passare - Tsikh

Moment map images (amoeba) of

$(z_1, z_2) \to (\log|z_1|^2, \log|z_2|^2)$

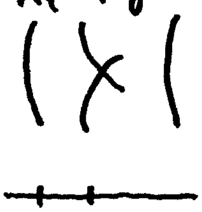


line



cubic curve.

High dim generalization (Diff geo)

C_g ($\subset \mathbb{CP}^2$)	Poincare metric	minimal geodesic
X ($\subset \mathbb{CP}^n$) $c_1(X) < 0$	Kähler-Einstein metric $Ric_g = -g$	minimal Lag submfd $L \subset (X, \omega_g)$
$X_t \to X_0$  $t \to 0$ Degeneration	Existance result: Aubin, Yau (cpt) Cheng-Yau (complete) (unique!)	$\omega_g\|_L = 0$ mean curv.$(L) = 0$ (hard PDE)

Expect:

① $(X_t, g_t^{KE}) \xrightarrow[\text{converge}]{\text{Cheeger-Gromov}} (X_0 \setminus Sing(X_0), g_0^{KE})$

② Existance (Construction) of minimal Lag rep vanishing cycle $\left(\begin{array}{l}\text{in middle dim} \\ \text{homology of } X_t\end{array}\right)$

What is the suitable generalization of Deligne-Mumford stable degeneration?

① Canonical degen. (Global condition)

Dualizing sheaf K_{X_0} : ample line bundle

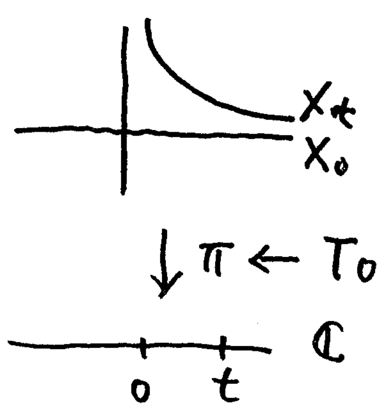
② Toroidal degen. (Singularity condition)

E.g. 1. Normal crossing:

$$X_t = \{z_1 \cdots z_n = t\} \longrightarrow X_0 = \{z_1 \cdots z_n = 0\} \subset \mathbb{C}^n$$

2. Complete intersection of normal crossing.

Local picture:



← Toric map

Total space X: Affine toric variety (convex polyhedron cone)

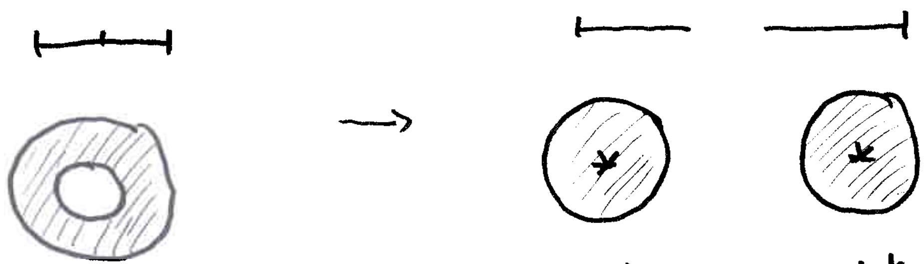
(E.g. $\mathbb{C}^n$: simplicial cone)

$$X_t \hookrightarrow X \setminus X_0 \cong (\mathbb{C}^*)^n$$

↑ subtorus ↑ cpx torus

Analogue of hyperbolic cylinder (local model).

$A = I \times S^1 \subset \mathbb{C}^* = \mathbb{R}_+ \times S^1$



A $\qquad$ D^* $\qquad$ $D^* = \{0 < |z| < 1\}$

For convex polyhedron $\Delta \subset \mathbb{R}^n$

$\Delta^{\mathbb{C}} = \Delta \times (S^1)^n \subset (\mathbb{C}^*)^n \simeq (\mathbb{R}_+)^n \times (S^1)^n$

↑ pseudo-convex domain.

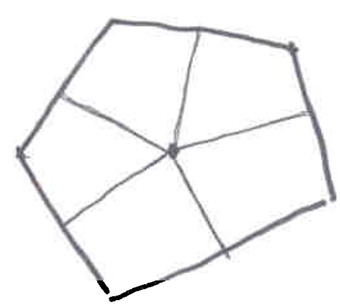
$(\Delta^{\mathbb{C}}_t, g^{KE}_t) \longrightarrow \cup\, ((D^*)^n, g^{KE}_0)$ → product of Poincare metrics.

$g^{KE}_t = \partial\bar\partial \log h_t$ ← Kähler potential.

$h_t : \Delta \to \mathbb{R}$ (convex)

unique min $x \in \Delta$

$\boxed{\pi_t^{-1}(X)}$: unique minimal Lag vanishing torus.



$\pi_t : \Delta^{\mathbb{C}}_t \to \Delta$

General result: $\left(\begin{array}{l}\text{math.DG/0303112,}\\ \text{0303113, 0309177.}\end{array}\right)$

① For toroidal canonical degen $\{X_t\}$

$$(X_t, g_t^{KE}) \xrightarrow{\quad} (X_0 \setminus \mathrm{Sing}(X_0),\ g_0^{KE})$$

in the sense of Cheeger–Gromov

② For each max degen pt $0 \in X_0$

$\exists$ minimal Lag torus family:

$$L_t \subset (X_t, \omega_t^{KE})$$

minimal Lag vanishing torus.

$$L_t \xrightarrow{t \to 0} 0$$

③ (X_t, ω_t^{KE}) possess H-minimal Lag torus fibration near L_t. $(t \neq 0)$

(In Calabi-Yau case:
H-minimal Lag fibration $\cong$ special Lag fibration)

Application: (math.DG/0311098)

Amoeba-like degeneration for hypersurfaces in cpx torus $(\mathbb{C}^*)^n$.

$$(X_t, g_t^{KE}) \xrightarrow[\text{toroidal degen}]{(CG)} (X_0, g_0^{KE}). \quad (X_t: \text{not toric})$$

$$X_0 = \bigcup \text{"pair of pants"}$$

$\uparrow$

(finite quotient of) hyperplane in $(\mathbb{C}^*)^n$

$\{z_1 + \cdots + z_n + 1 = 0\}$

$$\begin{pmatrix} \text{Minimal Lag} \\ \text{Vanishing torus} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{1-simplex in} \\ \text{Newton polyhydron} \\ \text{complex} \end{pmatrix}$$

For "Canonical pair of pants decomp of X_t $(t \neq 0)$"

Need H-minimal coisotropic vanishing cycles that form "fibration system".

Corres: (other simplex in Newton poly. complex.)

For $Y^k \subset (X^{2n}, \omega)$

$\mathrm{Rank}(\omega|_Y) = \min : \begin{cases} k < n, & Y : \text{isotropic submfd} \\ k = n & Y : \text{Lag} \end{cases}$

For $Y^k \subset (X^{2n}, \omega)$

$$\mathrm{Rank}(\omega|_Y) = \min : \begin{cases} k<n, & Y: \text{isotropic submfd} \\ k=n & Y: \text{Lag} \quad '' \\ k>n & Y: \text{Coisotropic} \quad '' \end{cases}$$

Coisotropic submfd:

* Part of mirror symmetry ? (Kapustin, Orlov)
* Formal deformation theory (Oh, Park)

  (Very non-trivial !)

  Complexity: isotropic foliation $\mathcal{F}$ of Y.

Integral coisotropic submfd ($\mathcal{F}$ is a fibration)

Deformation: unobstructed! (Simple !)

(R: math.SG/0312107)

H-minimal coisotropic submfd:

(local) minimum within Hamiltonian def class.

Vanishing cycle $\longrightarrow$ H-minimal integral coisotropic fibration system.

Ingredient: H-minimal integral coisotropic submfd

w/ " " boundary.