TOPOLOGICAL STRINGS AND INTEGRABLE HIERARCHIES

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Recall: 2 types of topological string theories

A-model topological string

Amplitudes in terms of holomorphic maps $\phi: \Sigma_g \to X$

$\bar{\partial}\phi = 0$

+ open version with boundaries on Lag. submanifolds

A-model top. string:

studied intensively in both physics and mathematics

E.g. recently: exact to all genera amplitudes when X is toric CY 3-fold

Based on a duality symmetry relating open and closed A-model strings.

B-model topological string on X

X is Kähler
+ $c_1(X)=0$

B-model is a point-particle theory.

= quantum theory
of variations of
complex structures on X

Kodaira-Spencer
theory of gravity

→ studied comparatively little.

In appropriate setting, topological B-model on Calabi-Yau $d_{\mathbb{C}}=3$ is in a natural way related to theory of integrable hierarchies and Hermitian matrix models.

Integrable hierarchies are known to have deep relation with A-model topological string

The results I'll present today (together with mirror symmetry) shed some light on this.

Today, I'll consider the case when X can be written as a hypersurface

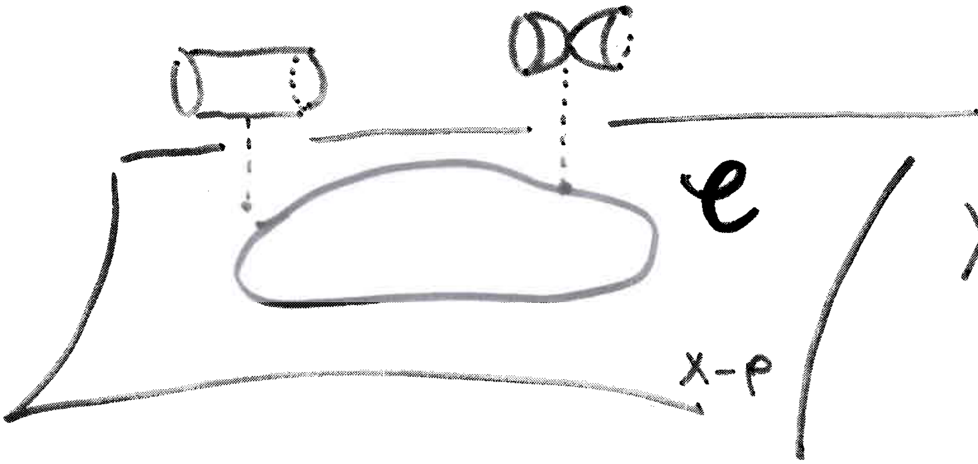
Calabi-Yau 3-fold

$$X: \quad zw = \mathcal{H}(x,p)$$

Hol. 3-form: $\Omega = \frac{dz}{z}\, dx\, dp$.

In this case KS theory simplifies significantly and can be solved exactly to all genera.

X can be viewed as



$X: \; zw = H(x,p)$

$$\mathcal{C}: \quad \underline{\mathcal{H}(x,p) = 0}$$

$$\underline{\int \Omega} = \int dp\, dx = \underline{\int p\, dx}$$

Deformations of complex structure:

$$\underline{\delta\Omega \to \delta p\, dx}$$

$$\bar{\partial}\, \delta p = 0 \Rightarrow \underline{\delta p\, dx = \partial \varphi(x)}$$

"chiral" scalar field φ , $\bar{\partial}_x \varphi = 0$

Where does this get us?

Riemann surface $\mathcal{C}$ typically has some # M of asymptotic regions $\to$ M boundaries.

KS theory on X associates to $\mathcal{C}$ a state

$$|V\rangle \in \mathcal{H}^{\otimes M}$$

$\mathcal{H}$ = Hilbert space of a boson on S^1

At every puncture : (x_i, p_i), $x_i \to \infty$

$$\delta p_i(x) = \partial_{x_i} \varphi_i(x) = \sum_{n>0} \alpha_{-n}^i x_i^{n-1} + \sum_{n>0} \alpha_{+n}^i x_i^{-n-1}$$

$$[\alpha_n^i, \alpha_m^j] = g_s^2 m \, \delta_{m+n,0} \, \delta^{ij}$$

$$\mathcal{H} = \left\{ \prod_{n>0} (\alpha_{-n}^i)^{k_n} |0\rangle \right\}, \quad \alpha_{n>0}^i |0\rangle = 0$$

It is useful to express $|V\rangle$ in a coherent state basis

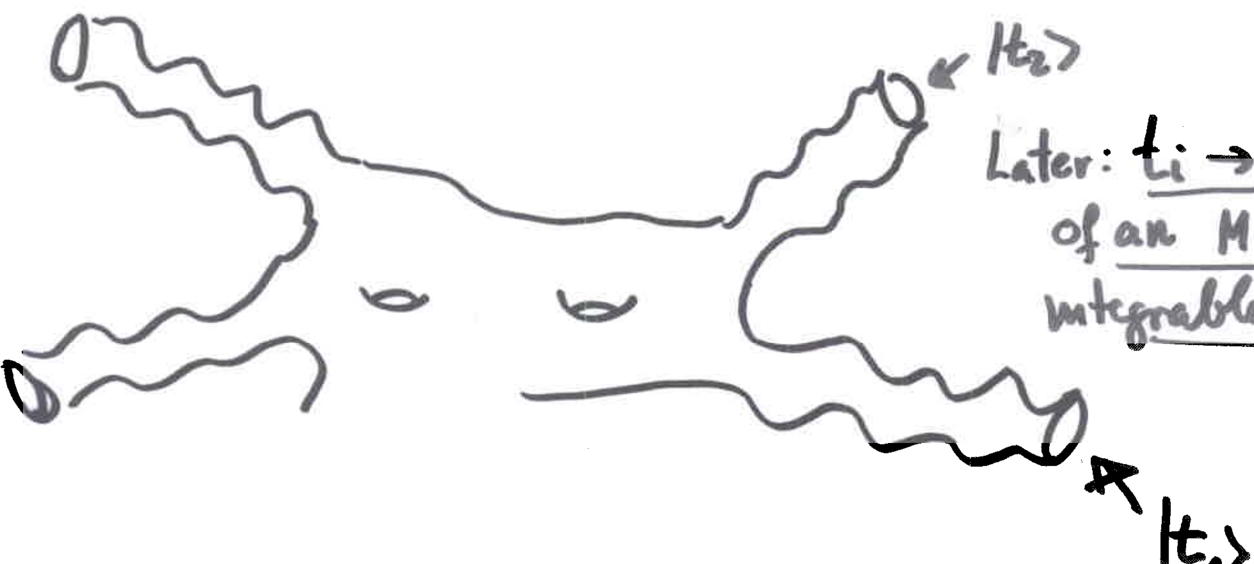
$$|t_i\rangle = e^{\sum_{n>0} \frac{t^i_n}{n} \alpha^i_{-n}} |0\rangle_i$$

$$\langle t_1, \ldots, t_M | V\rangle = e^{F(t_i;\, g_s)}$$

$$F(t_i; g_s) = \sum_g F_g(t_i)\, g_s^{2g-2}$$

$$\langle t_1, \ldots, t_M | \partial\varphi_i | V\rangle = \sum_{n>0} \frac{t_{i,n}}{4} x_i^{n-1} + n\frac{\partial}{\partial t^i_n} F\, x_i^{-n-1}$$

$\Rightarrow$ <u>t's</u> can be viewed as <u>non-normalizable deformations</u> of complex structure.



Later: <u>t_i</u> $\to$ <u>times</u> of <u>an M-component integrable hierarchy</u>.

What is $|V\rangle$?

$|V\rangle$ is highly constrained: KS theory has a huge symmetry group of symplectic diffeomorphisms preserving $dx \wedge dp$ and

$$\mathcal{H}(x,p) = 0$$

In fact: "Turning off" $\mathcal{H}(x,p) \to 0$

(by putting $\partial\varphi_{qu} \to \partial\varphi = p(x) + \partial\varphi_{qu}(x)$)

$\Rightarrow$ full $W_{1+\infty}$ algebra is realized

$$f(x,p): \quad \delta x = \frac{\partial}{\partial x} f(x,p)$$

$$\delta p = -\frac{\partial}{\partial p} f(x,p)$$

$$x^m p^n \to W_m^{n+1} = \oint x^m \frac{(\partial\varphi)^{n+1}}{n+1}$$

Globally, constrained by the deformations matching up:

Schematically: $\sum_i \langle \oint_{P_i} W \rangle = 0$

This crucially depends on how φ transforms ...

Recall:

B model admits boundaries on Holomorphic submanifolds of X

D-branes

KS on X has a class of D-branes which are related to φ by Bose-fermi correspondence

$$\psi(x) = e^{\varphi(x)/g_s}$$

inserts a D-brane at x

Moreover, on $\psi(x)$, p and x act as cannonically conjugate operators $[p, x] = g_s$

$\Rightarrow$ $\psi(x) = e^{\varphi(x)/g_s}$ transforms like a wave-function!

In terms of fermions $W_{1+\infty}$ algebra is generated by fermion bilinears:

$$x^m p^n \to \underline{W_m^{n+1}} = \oint \underline{\psi^\dagger(x)\, x^m \left(g_s \frac{\partial}{\partial x}\right)^n} \psi(x)$$

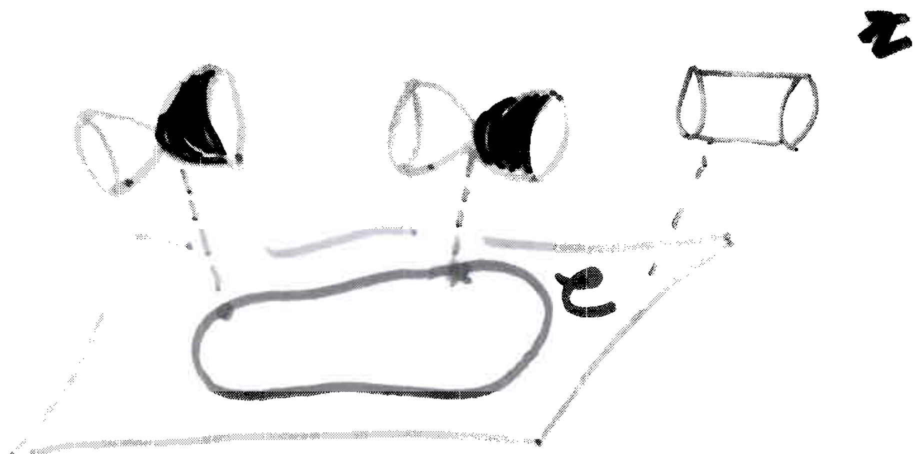
$W_{1+\infty}$ Symmetries:

$$\left\langle \oint_{P_i} \underline{\psi^\dagger(x_i)\, x_i^m\, p_i^n\, \psi(x_i)} \right\rangle = - \sum_{i \neq j} \left\langle \oint_{P_j} \psi^\dagger(x_j)\, \underline{x_i^m(x_j)\, p_i^n(x_j)\, \psi(x_j)}\right.$$

FIXES $|V\rangle$ COMPLETELY!

D-branes on X $\quad X: zw = H(x,p)$

$w=0$, $\underline{H(x,p)}=0$: $\mathcal{C}$



$$S_{ce} = \frac{1}{g_s}\int dz\, d\bar{z}\; p(z,\bar{z})\, \partial_{\bar{z}} x(z,\bar{z})$$

Reduction to zero modes

$\Rightarrow \quad [x, p] = g_s$

$\Rightarrow \quad S_{ce} = \int^{x} \frac{p(x)\,dx}{g_s}$

$$\Rightarrow \quad \langle \psi(x) \rangle = e^{\int^{x} \frac{p\,dx}{g_s}} + \cdots$$

background value of $\psi(x)/g_s$

Recall

$$\langle \Psi(x) \rangle = e^{\frac{1}{g_s}\int^x p\,dx + \dots}$$

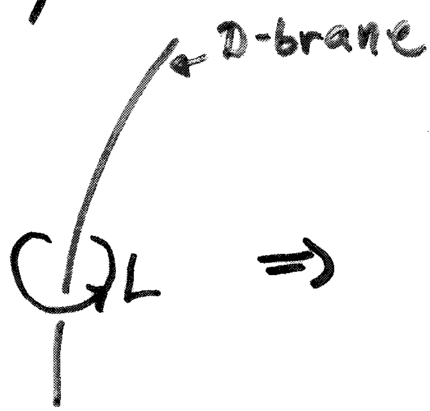
WKB approximation to

$$\mathcal{H}(x,p)\langle \Psi(x)\rangle = 0$$

$$p = g_s \frac{\partial}{\partial x}$$

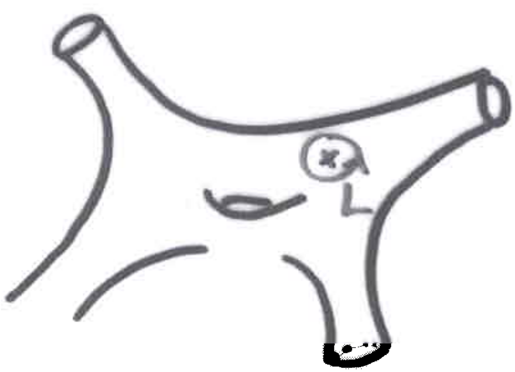
$\Rightarrow$ Riemann surface is the Hamiltonian!

The deep reason for why D-brane is a fermion is that B-branes deform B-model geometry:



$$\Rightarrow \quad \oint_L \Omega = g_s \neq 0$$

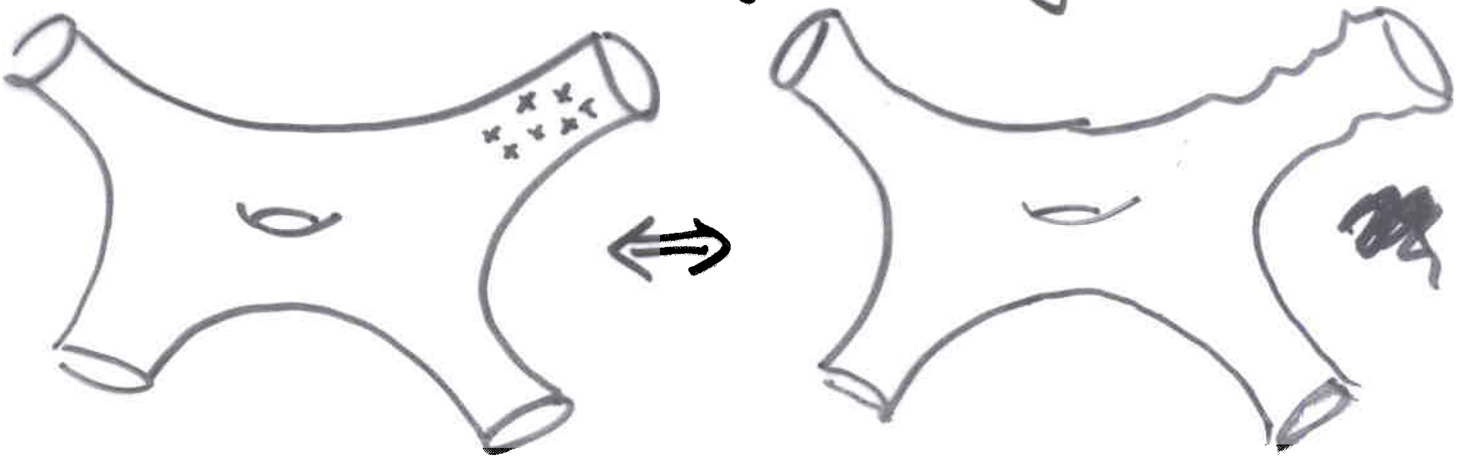
D-brane fixed to $\mathcal{C}$:



$$\oint_L \partial\varphi = g_s$$

$$\Rightarrow \quad \psi(x_L) = e^{\varphi(x)/g_s}$$

Open-Closed string duality:



$$\psi(x_1)\cdots\psi(x_N) = \prod_{i<j}(x_i - x_j)\, e^{\sum_i \varphi(x_i)/g_s}$$

$$\langle N|\underline{\psi(x_1)\cdots\psi(x_N)|}V\rangle = \prod_{i<j}(x_i - x_j)\,\langle e^{\sum_i \varphi(x_i)/g_s}|V\rangle$$

$$= \prod_{i<j}(x_i - x_j)\,\underline{\langle t|V\rangle}$$

$$t_n = g_s \sum_i \frac{(x_i)^{-n}}{n} = g_s \operatorname{Tr}\frac{X^{-n}}{n}$$

$Z_{closed}(M)\, Z_{open} = Z_{closed}(M$

$$X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix}$$

3 classes of examples,

in increasing level of complexity, depending on # of infinities

i) (1, m) minimal models + gravity

$$\mathcal{H}(x,p) = x^m + p$$

ii) A-model on $\mathbb{CP}^1$

$$\mathcal{H}(x,p) = p + e^x + e^{-x} q$$

iii) A-model on $\mathbb{C}^3$ with DBranes

→ "topological vertex"

$$\mathcal{H}(x,p) = e^x + e^{-p} + 1$$

$(1,m)$ minimal models have 2 different matrix model descriptions:

1) Kontsevich-type matrix <u>model</u>

2) <u>Double scaling matrix model</u>.

<u>Both</u> naturally arise in the context of B model on Calabi-Yau 3-fold.

Double scalling matrix model: Dijkgraaf, Vafa

Consider compact B-branes wrapping $\mathbb{P}^1$'s on CY3.

$\Rightarrow$ Ordinary Hermitian matrix models

Large N + double-scaling limits $\rightarrow$ theory with closed B-model strings on CY.

$zw = y^2 + (W')^2$

Recall: Hermitian Matrix Model

...en B model:

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int \mathcal{D}X_{N\times N}\, e^{\frac{\mathrm{Tr}\, W_n(X)}{g_s}}$$

$\downarrow$ 't Hooft Large N Limit

Riemann surface

...lanar limit $\rightarrow$

$$y^2 + (W_n'(x))^2 - f_{n-1}(x) = 0$$

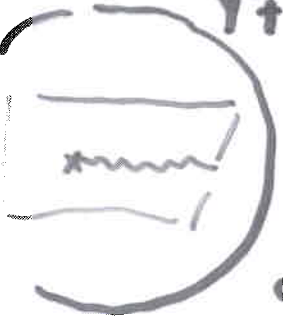
Comes from B-model on

$$zw = y^2 + (W_n'(x))^2 - f_{n-1}(x)$$

...ll Genus + geometric transitions

$$\Rightarrow \; y = \partial\varphi_{cl} + \partial\varphi_{qu}(x) \leftarrow \text{Quantum Kodaira-Spencer field}$$

$y^2+x=0$



Double scaling limits:

matrix model & geometry:

$$y^2+x^{2p+1}+\dots=0$$

$\Rightarrow$ Describes $(2,2p+1)$ bosonic minimal model

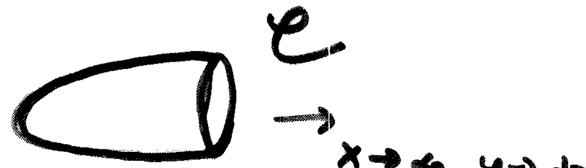
More generally

B model on $wz=y^p+x^q+\dots$ $\xLeftarrow{\text{double scaling}}$ Quiver matrix model $\xRightarrow{\text{double scaling}}$ (p,q) bosonic minimal model

(1,m) topol. minimal model (m ≥ 2)
+ topol. gravity

(1,2) – pure gravity

Calabi-Yau:
$wz = p + x^m$



$p + x^m = 0$

$$p = \partial\varphi(x) = x^m + \sum_{n>0} n\, t_n x^n + \partial_n F\, x^{-n}$$

↑ B branes in $\underline{x}$ coordinate

Ward identity

$$\langle \oint \psi^\dagger x^n \psi \rangle = 0$$
$$= \langle \oint x^n \partial\varphi(x) \rangle$$

$\Rightarrow \quad \partial_n F = 0$

<u>Classical</u> B-brane <u>action is exact</u>

$$S = \int^x p\, dx = \frac{x^{m+1}}{m+1}$$

Turning on desendents corresponds to putting B-branes at fixed p.

$$H = \left(g_s \frac{\partial}{\partial p}\right)^m + p$$

$H \to$ Lax Operator of KdV

$$H\, \Psi(p) = 0$$

$\langle \Psi(p) \rangle$ - Baker - Akhiezer function

$$\Psi(p) = \int dx\, e^{\frac{xp}{g_s}}\, \psi(x)$$

$$\langle N | \psi(x_1) \cdots \psi(x_N) | V \rangle = \prod_{i<j} (x_i - x_j)\, e^{\sum_i \frac{x_i^{m+1}}{(m+1) g_s}}$$

$$\langle N | \Psi(p_1) \cdots \Psi(p_N) | V \rangle = \int \mathcal{D}X_{N \times N}\, e^{\mathrm{Tr}\frac{Xp}{g_s} + \mathrm{Tr}\frac{X^{m+1}}{m+1}/g_s}$$

$\Big\}$ N branes at fixed p

Kontsevich matrix model for (1, m) top. minimal model!

Mirror of $\mathbb{CP}^1$

Landau - Ginsburg $\quad W(u) = e^u + q e^{-u}$

$q = e^{-t}$ ↖ size of $\mathbb{P}^1$

$\Leftrightarrow$ CY: $\quad \underline{wz = H(p,u)}$

$$\underline{\mathcal{H}(p,u) = p + e^u + q e^{-u}}$$

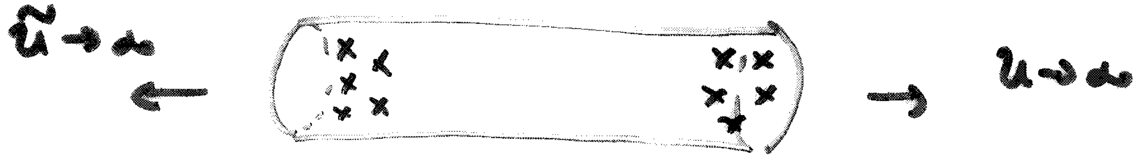
With $\quad u = g_s \frac{\partial}{\partial p} \rightarrow$ Lax operator of $\mathbb{P}^1$

$$\partial\varphi(p) = \partial\varphi_{cl} + \Sigma\, t_n\, p^n + \partial_n F\, p^{-n}$$

$t_n \rightarrow$ descendants of Kahler class

$\Rightarrow$ B-branes in p-patch

The theory is simple in the u-coordinate



$\tilde{u} = t - u$

$\to$ Free boson on a cylinder

$$Z = \langle s | q^{L_0} | \tilde{s} \rangle$$

Partition function of B-branes in u-patch.

From u-patch:

$$Z = \langle s | q^{L_0} | \tilde{s} \rangle$$

to p-patch:

$$\Downarrow$$

$$Z_{\mathbb{P}^1} = \langle s | W^{-1} q^{L_0} W^{+} | \tilde{s} \rangle$$

Okounkov & Pandharipande

W - "moves" branes from u to p

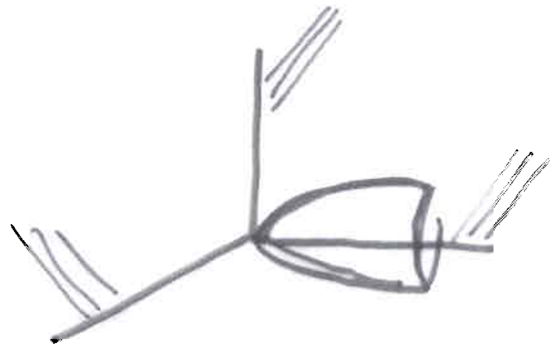
Conjecture:

$$Z_{\mathbb{P}^1} = \int \mathcal{D}_H u \; e^{\mathrm{Tr} \frac{pu}{g_s}} \; e^{\mathrm{Tr} \frac{e^{u} - q e^{-u}}{g_s}}$$

$\mathcal{D}_H u$ - Haar measure on tangent space to $U(N)$ group manifold

$$t_{n,w} = \frac{g_s}{n} \mathrm{Tr}\, p^{-n}$$

Topological Vertex

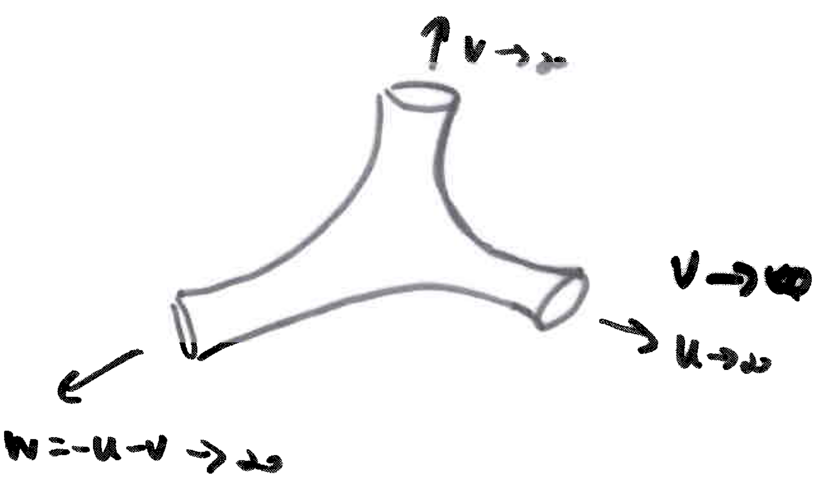


$\mathbb{C}^3$

Relation to Chern-Simons, Classical Chrystals, Quantum Foam ...

Mirror of $\mathbb{C}^3$

$$wz = e^{-u} + e^{v} + 1$$



$\mathbb{Z}_3$ symmetry generated by $u \to u + nv$

$$ST : \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} v \\ w \end{pmatrix} \qquad \tau = u$$

$$(ST)^3 = 1$$

* Ward identities

$$W_n^{m+1} = \oint \psi^\dagger(u)\, e^{nu} e^{mv}\, \psi(u)$$

$$\sum_{i=1}^{3} \langle 0| \oint \psi^\dagger W \psi \,|V\rangle = 0$$

$\Rightarrow$ Fixes $|V\rangle$ completely

*

$$\underline{\mathcal{H}(u,v) = e^{-u} + e^{g_s \frac{\partial}{\partial u}}} - 1$$

$$\mathcal{H}(u,v)\, \Psi(u) = 0$$

$\longrightarrow$ $$\Psi(u) = \exp\Big(\sum_{n>0} \frac{e^{-nu}}{n[n]} \Big)$$

* $|V\rangle$ is bilinear in fermions

$$|V\rangle = e^{\sum a_{mn}^{ij} \psi_i^n \psi_j^{m+}} |0\rangle$$