M. Hutchings: Embedded Contact Homology of T^3. (1)

1) E. C. H.

2) COMBINATORIAL CALCULATION FOR T^3

3) WHY IT'S INTERESTING

1) Y^3 CLOSED ORIENTED
ξ A CONTACT STR, $\xi = \ker \alpha$, $\alpha \wedge d\alpha > 0$
$h \in H_1(Y)$
Reeb VECTOR FIELD R: $\alpha(R) = 1$ $R \lrcorner\, d\alpha = 0$

ALMOST $\mathbb{C}$ STRUCTURE ON $\mathbb{R} \times Y$ S.T.
- J IS $\mathbb{R}$-INVT
- $J: \xi \circlearrowleft$
- $J(\partial_s) = R$

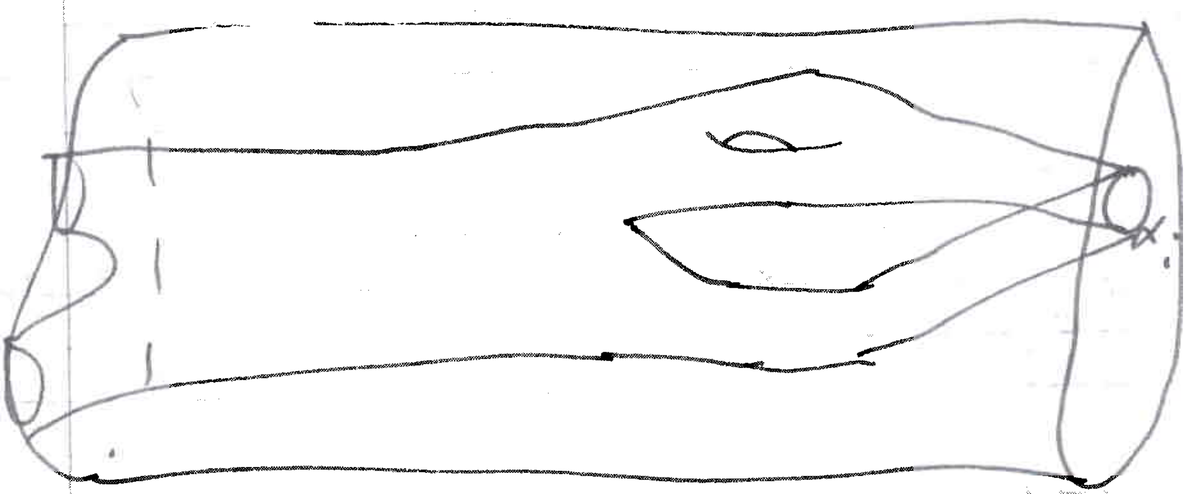
LIKE S.F.T., BUT ONLY FOR EMBEDDED CURVES & ONLY IN 3 DIMS.

DEFINE A CHAIN COMPLEX (C_*, ∂) OVER $\mathbb{Z}$

GENERATORS: FINITE SETS $\{(\alpha_i, m_i)\}$ S.T.
- α_i's ARE DISTINCT, EMBEDDED REEB ORBITS
- THE MULTIPLICITIES $m_i \in \{1, 2, \dots\}$
- $\sum m_i \alpha_i = h$
- $m_i = 1$ IF α_i HYPERBOLIC, i.e. if D(RETURN MAP) GIVES A SYMPLECTOMORPHISM OF THE PLANE $\perp$ REEB ORBIT; THIS MAP ~~SHOULD~~ HAS REAL EIGENVALUES

b) HUTCHINGS

COEFFICIENT $\langle \delta\alpha, \beta \rangle$, $\beta = \{(\beta_j, n_j)\}$ IS A SIGNED COUNT OF EMBEDDED J-HOLO CURVES C IN $\mathbb{R}\times Y$ ST C HAS POSITIVE ENDS AT α_i W/ TOTAL MULT. m_i & NEGATIVE ENDS AT β_j W/ TOTAL MULT. n_j



$\mathbb{R}$ COORD NEAR $+\infty$ ⤳ fixed $\mathbb{R}$ SLICES ARE BRAIDS NEAR α_i's W/ m_i STRANDS (& SIMILARLY FOR $-\infty$).

δ ONLY COUNTS CURVES OF MAXIMAL EXP DIM.

$$\delta\alpha = \sum_{\beta} \sum_{Z \text{ st } I(\alpha,\beta,Z)=1} \#\mathcal{M}(\alpha,\beta,Z)/\mathbb{R} \cdot \beta$$

expected max dim of curves w/

REL HOMOLOGY CLASS OF SURFACES CONNECTING α TO β

THIS SHOULD BE S-W FLOER HOMOLOGY ~~[illegible]~~ THEORY (SPECIFICALLY HF^+)

<u>CONJ</u> $H_*(C, \partial)$ DEPENDS ONLY ON Y, ξ, h.

M HUTCHINGS ③

THE "U" ACTION CORRESPONDS TO CURVES THROUGH A FIXED POINT.

EXAMPLE $Y = T^3 = (\mathbb{R}/2\pi\mathbb{Z})_\theta \times (\mathbb{R}^2/\mathbb{Z}^2)_{x,y}$, $h = 0$

Contact form: $\alpha = \cos\theta\, dx + \sin\theta\, dy$

Reeb V.F.: $R = \cos\theta\, \partial_x + \sin\theta\, \partial_y$

There is a circle of Reeb orbits $\forall\, \theta$ s.t. $\tan\theta \in \mathbb{Q} \cup \{\infty\}$.
PERTURB THIS TO 2 ORBITS, ONE ELLIPTIC, ONE HYPERBOLIC.
[TECHNICALITY: CAN'T DO ALL AT ONCE $\forall\theta$, SO USE A DIRECT LIMIT]

GENERATOR: CONVEX POLYGON IN $\mathbb{R}^2$ w/ CORNERS ON $\mathbb{Z}^2$, WITH EDGES LABELLED e OR h. (MODULO TRANSLATION).
TAKE COLLECTION OF ORBITS ($\leftrightarrow$ INTEGER VECTORS) IN INCREASING ANGLE ORDER TO GET A POLYGON.
IF AN EDGE IS LABELLED e, THEY'RE ALL ELLIPTIC; IF h, IT'S ALL BUT ONE ELLIPTIC & ONE HYPERBOLIC.
CHOOSE AN ORDERING OF 'h' EDGES, MODULO SIGNED PERMUTATIONS.

$\tilde{C}_*$ - SAME, BUT NOT MOD TRANSLATION OF POLYGONS
- IN FLOER THEORY $\leftrightarrow$ KEEPING TRACK OF HOM. CLASSES OF HOLO CURVES

M Hutchings

INDEX: $I(\alpha) = 2\,\text{Area} + \text{total mult of edges} - \#\text{'}h\text{' edges}$

$= 2\cdot\#(\text{lattice pts enclosed}) - 2 - \#\text{'}h\text{' edges}$

Pick's Thm

Rem. • IDX ≥ 0

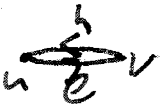
• ~~IDX 0 generators~~

u − empty set of Reeb orbits

$p(u)$

$h(u,v)$

IDX 1 generators

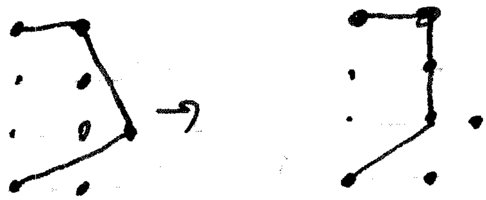


$e(u,v)$

$h(u,v,w)$

Differential: $\delta\alpha$ is the signed sum of ways to 'round a corner', locally losing one 'h'.

ex



Sign:

$+$

Exer $\partial^2 = 0$

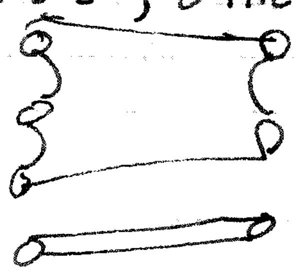
$-$

M HUTCHINGS

$\delta e(u,v) = p(u) - p(v)$

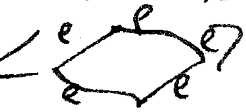
$\delta h(u,v,w) = h(u,v) + h(v,w) + h(w,u)$

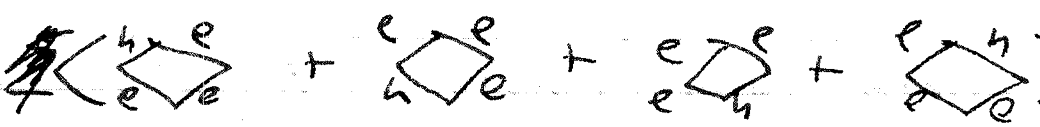
EDGES INVOLVED IN CORNERS → GERMS OF HOLO CURVE; OTHERS → TRIVIAL CYLINDERS



CONSTRAINTS ON ∂ FOR CURVES:
IF $\mathcal{M}(\alpha,\beta) \neq \emptyset$, POLYGON FOR β TO THE LEFT OF THE POLYGON FOR α.
i.e. THE RELATIVE HOMOLOGY CLASS DETERMINES THE RELATIVE POSITION OF THE POLYGONS

HOMOLOGY (W/O MODDING OUT BY TRANSLATION):
$\mathbb{Z}$ IN ALL POSITIVE DEGREES.

$\langle$  $\rangle$ IN EVEN DEGREE

IN ODD: $\langle$  $\rangle$

W/ MODDING OUT, $\mathbb{Z}^3$ IN ALL DEGREES ≥ 0
USING UNIV. COEFF. SPECTRAL SEQUENCE.

THIS AGREES WITH $HF^+(T^3)$ (IN OZSVATH-SZABO THEORY).

M HUTCHINGS:

6

VARIANTS

- $h \neq 0 \in H_1(T^3)$: HOMOLOGY VARIANTS
- CONTACT STR WHERE ~~REEB~~ V.F. ROTATES $n > 1$ TIMES. $\exists$ NEGATIVE IDX GENERATORS. SO FAR, SAME HOMOLOGY AS $n=1$.

PRECISE CONJECTURE

LET (Y^3, ξ) BE ANY CLOSED, ORIENTED CONTACT 3-MFD, $h \in H_1(Y^3)$. THEN

$$ECH(Y, \xi, h) \cong HF^+(Y, \mathfrak{s}_\xi(h))$$

spin$_c$ structure CORR TO h W/ ISOM DETERMINED BY ξ

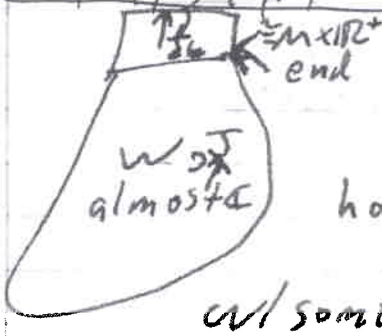
HOPE - 4-D FIELD THEORY BY COUNTING HOLO CURVES AGREEING W/ SW & OZ-SZ THEORY

QUESTIONS: $Aut^+(\mathbb{Z}^2)$ & CHAIN COMPLEX $\Leftrightarrow$ AN AUT OF CONTACT STRUCTURE

WHAT ABOUT A THEORY W/ $\leq k$ SINGULARITIES? HOW WOULD IT RELATE TO SYMPLECTIC FIELD THEORY.

SOME WORK BY M-L YAU, F BOURGEOIS

YASHA: SUBCRITICAL SYMPLECTIC MANIFOLDS (1)



ALMOST $\mathbb{C}$-STR IS $\mathbb{R}$-INVT AT END

FORCES $J\frac{\partial}{\partial t} = R \in TM$

holo curve: $S_g \setminus \{x_1,\dots,x_n\} \to (W, J)$
a holo proper map

W/ SOME ASSUMPTIONS, FORCES CURVE TO BE CYLINDRICAL AT ENDS.

QUOTIENT MODULI SPACES BY $\mathbb{R}$-ACTN

MANY POSSIBLE CONDITION ON ENDS OF W:

EXAMPLE: CONTACT TYPE – COMPACT BOUNDARY W/ LIOUVILLE V.F., & CAN GIVE SYMPLECTIZATION ~~...~~ AS CYLINDER

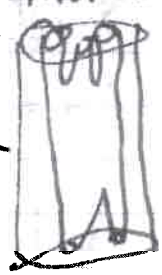
R WILL BE HAMILTONIAN V.F. HERE.

Floer Theory: $f: (M, \omega) \to (M, \omega) \rightsquigarrow$ ~~M~~$_f$ mapping torus comes with a canonical VF on it

$V_f \times \mathbb{R}$

Contact MFD: M^{2n-1}, $\xi = \ker\alpha$, $R_\alpha = \ker d\alpha$

$M\times\mathbb{R}$



$\rightsquigarrow$ Gromov-Witten POTENTIAL

$H(p_i, q_i, t, \hbar)$

$t \in H^* M$

THINK OF AS SYMBOL OF A DIFF. OPERATOR e.g. $p = \hbar\frac{\partial}{\partial q}$

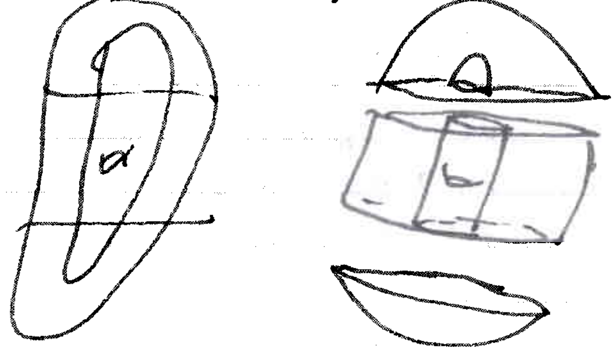
$H \circ H = 0 \rightsquigarrow$ CONTACT HOMOLOGY-INVT OF CONTACT MFLD.

NO MAXIMAL PRINCIPLE TO PREVENT ~~A~~ SPLITTING,

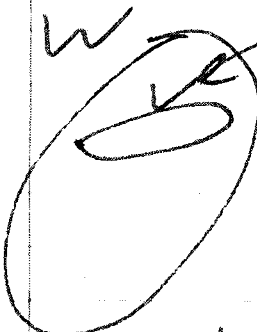
SO CYLINDERS ALONE $\not\Rightarrow \partial^2 = 0$, BUT CAN BE CLEVERER & GET A CYLINDRICAL CONTACT HOMOLOGY.

Yasha Eliashberg

Stretching necks, stretching cylinders



W



hyperplane section (possibly à la Donaldson)

subnbhd

$\hookrightarrow \Sigma = \partial U(V)$

V

$W \setminus V$ affine mfld admits exhausting plurisubharmonic fn φ

⇝ Kähler form $dd^c\varphi$

⇝ gradient v.f. X_φ

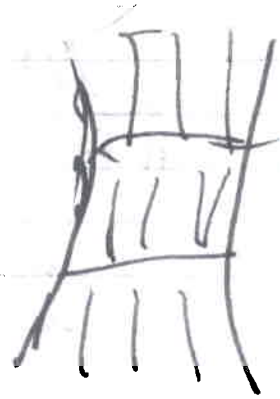
s.t. $L_{X_\varphi}\omega_\varphi = \omega_\varphi$ — Liouville v.f.

Flow $((X_\varphi)^t)^*\omega_\varphi = e^t\omega_\varphi$ — expanding

skeleton

$W \setminus V$ subcritical if $\dim(K) < \frac{1}{2}\dim(W)$.

$W = W_V \cup W_{middle}$



$\Sigma \times \mathbb{R}$

Yasha

(B)

<u>Thm</u> Any subcritical manifold is uniruled - $\forall$ cplx structure, $\exists$ holo sphere through each point.

<u>Hypotheses to be certain</u>

W monotone.

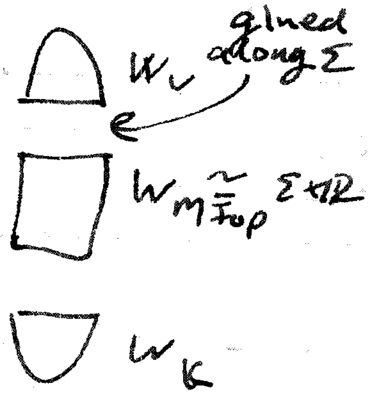
$\exists\ \alpha \in H^1(W)$ $\qquad \alpha > n$

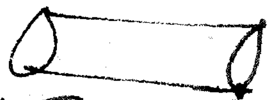
$\exists\ A \in H_2(W)$

$\langle \alpha, \mu \rangle_A \neq 0$

μ - generator of $H^{2n}(W)$

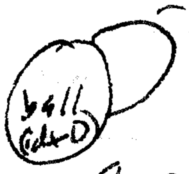
<u>Pf:</u>





$\forall\ \delta \in H_*(V)$

associate sequence $(\ldots \delta_k)$

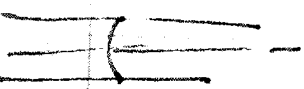


$W \setminus V$ admits self-idxing morse function

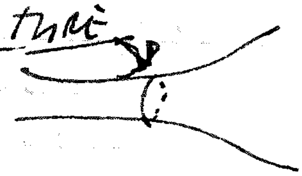
1 PER. ORBIT FOR EACH HANDLE, THESE ORBITS GENERATE CONTACT HOMOL.

MASLOV IDX DEPS ON IDX OF HANDLE

$\mathbb{R} \subset \mathbb{C}^2$, $\mathrm{link}(\mathbb{R}^2) \cong S^2$ - an $\mathbb{R}$ of orbits.



BUT IN HYPERBOLIC PICTURE JUST 1 ORBIT

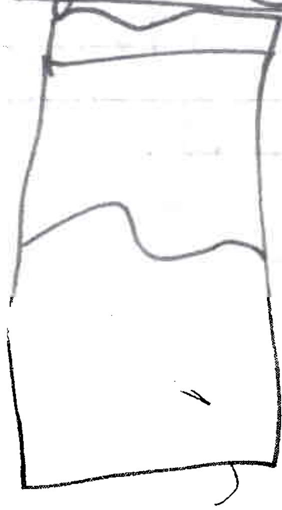


NEEDED SUBCRITICAL.

Yasha Eliashberg

(4)

RELATIVE G-W INVTS:

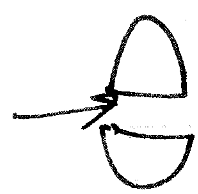


CYLINDER GIVES ISOMORPHISM OF CONTACT HOMOLOGIES

APPLICATION: TOPOLOGY OF LAGRANGIAN ~~FIBERS~~ SUBMANIFOLDS

$W = W_1 \cup T^*L$

$T^*_1 L$



both have cyl. ends over unit cot. bundle

ONE APPLICATION — ∄ EXACT LAGRANGIAN MFD PROVED DIRECTLY (UNLIKE ORIG PF OF GROMOV)

RECALL: $L \subset W$ IS EXACT IF FOR ANY RELATIVE CLASS D, $\int_D \omega = 0$

SPLIT UNIVALED MFD; PIECES HAVE SPHERES THROUGH P(S) BUT SOME HAVE TO TOUCH BOUNDARY, A CONTRADICTION

~~SECOND [illegible] EXACT LAGRANGIANS OF [illegible]~~

ARE HOMOLOGIC

$\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \hookrightarrow \mathbb{R}^{n-1} \times S^1$

1

Y-G Oh A_∞, Landau-Ginzburg, Fano Toric Case

work w/ C-H Cho

§1 A_∞- Algebras
§2 Linear σ-MODEL DESCRIPTION OF TORIC MFLDS
§3 COMPUTATIONS OF A_∞ ALGEBRAS
§4 Rel. to Landau-Ginzburg models

§1 Novikov Map

R: Ground ring: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{R}[q, q^{-1}]$, e.g.

Universal Novikov ring $\left\{ \sum_i a_i T^{\lambda_i} \mid \lambda_i < \lambda_{i+1}, \ \lambda_n \to \infty \right\} = \Lambda_{Nov}$

$\Lambda_{0,Nov} = \{ '' \mid \lambda_i \geq 0 \}$

$\Lambda_{+,Nov} = \{ '' \mid \lambda_i > 0 \}$

$\dfrac{\Lambda_{0,Nov}}{\Lambda_{+,Nov}} \cong R$

$L \subset (M, \omega)$ Lagrangian, J almost-$\mathbb{C}$ structure

$\beta \in \pi_2(M, L)$

$\mathcal{M}_{k+1}(\beta, L) = \{ (w, z_0, \ldots, z_k) \mid \bar\partial_J w = 0;\ z_0, z_1, \ldots, z_k \in \partial D^2 \}$

$\dim \mathcal{M}_{k+1}(\beta; L) = n + \mu(\beta) - 3 + (k+1)$

μ: Maslov idx

Given $[P_1, f_1], \ldots, [P_k, f_k]$, $f_j : P_j \to L$

Form Fiber product: $\underbrace{\mathcal{M}_{k+1}(\beta; L) \times_{ev_+} (P_1 \times \cdots \times P_k)}$

$ev_0 : \quad \downarrow \longrightarrow L$

Y-G Oh

dim (above fiber product) =

$$n+\mu(\beta)-2+\sum_{1}^{k}(\dim P_j+1-n)$$

$$n + \dim = \sum_{1}^{n}(\deg P_j-1)+2-\mu(\beta)$$

$(\deg P_j = n-\dim P_j)$

Define

$$m_{k,\beta}(P_1,\dots,P_k) = [\mathcal{M}_{k+1}(\beta;L)\times(P_1,\dots,P_k),\ ev_0]$$

$$m_k = \sum_{\beta\in\pi_2} m_{k,\beta}\, q^{\beta} \qquad \deg q^{\beta} = \mu(\beta)$$

$$\deg\,(m_k(P_1,\dots,P_k)-1) = \sum_{1}^{n}(\deg P_j - 1)+1$$

Consider shifted complex

$C[1]$: $C[1]^k = C^{k+1}$

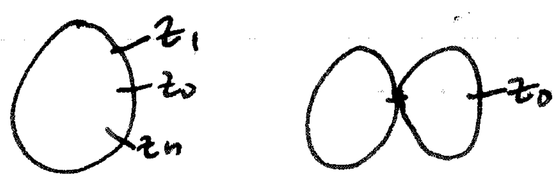
$m_k : C[1]^{\otimes k} \to C[1]$ is a deg. 1 map

Extend m_k to $BC[1] = \bigoplus_{0}^{\infty} C[1]^{\otimes k}$ as a coderivation

$\widehat{m}_k : BC[1] \circlearrowleft$

$$\widehat{m}_k(x_1\otimes\cdots\otimes x_n) = \sum_{1}^{n-k+1}(-1)^{|x_1|'+\dots+|x_{\ell+1}|'}\, x_1\otimes\cdots\otimes x_{\ell-1}\otimes m_k(x_\ell,\dots,x_{\ell+k-1})\otimes\cdots\otimes x_n$$

counts



Y-Gr Oh

Sum $\hat{d} = \sum_0^\infty \widehat{m_k}$

Prop $\hat{d} \circ \hat{d} = 0$ "A_∞-relation"

Ex
- $m_1(m_0(1)) = 0$
- $m_2(m_0(1), x) + (-1)^{|x|'}(x, m_0(1))$

$\quad + m_1 \circ m_1(x) = 0$

If $m_0(1) = 0$, $m_1 \circ m_1 = 0$, & we may define

$$HF(L; \Lambda_{0,nov}) = \ker m_1 / \operatorname{Im} m_1$$

Def When $m_0(1) = 0$, $(C[1], m = \{m_k\}_{k=1}^\infty)$ is an A_∞ filtered algebra.

Gauge transformation $b \in C[1]^0 = C^1$.

$e^b := 1 + b + b \otimes b + b \otimes b \otimes b + \ldots$

[$m_{0,\beta}(1) = [\mathcal{M}_1(\beta; L), ev_0]$ is the one pt invariant that is the obstruction to the floer cohomology of a pair.]

$m_k^b(x_1, \ldots, x_k) = m(e^b, x_1, e^b, x_2, \ldots, x_k, e^b)$

$= \sum m_{\ell_1}(b, \ldots, b, x_1, b, \ldots, b, \ldots, x_k, b, \ldots$

Prop $\hat{d}^b = \sum \hat{m}_k^b$

$\hat{d}^b \circ \hat{d}^b = 0$.

Y-G Oh

4 3

$m_0^b(1)=0 \iff \sum_0^\infty m_k(b,\dots,b)$

$m_0(1) + m_1(b) + \dots + m_k(\underbrace{b,\dots,b}_{k}) + \dots = 0$

Maurer-Cartan eqn.

$m_{0,\beta_0}(1)$ β_0 = homotopy class of smallest area holo disc

defines a cycle, the primary obstruction

§ Toric Mfd.

$$\begin{array}{c} X_\Sigma \\ \downarrow \pi \\ A \in P \subset (\mathbb{R}^n)^* \end{array}$$

natural cx structure
Kähler form

Classification Thm

$L = \pi^{-1}(A)$, $A \in \mathrm{Int}\, P$

1) ∃ Complete description of holo discs

2) All nonconstant holo discs have positive Maslov idx.

3) All singular strata (w/o sphere bubbles) are smooth.

4) $m_0(1) = \sum_1^N h^{v_j} T^{\mathrm{Area}(\beta_j)} [b] q$ h^{v_j} = holonomy of flat line bundle $\mathcal{L}$

$e^{i\langle v, v_j\rangle}$ = holonomy around $\partial D(v_j)$

Y-G Oh

($\delta = m_1$)

$$\delta \langle pt \rangle = \sum_j (-1)^{2} h^{v_j} T^{2\pi \, Area(\beta_j)} (v_j^1 e_1 + \dots + v_j^n e_n)$$

$\{e_i\}$ a $H_1(L;\mathbb{R})$ basis

§ Relations to mirror symmetry (Hori-Vafa)

Dual variables $Y_i \equiv Y_i + 2\pi i$ $\quad i = 1, \dots, N$

$Re(Y_i) \longleftrightarrow$ locates the position of X – Kähler class

$Im(Y_i) \longleftrightarrow$ represents some holonomy of a line bundle on X.

$$\sum_1^N Q_{ia} Y_i = t_a = r_a - i\theta_a$$

from support fn of moment polytope — generator in toric construction

General soln of ↑ : $Y_i = y_i + \langle \Theta, v_j \rangle$

$$W = \sum_1^N e^{-Y_i} = \sum_1^N \exp(-y_i - \langle \Theta, v_i \rangle)$$

Thm $HF^*(L; \Lambda_{0,nov}) \neq 0$ iff $\delta_2(pt) = 0$

$$HF^*(L, \Lambda_{0,nov}) \simeq H^*(L) \otimes \Lambda_{0,nov}$$

Thm $Area(\beta_j) = 2\pi(\langle A, v_j \rangle - \lambda_j)$

Thm Under replacement $T^{2\pi} = e^{-1}$

$m_0(1) \longleftrightarrow W$ $\qquad \delta_2(pt) \longleftrightarrow \frac{\partial W}{\partial \theta}$ g

Cho $\frac{\partial^2 W}{\partial \theta \partial \theta}$ (Hessian) $\rightsquigarrow$ product on Floer cohomology in terms of Clifford algebras

Givental: Quantum Cobordisms & Formal Group Laws (w/ T. Coates) ①

$\overline{\mathcal{M}}_{0,n}$ – $(n-3)$ dim'l cpt cplx mflds. D-M ~~compacts~~

$[\overline{\mathcal{M}}_{0,n}] \in U^*(pt) \simeq \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \ldots]$ ← Thom cobordism ring

deg 2 4 ...

Generalizations

1) $U^*(\overline{\mathcal{M}}_{0,n}) \to U^*(pt)$ (pushforward map)

2) $\overline{\mathcal{M}}_{g,n}$ (an orbifold)

3) $[\overline{\mathcal{M}}_{g,n}(X,d)]^{virt}$ T^{virt}

$\overline{\mathcal{M}}_{0,n}$'s Cohomological Int. Theory

universal curve $\begin{array}{c}\overline{\mathcal{M}}_{0,n+1} \\ \downarrow \uparrow \sigma_1 \ldots \uparrow \sigma_n \\ \overline{\mathcal{M}}_{0,n}\end{array}$ $\psi_i = c_1(L_i)$ ← universal cotangent lines at i^{th} marked pt

$L_i =$ "$T^*_{\sigma_i}\Sigma$" (conormal to section pulled back to base)

$$\langle \psi_1^{k_1}, \ldots, \psi_n^{k_n} \rangle = \int_{[\overline{\mathcal{M}}_{0,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

Encode in generating function:

$$\mathcal{F}(t_0, t_1, \ldots) = \sum_{\text{all } k_i \geq 0} \langle \psi_1^{k_1} \ldots \psi_n^{k_n} \rangle_{0,n} \frac{t_{k_1} \cdots t_{k_n}}{n!}$$

$\mathcal{H} = \mathbb{Q}((\frac{1}{z})) \ni f, g$

$$\Omega(f,g) = \frac{1}{2\pi i} \oint f(-z) g(z)\, dz$$

$f = q_0 + q_1 z + \ldots + \frac{p_1}{z} + \frac{p_2}{z^2} + \ldots$ (p_i, q_i) Darboux coords

2

Givental

$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = T^*\mathcal{H}_+$ polarization

$\mathcal{H}_+ = \mathbb{Q}[z]$

$\mathcal{H}_- = z^{-1}\mathbb{Q}[[z^{-1}]]$

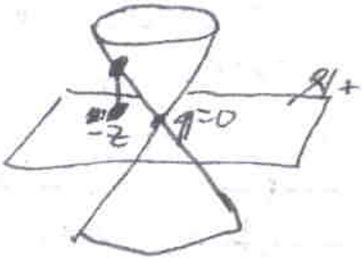
Consider coords $(t_0, t_1, \dots)$ as coefficients of a polynomial $t_0 + t_1 z + t_2 z^2 + \dots \equiv \mathbf{t}$

$$\mathcal{L} = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p} = d_{z+\mathbf{q}(z)}\mathcal{F}\} \subset T^*\mathcal{H}_+$$

← Lagrangian section of $\mathcal{F}$

dilaton shift

$\mathcal{L}$ is a cone:



$$\mathcal{L} = \{ e^{\tau/z} z\, \mathbf{q}(z) \mid \tau \in \mathbb{Q}, \mathbf{q} \in \mathbb{Q}[[z]] \}$$

$$L_\tau = e^{\tau/z}\mathcal{H}_+$$

zL_τ

$$\sum q_i \frac{\partial}{\partial t_i}\mathcal{F} = 2\mathcal{F} \qquad \sum t_i \frac{\partial}{\partial t_i}\mathcal{F} = 2\mathcal{F} + \frac{\partial \mathcal{F}}{\partial t_1}$$

$\boxed{\mathbf{t} = \mathbf{q} \text{ except } t_1 = q_1 + 1}$ Dilaton eqn

$\boxed{\text{COBORDISM THEORY}}$

Thom space

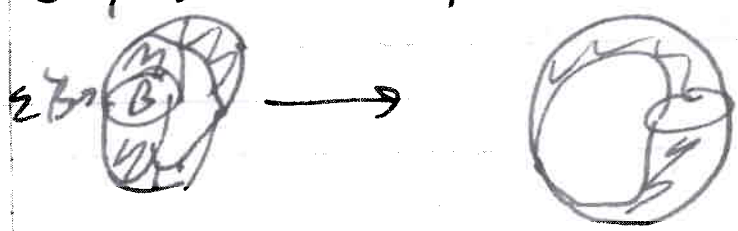
$$U^*(B) = \lim_{k\to\infty} \pi(\Sigma^k B, MU(\tfrac{n+k}{2}))$$

Givental

$U_n(B) = \{\text{Maps } Z^n \to B\}/\text{bordism}$

stably almost-$\mathbb{C}$ m

Say B a stably almost-$\mathbb{C}$ mfld



$MU(--)$

bordism class is preimage of 0

Thom-Pontryagin gives Poincare isomorphism between bordism & cobordism.

Chern Classes of complex V.B.'s with values in CX cobordisms

$$\begin{array}{ccc} L = \mathcal{O}(1) & \leftarrow & \text{univ line bundle} \\ \downarrow & & \\ \mathbb{C}P^\infty & & \end{array}$$

What's $u := c_1^U(L)$?

Hyperplane section $\mathbb{C}P^{\infty-1} \subset \mathbb{C}P^\infty$ is Poincare dual to 1^{st} chern class.

$$\begin{array}{ccc} L_1 \otimes L_2 & \longrightarrow & L \\ \downarrow & & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

1^{st} Chern class of $\uparrow$ in terms of c_1 of factors

$F(v, w) = v + w + \sum F_{k\ell} v^k w^\ell \in U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$

$\rightsquigarrow$ formal group law.

Givental

Chern-Dold character

$$U^*(B) \longrightarrow H^*(B, U^*(pt))$$

$\langle z \rangle = H^2(\mathbb{C}P^\infty)$

$U^*(\mathbb{C}P^\infty) \ni u.$ $\underline{ch}(u) = z + a_1 z^2 + a_2 z^3 + \dots$

$a_1, a_2, \dots$ are alternate generators for $\mathbb{Q}[\mathbb{C}P^1_*]\dots$

Inverse series

$$z(u) = u + [\mathbb{C}P^1]\, u^2/2 + [\mathbb{C}P^2]\, u^3/3 + \dots + [\mathbb{C}P^{n-1}] \frac{u^n}{n}$$

logarithm of formal group

$$F(u(x), u(y)) = u(x+y)$$

$$F(v,w) = u(z(v) + z(w))$$

$\boxed{\text{HRR Formula}}$

stably almost-$\mathbb{C}$

$B \downarrow pt$ $\quad \boxed{ch(\pi_*(a)) = \int_B ch(a)\, Td(TB)}$ $\quad (a \in U^*(B))$

$$Td(\cdot) = e^{\sum s_k ch_k(\cdot)}$$

To compute: $\frac{z}{u(z)} = Td\left(\begin{smallmatrix} L \\ \downarrow \\ \mathbb{C}P^\infty \end{smallmatrix}\right)$

$= \exp\left(\sum s_k z^k/k!\right)$

$$\mathcal{F}^U = \sum \langle \psi_1^{k_1} \cdots \psi_n^{k_n} \rangle_{0,n} \frac{t_{k_1} \cdots t_{k_n}}{n!} = \pi_*(-) \in U^*(pt)$$

$\psi_i = c_1^U(L_i)$

$\mathcal{H} = U^*(pt)\{\{\frac{1}{u}\}\}$ ← formal completion – coeffs → 0 in pos direction

$$\Omega^U(f,g) = \sum_{k=0}^{\infty} \frac{[\mathbb{C}P^k]}{2\pi i} \oint f(u^*) g(u)\, u^k\, du$$

$\boxed{3/23}$ Givental 5

$\{u^*(u(+z)) = u(-z)\}$ u^* replaces $-z$ in cohomological situation

$u^k du$

$$q\,Ch : U \to \mathcal{H} \otimes U^*(pt)$$

$$\sum f_k u^k \mapsto Ch(f_k)\, u(z)^k$$

$$\Omega^u = q\,Ch^* \Omega$$

$$\mathcal{F}^u \rightsquigarrow \mathcal{L}^u \subset \mathcal{U}, \Omega^u \xrightarrow[qCh]{} (\mathcal{L}^u) \in$$

$$\mathcal{L} \subset \mathcal{H} \hat{\otimes} U^*(pt)$$

Thm $q\,Ch(\mathcal{L}^u) = \mathcal{L}$

Rem $\mathcal{F}^u \neq \mathcal{F}$ b/c of polarization and dilaton shift

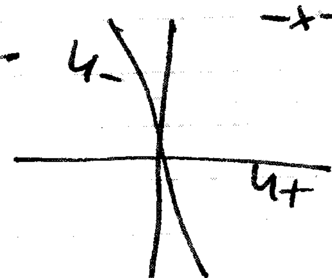
$$\mathcal{U} \to \mathcal{U}_+ = U^*[u]$$

$$\frac{1}{2\pi i}\oint \frac{dz}{u(z-x)\,u(-z-y)} = \begin{cases} \frac{1}{u(-x-y)} & |x|<|z|<|y| \\ -\text{same} & |y|<|z|<|x| \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{u(-x-y)} = \sum_{|x|<|y|} u(x)^k\, v_k(y) \qquad \frac{1}{-x-y} = \sum x^k \left(\frac{1}{y}\right)^{k+1}$$

$\Downarrow$

Darboux basis



$\subset \mathcal{H}$

6

Mivental

$d_{u^*} + q(u)$

$$\mathcal{L}^u = \text{graph}\left(d_{u^*} + q(u)\, \mathcal{F}^u\right)$$

$$H(u) + u^*(u) = q(u)$$

$$H(z) - z = q(z)$$

Relation bet. Quantum Group Laws and geometry of DM spaces still mysterious

McDuff
Work w/ Kedra

Recall $\mathrm{Ham} \subset \mathrm{Symp}_0$

Want extensions $\mathrm{Ham} \subseteq \mathrm{Ham}^s \subseteq \mathrm{Symp}$

~~Recall $H_t \mapsto \varphi_1^H \in \mathrm{Ham}$~~ ~~$\pi_1(M) \to \mathbb{R}$~~ $\to \Gamma$ (Flux. gp)

$$\mathrm{Ham} \to \widetilde{\mathrm{Symp}_0} \xrightarrow{\text{Flux}} H^1(M;\mathbb{R})$$

$$\mathrm{Id} \xrightarrow{\{\varphi_t\}} \varphi \longmapsto \omega\Big($$  $$\Big)$$

$\partial \quad \{\varphi_t(\partial)\}$

ex  $X = \dot{\varphi}_t = \frac{\partial}{\partial x} \overset{\omega}{\longleftrightarrow} dy \neq 0$ in H^1.

Not Hamiltonian

$$\underset{\ker \mathrm{Flux}}{\overset{\shortparallel}{\mathrm{Ham}}} \to \mathrm{Symp}_0 \to H^1(M;\mathbb{R})/\Gamma$$

Unknown if Γ is discrete so that Ham C^1-closed

Prop $(M,\omega) \to P$ symp. bundle
$\downarrow$
B cw

When does ω extend to Ω closed on P? ⟺ When structure gp reduces to Ham from $\mathrm{Symp}(M,\omega)$

Also require $\pi_0 \mathrm{Ham} = 0$ i.e. P symp. trivial over 1-skeleton

Given (P,Ω), can define fibrewise gromov-witten invariants & thus char. classes $\in H^*(B\mathrm{Ham})$

McDuff

②

How to remove hypothesis that P is trivial over B?? Hopefully, extension of Ham makes it possible.

ex $(M,\omega) \to P = (D_+ \times M) \cup (D_- \times M)$
$\downarrow \pi \quad (t, x_+) \sim (t, \varphi_t x_-)$
$\{\varphi_t\}$ a loop in Symp
$S^2 = D_+ \cup D_-$

Claim ~~Ω~~ ω extends iff $\{\varphi_t\}$ is homotopic to a loop in Ham.
$\Leftrightarrow \mathrm{Flux}\{\varphi_t\} = 0$

Want to construct ~~Ω~~ Ω.
Take $pr_M^* \omega$

$$\psi^*(pr_M^*\omega) = \omega + \underline{(\mathcal{L}_{\dot\varphi_t}\omega)\,dt} \quad \text{on } (\partial D_+) \times M$$

want to extend this over D_+

t angular coordinate on ∂D_+

Obstruction to extension is
$$\int \mathcal{L}_{\dot\varphi_t}\omega = \mathrm{Flux}(\varphi_t)$$

$\mathrm{Ham} = \ker\left(\mathrm{flux} : \mathrm{Symp}_0 \to H^1(M;\mathbb{R})/\Gamma\right)$

Flux is eqvt w.r.t. $\pi_0(\mathrm{Symp})$ action

Kotschick-Morita extended flux as a crossed homomorphism i.e. $F(g \cdot h) = h \cdot F(g) + F(h)$
$F : G \to A$

McDuff

(3)

For $M = \Sigma_g$ (monotone) $\exists$ flux : Symp $\to H^1(M;\mathbb{R})$

In general,

$$\exists \hat{F}_s : \text{Symp} \to H^1(M; \mathbb{R}/P_\omega)$$

$P_\omega = \{$values of ω on $H_2(M;\mathbb{Z})\}$

Polterovich's idea: 'Strange homology gp'

$$0 \to \mathbb{R}/P_\omega \to SH_1(M) \equiv \frac{(1\text{-cycles})}{(l_1 \sim l_2 \Leftrightarrow \exists W : \partial W = l_1 - l_2,\ \int_W \omega = 0)} \to H_1(M) \to 0$$

$$0 \to \mathbb{R}/P_\omega \to SH_1(\omega) \underset{s \text{ - a splitting}}{\rightleftarrows} H_1(M) \to 0$$

assume no torsion

$\pi_0(\text{Symp})$ equivariant.

s splitting: $[l_1], \ldots, [l_k]$ a basis

Choose reps $l_1, \ldots, l_k$

$\hat{F}_s(g)([l]) = g(s[l]) - s[gl]$ $\quad \hat{F} : \text{Symp} \to H^1(M; \mathbb{R}/P_\omega)$

If $[gl] = [l]$, it's area $\Big) / P_\omega$

Can Check: $\hat{F}_s$ is a crossed homomorphism extending Flux.

McDuff

If $g \in Symp_0$ $\hat{F}_S(g) = pr(Flux(g)) \in H^1(M) \to H^1(M)/\Gamma$

$Ham^S = \ker \hat{F}_S$

$\circledast$ $Ham^S \cap (Symp_0) \supsetneq Ham$ in general

$Symp \circlearrowright \{splittings\}$ transitively, so all Ham^S are conjugate

$Ham^S \cap Symp^H$ is indep of H. ← subgp of Symp that acts trivially on H^1

$\circledast$ is odd, but maybe inevitable:

$\hat{F}_S : Symp^H \to H^1(\cdot,\cdot)$ is a hom,

& $[Symp^H, Symp^H] \subseteq \ker \hat{F}_S = Ham^S$

but $[Symp^H, Symp^H] \cap Symp_0 \overset{?}{=} Ham$, & ↑ must be contained in any extension

[Rem $[Symp_0, Symp_0] = Ham$.

Prop $(M, \omega) \to P$ has a closed extension of ω iff the structure gp reduces to Ham^S.

∃ theory of symplectic connections.

Extension of $\omega \leadsto$ symp section: τ extends ω, $(T^{vert}P)^{\perp_\tau}$ = Horiz distribution

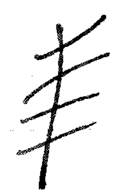
What is translation like?

Is holonomy symplectic?

McDuff (5)

To get symp holonomy,

$\tau|_{\pi^{-1}(\text{path in } B)}$ is closed



horiz = ker τ → std symp transport

To get a Hamiltonian connection:

τ closed $\Longleftrightarrow$ holonomy around contractible loops is Ham

If str. gp of bundle is Ham, $\exists$ connection w/ hol around contractible loops is $(\mathrm{Ham}^s)_0$.

$$\begin{tikzcd} B\mathrm{Ham}^s \arrow[r] & B\,\mathrm{Symp} \\ & B \arrow[ul, dotted] \arrow[u] \end{tikzcd}$$

($\exists$ lift ~~to~~ symp bundle is ~~a~~ Ham bundle)

$\exists \theta \in H^2(B\mathrm{symp}, \xi \cdot 3)$

$f^*(\theta) = 0 \Longleftrightarrow \exists$ lift to $B\mathrm{Ham}^s$

Say $\Gamma = 0$. $\mathrm{Ham} \hookrightarrow \mathrm{Symp}_0 \to H^1(M)$

Is $\theta = 0$??

Kotschick–Morita $\Sigma_g = M$, $g \geq 1$

$$\begin{array}{c} M_{\mathrm{symp}} \\ \downarrow \\ B\,\mathrm{Symp}(M) \underset{h.e.}{\simeq} B(\pi_0 \mathrm{Symp}\, M) \end{array}$$

⑥

McDuff

Consider

$$\begin{tikzcd} M_{symp}^{\delta} \arrow[r] \arrow[d] & M_{symp} \arrow[d] \\ BSymp^{\delta} M \arrow[r, "\text{discrete}"] & BSymp\, M \simeq B(\pi_0(Symp)) \end{tikzcd}$$

M_{symp}^{δ} is foliated

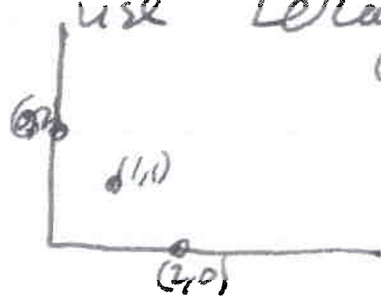
$\exists$ closed extension $\tilde{\omega}$ of ω that's 0 on leaves.

$\exists$ another closed ext of ω:

$\lambda\, c_1^{vert}$

$[\tilde{\omega}] - \lambda c_1^{vert} \in H^2(M_{symp}^{\delta})$

Can use Leray-Serre S-S-



$\downarrow$ proj

$0 \in E_{\infty}^{0,2}$

b/c class vanishes on fiber

But $E_{\infty}^{1,1} = H^1(BSymp^{\delta}, \{H^1(M;\mathbb{R})\})$

$\qquad \parallel$ gp cohomology

$\qquad H^1(Symp^{\delta}, \{\ \})$

$[\tilde{\omega}] - (\lambda c_1^{vert}) = \tilde{F}$

F is equiv class of crossed homs.

$g \in Symp_0$

$\exists$ lift $\tilde{F}(g) \in H^1(M;\mathbb{R})$