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w/ L. Kessler & M. Pinsonnault

$\boxed{\text{Torus Actions on Blowups of } \mathbb{C}P^2}$ ①

$(S^1)^k \cong T \circlearrowright (M, \omega)$

$\rho : T \to Symp(M, \omega)$

st $T \times M \to M$ is smooth

Image $\rho \subset Symp(M, \omega)$ is a subtorus

T actions on (M, ω) / eqvt. symplecto Aut T

$=$ k-dim'l subtori of $Symp(M, \omega)$ / Conjugacy

Finiteness Thm

If (M^4, ω) cpt, $\pi_1 = 0$, then

$$\frac{\#\{2\text{-dim'l subtori of } Symp(M, \omega)\}}{\text{conjugation}}$$

is finite

HAMILTONIAN TORUS ACTIONS

$T \circlearrowright (M, \omega)$ $\quad \pi_1 = 0 \Rightarrow \exists\ \varphi : M \to \mathfrak{t}^*$

$\forall \xi \in \mathfrak{t} \xrightarrow[\rho_*]{} \xi_M \in V.F.(M)$

$d\varphi^\xi = -i_{\xi_M}\omega$

(M, φ, ω) DETERMINES $T \circlearrowright M$

M CPT, $\Delta := \mathrm{Im}(\varphi) \subseteq \mathfrak{t}^*$.

<u>CONVEXITY THM</u> Δ IS A CONVEX POLYTOPE.

ASSUME: $\dim T = \frac{1}{2}\dim(M)$ (WHICH IS MAXIMAL)

<u>DELZANT</u>: Δ DETERMINES (M, ω, φ)
UP TO AN EQVT. SYMPLECTO PRESERVING φ.
$\varphi \rightsquigarrow \varphi +$ CONST — SHIFTS POLYTOPE
REPARAMETERIZING $T \rightsquigarrow$ TRANSFORM Δ BY $GL_n(\mathbb{Z})$

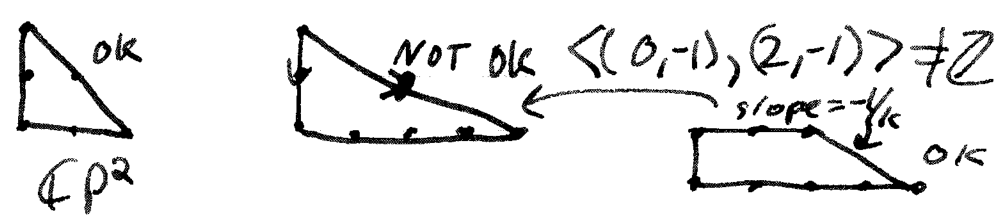
<u>REM</u> $\mathfrak{t}^* = \mathbb{R}^n$
$\ell^* = \mathbb{Z}^n$

<u>POSSIBLE Δ'S</u>

$\forall$ VERTEX v, $\exists$ $\mathbb{Z}$-BASIS TO THE LATTICE $\alpha_1, \ldots, \alpha_n$ & NBHD U OF v ST.

$\Delta \cap U = \{ v + \sum_1^n t_j \alpha_j \mid t_i \geq 0\ \forall i \}$

<u>EX</u>



$\langle (0,-1), (2,-1) \rangle \neq \mathbb{Z}^2$

③

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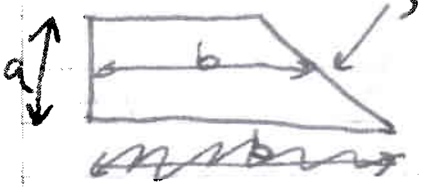
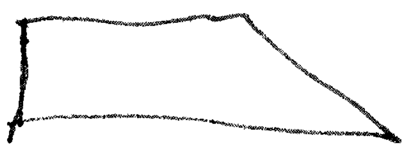
slope $-1/k$, $k \in \mathbb{Z}_{\geq 0}$



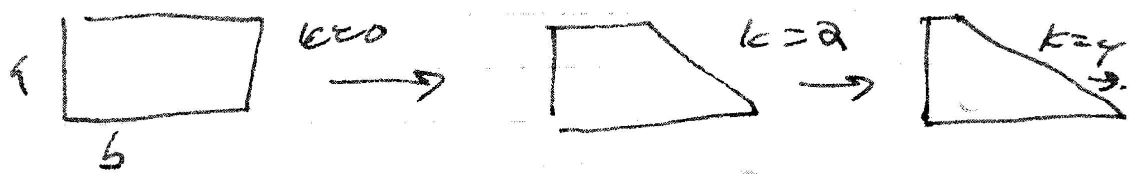
HIRZEBRUCH SURFACE

$\wr$ SYMPLECTOMORPHIC (NON-EQUIV'LY)



SAME a, b
$k \to k-2$

Ex BEHIN on $S^2 \times S^2$



$\left[\frac{b}{a}\right]$ SUCH TRAPEZOIDS $\rightsquigarrow$ $\left[\frac{b}{a}\right]$ ACTIONS ON $S^2 \times S^2$

THM THESE ARE ALL THE ACTIONS

$\#$ EDGES OF $\Delta = 2 + \dim H^2(M_\Delta)$

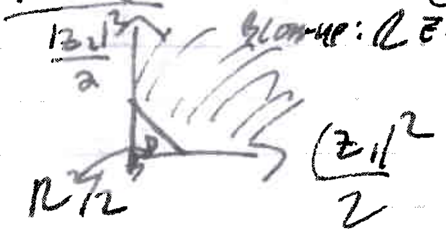
SYMPLECTIC BLOWUP

STD BLOWUP OF $\mathbb{C}^n$ OF SIZE $R^2/2$ IS:

$$\{|z| > R\} \sqcup \{|z| = R\} \big/ S^1$$

$R^2/2$

PICTURES $n=2$. $(S^1)^2 \circlearrowright \mathbb{C}^2$ $\quad (z_1, z_2) \mapsto \frac{1}{2}(|z_1|^2, |z_2|^2)$



BLOWUP: REMOVE $\frac{|z_1|^2}{2} + \frac{|z_2|^2}{2} \leq R^2/2$

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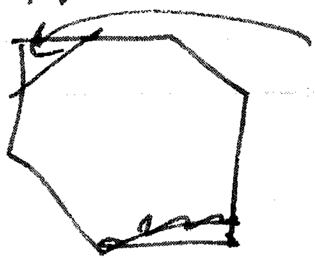
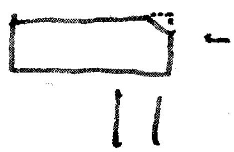
EQVT. SYMP. BLOWUP OF SIZE ε, OF A SYMPLECTIC TORIC MFLD



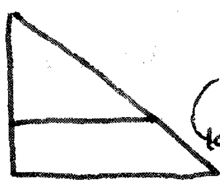
CHOP OFF A CORNER OF SIZE ε.

ex BLOWUP OF $S^2 \times S^2$



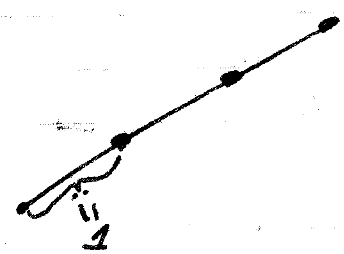
$=$ 2 BLOWUPS OF $\mathbb{C}P^2$

e.g.



(HIRZEBRUCH $k=1$) = BLOWUP OF $\mathbb{C}P^2$

SIZE = RATIONAL LENGTH



$$\text{SIZE}(C) = \frac{\text{SYMP AREA}}{2\pi}$$

$C^2 \subseteq M$

$$\underset{\text{SIZE}}{\varphi^{-1}(\text{EDGE})} = \underset{\text{SIZE}}{C \cong S^2} \subseteq M$$

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ASSUME $n=2$

DELZANT CONDITION $\Rightarrow$ Δ OBTAINED FROM $\mathbb{CP}^2$ OR A HIRZEBRUCH SURFACE BY A SEQUENCE OF CHOPPINGS

$\Rightarrow$ $(M_\Delta, \omega_\Delta, \varphi_\Delta)$ IS $\mathbb{CP}^2$ OR OBTAINED FROM A HIRZEBRUCH SURFACE BY A SEQ OF EQUIV. BLOWUPS

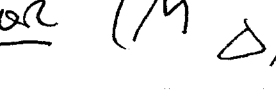
COR $(M_\Delta, \omega_\Delta) \cong S^2 \times S^2$ OR $\mathbb{CP}^2$ BLOWNUP

PROBLEM DETERMINE WHICH BLOWUPS OF $\mathbb{CP}^2$ ARE SYMPLECTOMORPHIC.

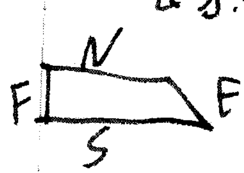
NOTE [BIRAN-MCDUFF]: A BLOWUP OF $\mathbb{CP}^2$ IS UNIQUELY DETERMINED UP TO SYMPLECTO BY SIZES OF $\mathbb{CP}^2$ & OF THE BLOWUP

FINITENESS THM FOLLOWS FROM THIS:

FIX (M,ω) LET $\dim H^2(M) = 2+s$, $s \geq 0$. SET OF ALL TUPLES $(a, b, k; \delta_1, \dots, \delta_s)$ s.t. CAN START WITH



& CHOP CORNERS OF SIZES $\delta_1, \dots \delta_n$ & s.t. THE RESULTING $(M_\Delta, \omega_\Delta) \cong (M, \omega)$ IS FINITE.



PREIMAGES OF EDGES — $S^2 \subset M$, SIZE(S^2) = (LAT'L) LENGTH OF EDGE

$N, F, S \in H_2^{\#}$, $S = N + kF$

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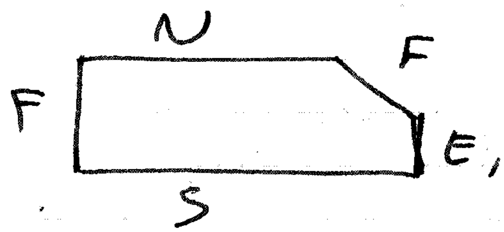
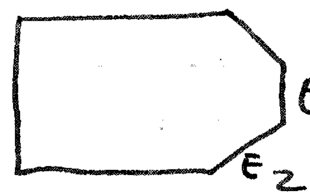
BLOWUP





WHEN REACH $M_s \cong M$,

GET $N, F, S, E_1, \dots, E_s \in H_2(M)$.

PREIMAGE OF WHOLE BOUNDARY IS POINCARE DUAL TO $c_1(M)$.
∴ PERIMETER OF POLYTOPE = SYMP INV = $\langle \omega, c_1(M) \rangle$
SO AREAS OF EDGES HAVE UNIFORM BOUND

~~PROPERTIES~~: EACH OF $S, F, N, E_1, \dots, E_s$ REP'D BY A CHAIN OF S^2's IN M INTERSECTING NICELY, AND ITS ~~SELF INTERS.~~ SYMPLECTIC AREA IS $\leq \langle \omega, c_1(M) \rangle$ & POSITIVE.

- $F \cdot F = 0$, $E_i \cdot E_j = -1$
- $S = N + kF$, $k \geq 0$

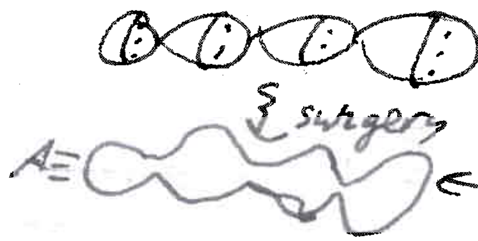
ENOUGH TO SHOW: ALL SUCH $\{(k, N, F, S, E_1, \dots, E_s)\}$ IS FINITE.

$\boxed{\text{CARSHON}}$

STEP 1

SET OF $A \in H_2(M)$ OCCURRING AS F OR E_j ABOVE IS FINITE.

Pf: PROPERTY 1:



$A \cdot A = 0$ OR -1.

LEMMA $\forall$ GENERIC ω-COMPAT. ALMOST $\mathbb{C}$ STR. ON M, $\exists$ EMBEDDED J-HOLO SPHERE IN THE CLASS A.

Pf: GW THEORY:

ASSUME ∞'LY MANY DISTINCT $A_m \in H_2(M)$

CHOOSE J GENERIC.

LEMMA $\Rightarrow$ $\exists$ J-HOLO SPHERE C_m, IMBEDDED, IN EACH CLASS.

AREA $\int_{C_m} \omega = \langle \omega, A_m \rangle \le \langle \omega, G(M) \rangle$

APPLY GROMOV COMPACTNESS $\Rightarrow$ $\exists$ SUBSEQUENCE $m_\ell \to \infty$ s.t. C_{m_ℓ} CONVERGES $\Rightarrow$ $A_{m_\ell} \in H_2(M;\mathbb{R})$ CONVERGES; AS $H_2(M;\mathbb{Z})$ IS DISCRETE, A_{m_ℓ} STABILIZES, CONTRADICTION ASSUMPTION OF DISTINCTNESS

COR $\{$ VALUES OF $(a, \delta_1, \dots, \delta_5) \}$ IS FINITE.

STEP 2 AREA $(\Delta) = ab - \frac{1}{2} \sum_1^5 \delta_i^2$

$\quad \parallel$

$\frac{VOL(M)}{2\pi^2}$ ← SYMP. INV.

$\Rightarrow a, \delta_1, \dots, \delta_5$ ~~DETER~~ DETERMINE b.

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(8)

STEP 3 $\quad k \geq 0 \quad k \in \mathbb{Z}$

$$S = N + kF$$

$$\underbrace{\langle \omega, S \rangle}_{\text{BDD ABOVE}} \geq k \underbrace{\langle \omega, F \rangle}_{q}$$

FINITELY MANY POSSIBILITIES $\Rightarrow$ BDD FROM BELOW

$\Rightarrow$ # k's IS FINITE. $\blacksquare$