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Univariate representations and algebraic sets over quadratic maps.

The setup:

$$\{\mathbf{x} \in \mathbb{R}^n : p_1(Q(x)) \geq 0, \dots, p_s(Q(x)) \geq 0\}$$ -

a semialgebraic set.

$Q : \mathbb{R}^n \to \mathbb{R}^k$, $\quad n \gg k$,

$p_i \in \underset{\mathbb{R}}{\overset{\cap}{D}}[Y_1, \dots, Y_k] = D[Y]$, (or even a real closed field R)

D is some "nice" subring of $\mathbb{R}$.

A special case: $p = Y_1^2 + \dots + Y_k^2$ and we consider the set

$$S = \{x : p(Q(x)) = 0\}.$$

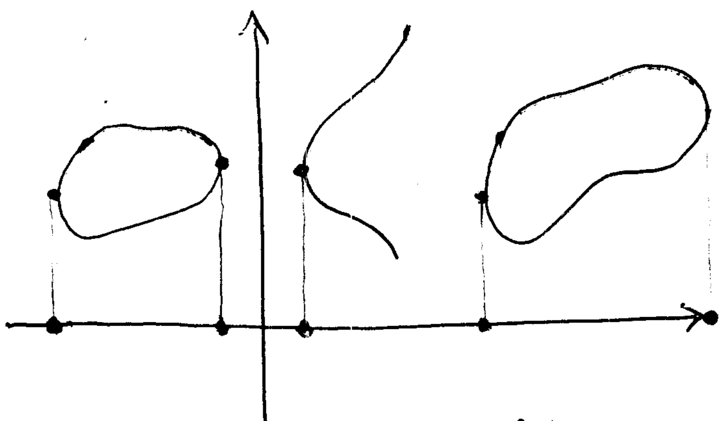
Here $Q : x \mapsto (Q_1(x), \dots, Q_k(x))$.

The results are available at arxiv.org, cs.sc/0403008.

There is an upper bound for the Betti numbers of the set S:

$$b(S) \leq (s \cdot n \cdot d)^{O(k)}.$$

To obtain this estimate, we count the number of critical points of the projection $X \mapsto X_1$, after deforming



$p(Q_{\varepsilon_2}(x)) = \varepsilon_1$ and working over the field $R((\varepsilon^{1/\infty}))$ of Puseux series in ε over R.

We obtain the following system of equations:

$$\frac{\partial P(Q(x))}{\partial x_i} = 0, \qquad i = 1, \dots, n,$$

$$\sum_{j=1}^{k} \left(\frac{\partial P}{\partial Y_j} \right) (Q(x))\, e_i^T (H_j x + b_j) = 0,$$

$$Q_j(x) = x^T H_j x + b_j^T x + c_j.$$

$$(Q_j)_{\varepsilon_2}(x) = Q_j(x) + \varepsilon_2 \, \mathrm{diag}(1^j, 2^j, \dots, n^j)$$

We get matrix $A(Y)$ such that

$Y = Q(x)$, $\quad A(Y) = \sum_j \left(\frac{\partial P}{\partial Y_j}\right)(Y) H_j$

$A(Y) \cdot x = b(Y)$.

One can show that

$\operatorname{rank}(A(Y)) \geq n-k$, $\forall\, Y \neq 0$.

Thus we have $(dn)^{O(k)}$ invertible submatrices, parametrized by $U, W \subseteq \{1, .. n\}$, $|U| = |W| \geq n-k$.

Let us define

$$x_W = A(Y)^{-1}_{UW} \cdot \left(b(Y)_U - A(Y)_{U\overline{W}}\, x_{\overline{W}}\right). \quad (*)$$

We reduced our problem to the case of the sets of the form

$$V = \{z_1, \dots, z_{2k} : g_j(z) = 0\}, \text{ for some}$$

$$g_j \in D[\varepsilon][z], \quad j = 1, \dots, 2k.$$

There exists a polynomial map given by $(*)$ $\varphi : R_\varepsilon^{2k} \to R_\varepsilon^{n}$. We would like to compute the limit $\lim_{\varepsilon \to 0} \varphi(V)$.

Let $a = \sum_i d_i \varphi_i(z)$. Consider the characteristic polynomial $\chi_t(a(z) + s\varphi_i(z)) =$

$$= \prod_{z \in V} \left(t - a(z) - s\varphi_i(z)\right)^{\mu(z)}.$$

of the linear transformation induced on $R((\varepsilon^{1/\infty}))/(g_1, \dots g_k)$ via multiplication by $a(z) + s\varphi_i(z)$.

Let $q_i(t) = \left.\dfrac{\partial \chi_t}{\partial s}\right|_{s=0}$

One can prove that

$$\frac{\hat{q}_i^{(\dots)}(t)}{\hat{q}_0^{(\dots)}(t)} = x_i \;,$$ for t such that

$\hat{\chi}_t(a(z)) = 0$, where $\hat{f}(t)$ denotes the appropriate limit of $f(t)$ w.r.t. $\varepsilon \to 0$.