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Dynamics and Tropical Varieties

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Basic References:

- Klaus Schmidt, "Dynamical Systems of Algebraic Origin", Birkhauser, 1995
- Einsiedler, Lind, Miles, Ward, "Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions", Ergodic Th. & Dyn. Syst. 21 (2001), 1695-1729
- Einsiedler, Kapranov, Lind, "Non-archimedean amoebas and tropical varieties", preprint RSN.

Set-up:

X: compact abelian group

ρ: metric on X (all equivalent)

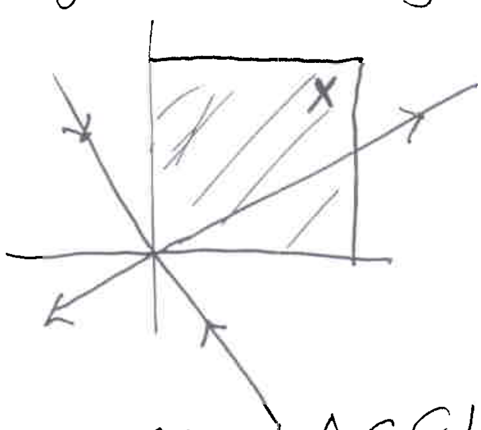
α: action of $\mathbb{Z}^d$ on X by (cts) gp autos

$\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$

$\alpha^{\underline{n}}$: elt of α corresp to $\underline{n} \in \mathbb{Z}^d$

Def: α is expansive if $\exists\, \delta > 0$ s.t. if $x \neq y \in X$, then $\sup_{\underline{n} \in \mathbb{Z}^d} \rho(\alpha^{\underline{n}}(x), \alpha^{\underline{n}}(y)) \geq \delta$.

Ex ①: $d = 1$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $X = \mathbb{T}^2$, and α generated by $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2$.



α is expansive.

[General: $A \in GL(n, \mathbb{Z})$ is expansive on $\mathbb{T}^n$ iff A has no eigenvalues λ with $|\lambda| = 1$.]

Ex 2: $d = 1$, $X = \mathbb{T}^{\mathbb{Z}}$, α generated by left shift $\sigma : \leftarrow$ on X

$$\sigma(u)_n = u_{n+1}$$

Expansive?

Ex 3: $X \subset \mathbb{T}^{\mathbb{Z}}$: $u_{n+2} - u_{n+1} - u_n = 0$ all n

$\alpha = \text{shift}|_X$

Expansive? Yes!

u: $\cdots \boxed{\overset{u_0}{\bullet} \ \overset{u_1}{\bullet}} \cdots$

$$\updownarrow \phi$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

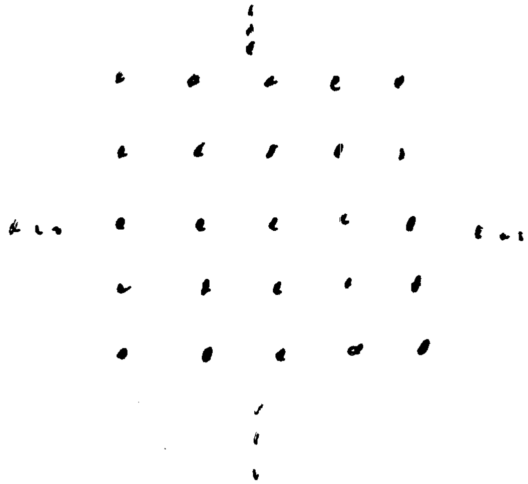
$$\begin{tikzcd} \in X \arrow[r, "\sigma"] \arrow[d, "\phi"] & X \arrow[d, "\phi"] \\ \in \mathbb{T}^2 \arrow[r, "{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}"'] & \mathbb{T}^2 \end{tikzcd}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_0 + u_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \checkmark$$

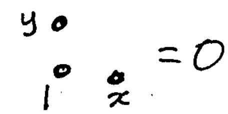
The relation $u_{n+2} - u_{n+1} - u_n$ all n corresponds to the polynomial $x^2 - x - 1$.

This sort of example works for <u>any</u> polynomial $f(x)$ in $\mathbb{Z}[x^{\pm 1}]$.

Ex 4: $d = 2$, $X = \mathbb{T}^{\mathbb{Z}^2}$, α gen by $\leftarrow$, $\downarrow$



Not expansive.



Ex 5: $d = 2$, $X \subset \mathbb{T}^{\mathbb{Z}^2}$: $u_{m,n} + u_{m+1,n} + u_{m,n+1} = 0$

for all $(m,n) \in \mathbb{Z}^2$, $\alpha = \text{shift}|_X$

Corresponds to the Laurent polynomial

$1 + x + y \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$

Not expansive (harder to see)

Ex 6: $d = 2$, $X \subset \mathbb{T}^{\mathbb{Z}^2}$: $2u_{m,n} = 0 \quad \forall m,n.$

Corresponds to $2 \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$

$X = \{0, \frac{1}{2}\}^{\mathbb{Z}^2}$, α is expansive

$\alpha = \text{shift}|_X$

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Ex 7: $d=2$, $X \subset \mathbb{T}^{\mathbb{Z}^2}$:

both $\begin{cases} u_{m,n} + u_{m+1,n} + u_{m,n+1} = 0 \\ 2u_{m,n} = 0 \end{cases}$ all n

Corresponds to two polynomials $f(x,y) = 1+x+y$ and $g(x,y) = 2$ in $\mathbb{Z}[x^{\pm1}, y^{\pm1}]$ and all polynomial combinations of these as well, i.e. to the ideal $\mathfrak{a} = \langle 2, 1+x+y \rangle$ in

$R_2 := \mathbb{Z}[x^{\pm1}, y^{\pm1}]$.

Using $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ insted of $\{0, \frac{1}{2}\}$;

$$X = \{ u \in \{0,1\}^{\mathbb{Z}^2} : u_{m,n} + u_{m+1,n} + u_{m,n+1} \equiv 0 \pmod 2 \}$$

$\ldots\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ \ldots$

This works for any ideal $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$ in R_2 (or d), yielding a group $X_{\mathfrak{a}} \subset \mathbb{T}^{\mathbb{Z}^d}$ and a $\mathbb{Z}^d$-action $\alpha_{\mathfrak{a}} = \text{shift}|_{X_{\mathfrak{a}}}$.

$$\mathfrak{a} \rightsquigarrow (X_{\mathfrak{a}}, \alpha_{\mathfrak{a}})$$

$$\{\text{ideals in } R_d\} \rightsquigarrow \{\text{algebraic } \mathbb{Z}^d\text{-actions}\}$$

[Behind the scenes: Pontryagin duality: dual group of $X_{\mathfrak{a}}$ is $R_d/\mathfrak{a}$, and the $\mathbb{Z}^d$ shift-action on $X_{\mathfrak{a}}$ dualizes to R_d-module structure on $R_d/\mathfrak{a}$]

¿ When is $\alpha_{\mathfrak{a}}$ expansive?

$$V_{\mathbb{C}}(\mathfrak{a}) = \{\underline{z} \in (\mathbb{C}^\times)^d : f(\underline{z}) = 0 \text{ all } f \in \mathfrak{a}\}$$

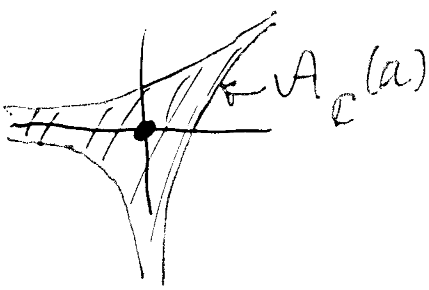
$$\mathcal{A}_{\mathbb{C}}(\mathfrak{a}) = \{(\log|z_1|, \dots, \log|z_d|) : \underline{z} \in V_{\mathbb{C}}(\mathfrak{a})\}$$

$$= \text{complex amoeba of } \mathfrak{a}.$$

<u>Thm</u> (Schmidt): $\alpha_{\mathfrak{a}}$ is expansive iff $\underline{0} \notin \mathcal{A}_{\mathbb{C}}(\mathfrak{a})$.

<u>Ex</u>: $d=1$, $\mathfrak{a} = \langle x^2 - x - 1 \rangle$, $\mathcal{A}_{\mathbb{C}}(\mathfrak{a}) = \{\pm \log(\frac{1+\sqrt{5}}{2})\} \not\ni 0$, so expansive.

<u>Ex</u>: $d=2$, $\mathfrak{a} = \langle 1 + x + y \rangle$, $\underline{0} \in \mathcal{A}_{\mathbb{C}}(\mathfrak{a})$, so $\alpha_{\mathfrak{a}}$ is <u>not</u> expansive.



$\left[\omega = \frac{-1+\sqrt{3}}{2}, \quad (\omega, \omega^2) \in V_{\mathbb{C}}(\mathfrak{a})\right]$

<u>Problem</u>: Find an algorithm to decide, given $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$, whether or not $\underline{0} \in \mathcal{A}_{\mathbb{C}}(\mathfrak{a})$.

Recall Kronecker's Thm: If $f(x) = x^r + c_{r-1}x^{r-1} + \dots + c_0 \in \mathbb{Z}[x]$, and all roots of f are on the unit circle in $\mathbb{C}$, then f is cyclotomic.

⑧

Crazy reformulation:

$$\mathcal{A}_{\mathbb{C}}(f) \text{ spans a proper subspace of } \mathbb{R}$$
$$\Longleftrightarrow x^n - 1 \in \langle f \rangle \text{ for some } n \neq 0.$$

<u>Thm</u> (Einsieder): Suppose $\mathfrak{a}$ is an ideal in R_d with $\mathfrak{a} \cap \mathbb{Z} = \{0\}$. Then

$$\mathcal{A}_{\mathbb{C}}(f) \text{ spans a proper subspace of } \mathbb{R}^d$$
$$\Longleftrightarrow \underline{x}^{\underline{n}} - 1 \in \mathfrak{a} \text{ for some } \underline{n} \neq \underline{0}$$

[$\Leftrightarrow \alpha_{\mathfrak{a}}$ is mixing]

<u>Problem</u>: Find an algorithm (if it exists!) to decide, given $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$, whether or not $\mathfrak{a}$ contains $\underline{x}^{\underline{n}} - 1$ for some $\underline{n} \neq \underline{0}$.

Even if $\alpha_{\mathfrak{a}}$ is expansive, this is not the whole story. Need to know the dynamical behavior of $\alpha_{\mathfrak{a}}$ along half-spaces as well.

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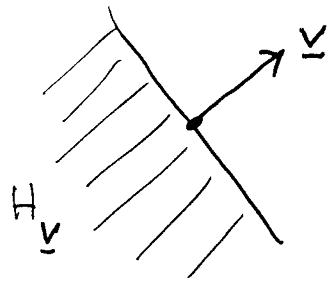
$S_{d-1} = \{\underline{v} \in \mathbb{R}^d : \|\underline{v}\| = 1\}$ = unit $(d-1)$-sphere

$\underline{v} \in S_d$: $H_{\underline{v}} = \{\underline{w} \in \mathbb{R}^d : \underline{v} \cdot \underline{w} \leq 0\}$



Def: α is expansive along $H_{\underline{v}}$ if $\exists\, \delta > 0$ s.t. if $u \neq v \in X$, then $\sup_{\underline{n} \in \mathbb{Z}^d \cap H_{\underline{v}}} \rho(\alpha^{\underline{n}}(x), \alpha^{\underline{n}}(y)) \geq \delta$.

$N(\alpha) := \{\underline{v} \in S_{d-1} : \alpha$ is not expansive along $H_{\underline{v}}\}$, a compact subset of S_{d-1}.

Why is $N(\alpha)$ important?

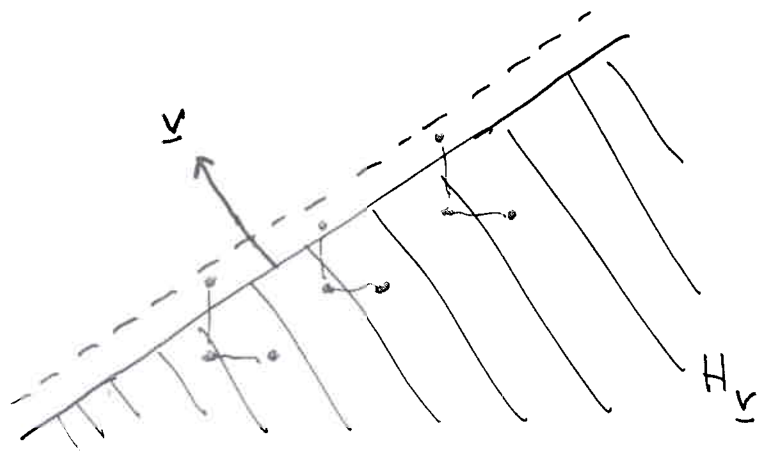
Subdynamics Philosophy [Boyle-L, "Expansive subdynamics", TAMS 349, 55-102]:

For $\underline{v}$'s within a connected component of $S_{d-1} \setminus N(\alpha)$, dynamical properties of α along $H_{\underline{v}}$ vary nicely, but change abruptly when passing from one component to another ["phase transition"]

Ex: $d=2$, Ledrappier's example: for $\mathbb{Z}/2\mathbb{Z}$

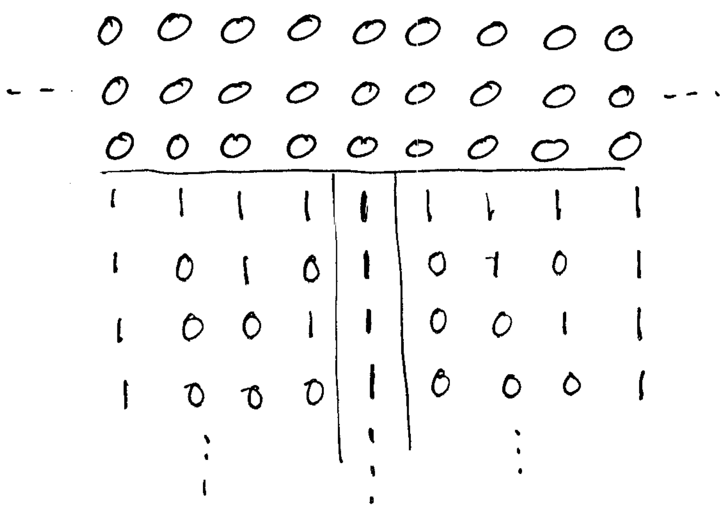
α is expansive along H means:

if $u \in X$ and you know its coordinates $u_{\underline{n}}$ for all $\underline{n} \in \mathbb{Z}^2 \cap H$, Then you also can find out The rest of The coordinates.

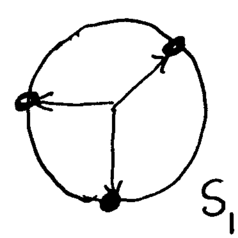


So $H_{\underline{v}}$ is expansive.

But This argument fails for $\underline{v} = \downarrow$



$N(\alpha)$:



Problem: Given $\mathfrak{a} \subset R_d$, compute the compact set $N(\alpha_{\mathfrak{a}}) \subset S_{d-1}$.

Reduction to prime ideal case:

$$N(\alpha_{\mathfrak{a}}) = \bigcup_{\mathfrak{p} \text{ min over } \mathfrak{a}} N(\alpha_{\mathfrak{p}})$$

↑ finite union.

Two types of prime ideals:

$$\mathfrak{p} \cap \mathbb{Z} = \begin{cases} 0 & \Leftrightarrow \text{char}(R_d/\mathfrak{p}) = 0 \\ p\mathbb{Z} & \Leftrightarrow \text{positive char} \end{cases}$$

Solution in char 0 case:

$$V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) = \{ \underline{z} \in (\overline{\mathbb{Q}}_p^{\times})^d : f(\underline{z}) = 0 \quad \forall f \in \mathfrak{p} \}$$

$$\mathcal{A}_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) = \{ (\log|z_1|_p, \ldots, \log|z_d|_p) : \underline{z} \in V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \}$$

"p-adic amoeba" = "tropical variety of $\mathfrak{p}$"

Convention: $\overline{\mathbb{Q}}_\infty = \mathbb{C}$

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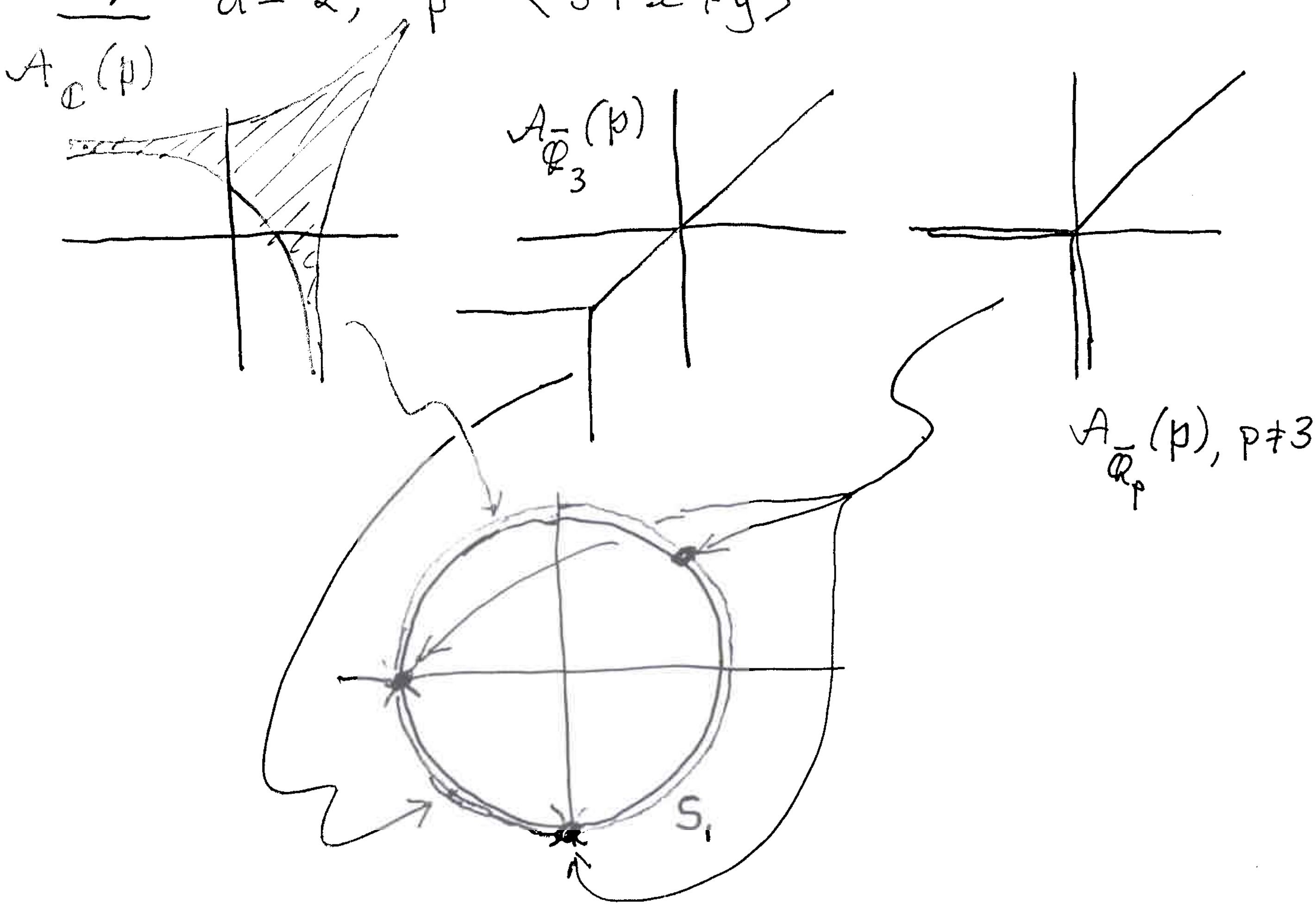
Thm (ELMW): If $\operatorname{char}(R_d/\mathfrak{p})=0$, then

$$N(\alpha_{\mathfrak{p}}) = \text{radial projection to } S_{d-1} \text{ of } \bigcup_{p\le\infty} \mathcal{A}_{\overline{\mathbb{Q}}_p}(\mathfrak{p})$$

Ex: $d=2$, $\mathfrak{p}=\langle 3+x+y\rangle$

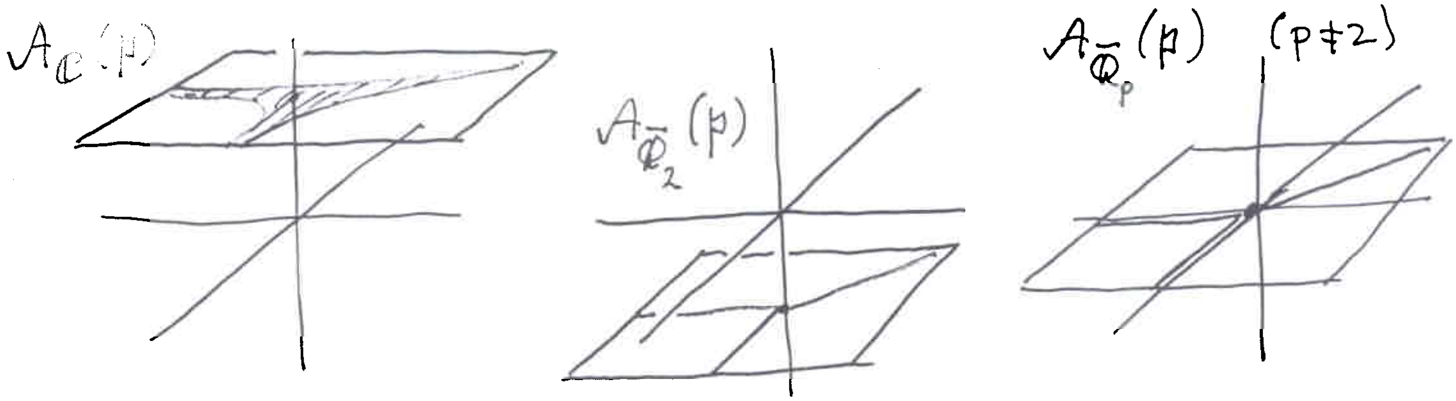


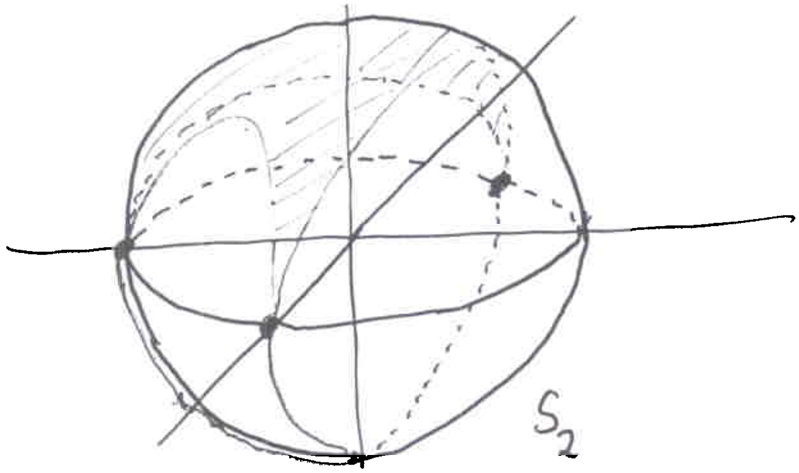
Note: $N(\alpha_{\mathfrak{p}})=S_1$; This is not an accident.

Thm: $N(\alpha_{\langle f\rangle}) = S_{d-1}$

[First discovered dynamically using homoclinic points]

Ex ("Space helmet") $d = 3$, $\mathfrak{p} = \langle 1+x+y,\ z-2 \rangle$





Note how the amoebas "cooperate" to give a closed set under radial projection.

There appears to be some notion of "adelic amoeba" for ideals in R_d, and some sort of local/global principle (e.g. the product formula)

What about the positive char case?

Similar results, but instead of $\mathbb{C}$ and $\bar{\mathbb{Q}}_p$, we use the algebraic closure of
$\mathbb{F}_p((t))$ = field of Laurent power series in t
[Classification of locally compact fields]

If $\mathfrak{p}$ is a prime ideal, then $V_{\mathbb{C}}(\mathfrak{p})$ is connected $(d>1)$, so $\mathcal{A}_{\mathbb{C}}(\mathfrak{p})$ is connected.
True for $\mathcal{A}_{\bar{\mathbb{Q}}_p}(\mathfrak{p})$ (or tropical varieties)?

<u>Thm</u> (EKL). If $\mathfrak{p}$ is a prime ideal in R_d with $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, then $\mathcal{A}_{\bar{\mathbb{Q}}_p}(\mathfrak{p})$ is connected for all $p \leq \infty$.

Proof uses "rigid analytic spaces", "affinoid algebras", and recent work of Brian Conrad.