What is Tropicalization?

 $K = \mathbb{C}((t^{\mathbb{R}}))$ $v(at^{\alpha} + \text{ higher order terms }) = \alpha$ If $I \subset K[x_1, \ldots, x_n]$ is an ideal, we define Trop $I = v((K^*)^n \cap \operatorname{Var}(I)).$ If $w \in \mathbb{R}^n$, $f \in K[x_1, \ldots, x_n] \setminus \{0\}$, write $f(x_1,\ldots,x_n) = t^W \operatorname{in}_w f(t^{w_1}x_1,\ldots,t^{w_n}x_n)$ $+ \sum t^{W_j} g_j(t^{w_1} x_1, \dots, t^{w_n} x_n)$ with $W_i > W$, $\operatorname{in}_w f \in \mathbb{C}[x_1, \ldots, x_n] \setminus \{0\}$. This defines $\operatorname{in}_w(f)$. $\operatorname{in}_w I = < \operatorname{in}_w(f) : f \in I >$

Theorem (Kapranov).

Trop $I = \{ w \in \mathbb{R}^n : in_w I \text{ contains no monomial} \}.$

From now on, we are interested only in the case that I is the ideal of a linear space L of rank k in n-space. Write

$$L = \text{RowSpan} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k1} & x_{k2} & x_{k3} & \cdots & x_{kn} \end{pmatrix}$$

The Plücker coordinates of L are

$$p_{i_1\dots i_k}(I) = \det \begin{pmatrix} x_{1i_1} & \cdots & x_{1i_k} \\ \vdots & \vdots & \vdots \\ x_{ki_1} & \cdots & x_{ki_k} \end{pmatrix}.$$

For simplicity, assume all the $p_{i_1...i_k}(I)$ are nonzero.

Proposition. $\operatorname{in}_w I$ is also the ideal of a linear space. We have $p_{i_1...i_k}(\operatorname{in}_w I) \neq 0$ iff, for all $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$,

$$v(p_{i_1...i_k}(I)) - w_{i_1} - \dots - w_{i_k} \le v(p_{j_1...j_k}(I)) - w_{j_1} - \dots - w_{j_k}.$$

The previous proposition can be viewed geometrically. Define

$$\Delta(k,n) = \text{ConvexHull} \ (e_{i_1} + \cdots + e_{i_k}) \subset \mathbb{R}^n.$$

Consider the lower convex hull of

$$\{(e_{i_1} + \dots + e_{i_k}, v(p_{i_1 \dots i_k}))\} \subset \Delta(k, n) \times \mathbb{R}.$$

Project this lower hull to $\Delta(k, n)$ to get a subdivision of $\Delta(k, n)$. Trop *I* will be a subcomplex of the dual complex to this subdivision. A face *F* is dual to Trop *I* iff *F* does not lie in any of the *n* facets

ConvexHull
$$(e_{i_1} + \cdots + e_{i_k} : j \notin \{i_1, \dots, i_k\})$$

for $1 \leq j \leq n$. A face F is dual to a bounded face of Trop I iff F is in the interior of $\Delta(k, n)$. A matroid is Series-Parallel if it corresponds to a graph which can be obtained from a single edge by the composition of series and parallel extensions as below.



The Series and Parallel Extensions

Conjecture. Let f_i be the number of dimension *i* bounded faces of a tropical linear space of dimension *k* in *n*-space. Then

$$f_i \le \binom{n-i-2}{i} \binom{n-2i-2}{k-i-1}$$

with equality iff every facet of the corresponding decomposition of $\Delta(k,n)$ is corresponds to a Series-Parallel matroid.

Theorem. The collection of Series-Parallel tropical linear spaces is closed under transverse intersections and dualization. In particular, the intersection of n - k transverse hyperplanes (a "tropical complete intersection") is Series-Parallel.

Random sampling of complete intersections with (k, n) = (3, 7), (3, 8), (3, 9), (4, 8) and (4, 9) suggests the conjecture is valid.

Theorem. Let $t_M(z, w)$ denote the Tutte polynomial of a matroid M. We abuse notations by identifying a matroid with its corresponding polytope. For $|M| \ge 2$, the β invariant of M is defined by

$$t_M(z,w) = \beta(M)(z+w) + \dots$$

Let \mathcal{D} be a decomposition of $\Delta(k, n)$ into matroidal polytopes and let $\mathring{\mathcal{D}}$ denote the internal faces of \mathcal{D} .

$$t_{\Delta(k,n)}(z,w) = \sum_{M \in \mathring{\mathcal{D}}} (-1)^{\operatorname{codim}(M)} t_M(z,w).$$
$$\binom{n-2}{k-1} = \beta(\Delta(k,n)) = \sum_{F \text{ a facet of } \mathcal{D}} \beta(F).$$

Corollary. $f_0 \leq \binom{n-2}{k-1}$ with equality iff $\beta(F) = 1$ for all facets of \mathcal{D} iff \mathcal{D} is Series-Parallel.

The second equivalence is due to Crapo.

Theorem. Suppose (x_1, \ldots, x_n) and $(y_1, \ldots, y_n) \in K^n$ are such that

Trop RowSpan
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

is a trivalent tree. Then

Trop RowSpan
$$\begin{pmatrix} x_1^{k-1} & x_2^{k-1} & \cdots & x_n^{k-1} \\ x_1^{k-2}y_1 & x_2^{k-2}y_2 & \cdots & x_n^{k-2}y_2 \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{k-1} & y_2^{k-1} & \cdots & y_n^{k-1} \end{pmatrix}$$

achieves the bound in the conjecture.







В A 2 1 2 3 4 5 6 6 D In the above diagrams, a graph represents the matroid arising as the rowspan of the adjacency matrix.

