What is Tropicalization?

$$K = \mathbb{C}((t^{\mathbb{R}})) \quad v(at^{\alpha} + \text{ higher order terms }) = \alpha$$

If $I \subset K[x_1, \ldots, x_n]$ is an ideal, we define

$$\text{Trop } I = v((K^*)^n \cap \text{Var}(I)).$$

If $w \in \mathbb{R}^n$, $f \in K[x_1, \ldots, x_n] \setminus \{0\}$, write

$$f(x_1, \ldots, x_n) = t^W \text{in}_w f(t^{w_1} x_1, \ldots, t^{w_n} x_n) + \sum t^{W_j} g_j(t^{w_1} x_1, \ldots, t^{w_n} x_n)$$

with $W_j > W$, $\text{in}_w f \in \mathbb{C}[x_1, \ldots, x_n] \setminus \{0\}$. This defines $\text{in}_w(f)$.

$$\text{in}_w I =< \text{in}_w(f) : f \in I >$$

Theorem (Kapranov).

$$\text{Trop } I = \{w \in \mathbb{R}^n : \text{in}_w I \text{ contains no monomial}\}.$$

From now on, we are interested only in the case that I is the ideal of a linear space L of rank k in n-space. Write

$$L = \text{RowSpan} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k1} & x_{k2} & x_{k3} & \cdots & x_{kn} \end{pmatrix}.$$

The Plücker coordinates of L are

$$p_{i_1 \ldots i_k}(I) = \det \begin{pmatrix} x_{1i_1} & \cdots & x_{1i_k} \\ \vdots & \vdots & \vdots \\ x_{ki_1} & \cdots & x_{ki_k} \end{pmatrix}.$$

For simplicity, assume all the $p_{i_1 \ldots i_k}(I)$ are nonzero.

Proposition. $\text{in}_w I$ *is also the ideal of a linear space. We have* $p_{i_1 \ldots i_k}(\text{in}_w I) \neq 0$ *iff, for all* $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$,

$$v(p_{i_1 \ldots i_k}(I)) - w_{i_1} - \cdots - w_{i_k} \leq v(p_{j_1 \ldots j_k}(I)) - w_{j_1} - \cdots - w_{j_k}.$$

The previous proposition can be viewed geometrically. Define

$$\Delta(k,n) = \text{ConvexHull}\ (e_{i_1} + \cdots e_{i_k}) \subset \mathbb{R}^n.$$

Consider the lower convex hull of

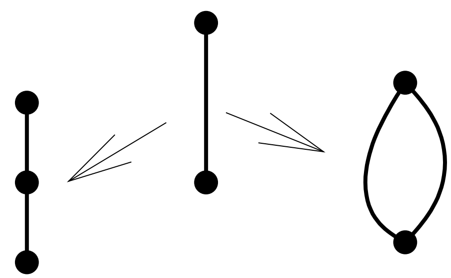
$$\{(e_{i_1} + \cdots + e_{i_k}, v(p_{i_1 \ldots i_k}))\} \subset \Delta(k,n) \times \mathbb{R}.$$

Project this lower hull to $\Delta(k,n)$ to get a subdivision of $\Delta(k,n)$. Trop I will be a subcomplex of the dual complex to this subdivision. A face F is dual to Trop I iff F does not lie in any of the n facets

$$\text{ConvexHull}\ (e_{i_1} + \cdots e_{i_k} : j \notin \{i_1, \ldots i_k\})$$

for $1 \leq j \leq n$. A face F is dual to a bounded face of Trop I iff F is in the interior of $\Delta(k,n)$.

A matroid is Series-Parallel if it corresponds to a graph which can be obtained from a single edge by the composition of series and parallel extensions as below.



The Series and Parallel Extensions

Conjecture. *Let f_i be the number of dimension i bounded faces of a tropical linear space of dimension k in n-space. Then*

$$f_i \leq \binom{n-i-2}{i}\binom{n-2i-2}{k-i-1}$$

with equality iff every facet of the corresponding decomposition of $\Delta(k,n)$ is corresponds to a Series-Parallel matroid.

Theorem. *The collection of Series-Parallel tropical linear spaces is closed under transverse intersections and dualization. In particular, the intersection of* $n-k$ *transverse hyperplanes (a "tropical complete intersection") is Series-Parallel.*

Random sampling of complete intersections with $(k,n) = (3,7)$, $(3,8)$, $(3,9)$, $(4,8)$ and $(4,9)$ suggests the conjecture is valid.

Theorem. *Let* $t_M(z,w)$ *denote the Tutte polynomial of a matroid* M*. We abuse notations by identifying a matroid with its corresponding polytope. For* $|M| \geq 2$*, the* β *invariant of* M *is defined by*

$$t_M(z,w) = \beta(M)(z+w) + \dots.$$

Let $\mathcal{D}$ *be a decomposition of* $\Delta(k,n)$ *into matroidal polytopes and let* $\mathring{\mathcal{D}}$ *denote the internal faces of* $\mathcal{D}$*.*

$$t_{\Delta(k,n)}(z,w) = \sum_{M \in \mathring{\mathcal{D}}} (-1)^{\mathrm{codim}(M)} t_M(z,w).$$

$$\binom{n-2}{k-1} = \beta(\Delta(k,n)) = \sum_{F \text{ a facet of } \mathcal{D}} \beta(F).$$

Corollary. $f_0 \leq \binom{n-2}{k-1}$ *with equality iff* $\beta(F) = 1$ *for all facets of* $\mathcal{D}$ *iff* $\mathcal{D}$ *is Series-Parallel.*

The second equivalence is due to Crapo.

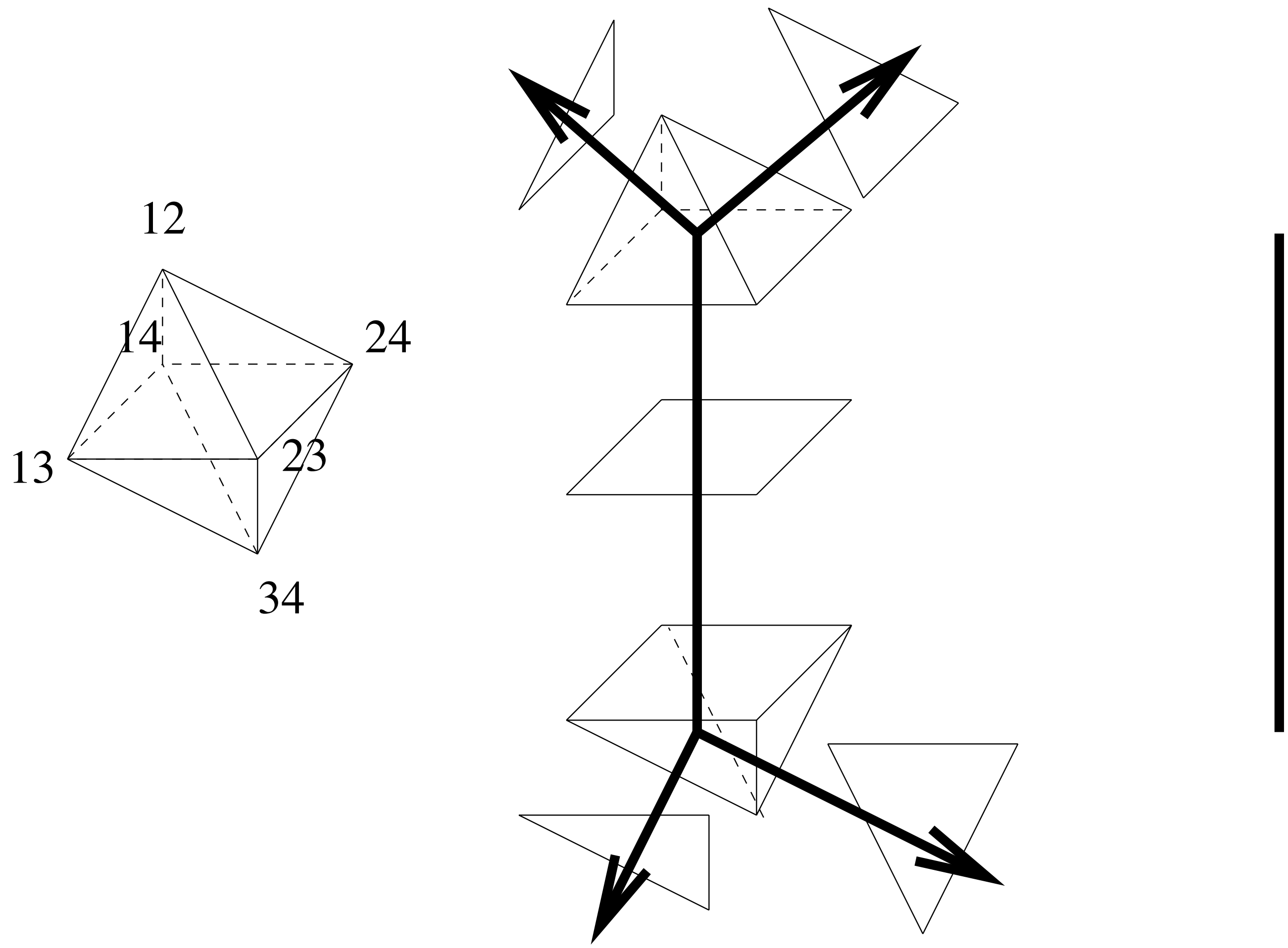
Theorem. *Suppose* $(x_1, \ldots x_n)$ *and* $(y_1, \ldots, y_n) \in K^n$ *are such that*

$$\text{Trop RowSpan} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

is a trivalent tree. Then

$$\text{Trop RowSpan} \begin{pmatrix} x_1^{k-1} & x_2^{k-1} & \cdots & x_n^{k-1} \\ x_1^{k-2}y_1 & x_2^{k-2}y_2 & \cdots & x_n^{k-2}y_2 \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{k-1} & y_2^{k-1} & \cdots & y_n^{k-1} \end{pmatrix}$$

achieves the bound in the conjecture.


12
14
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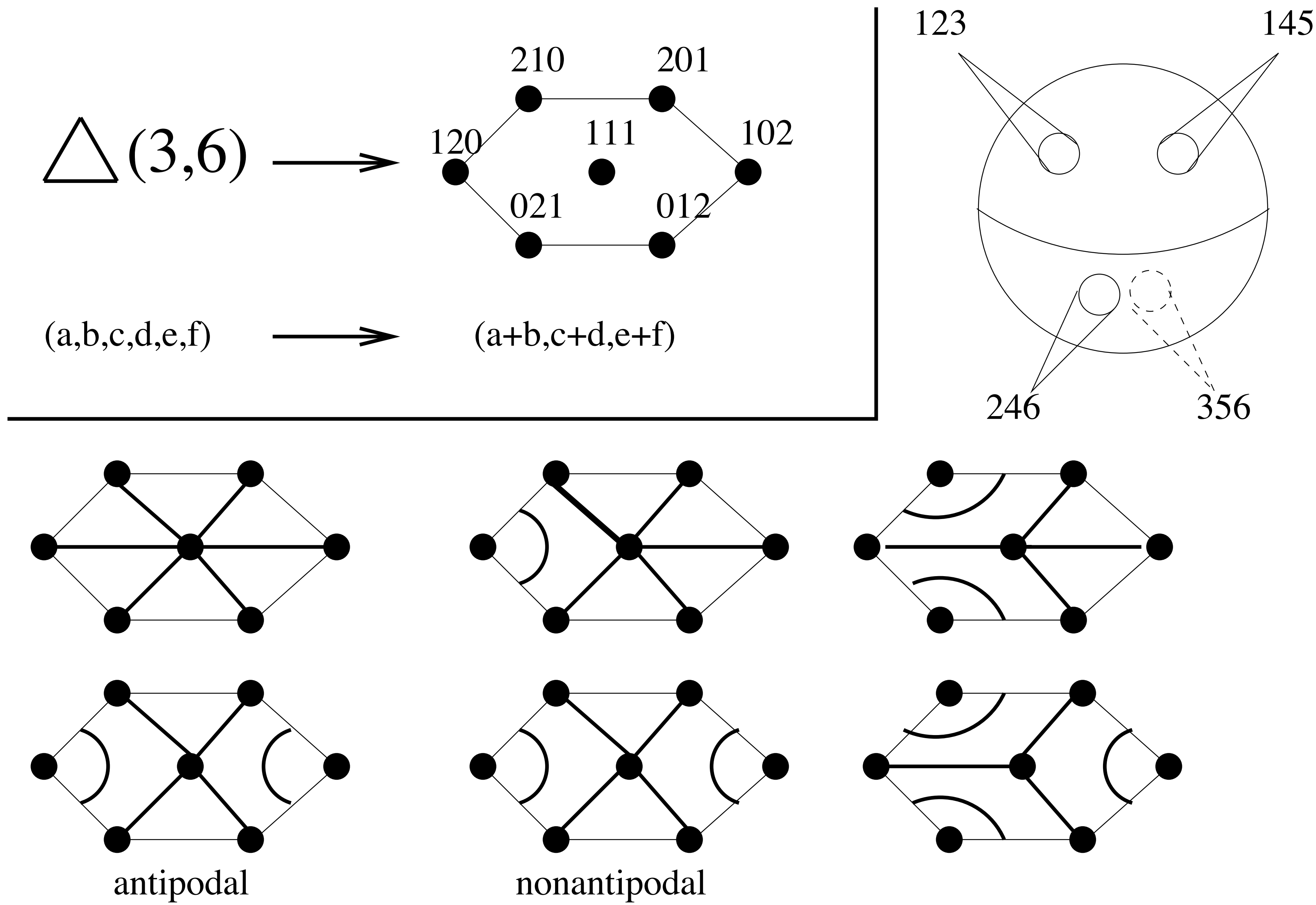
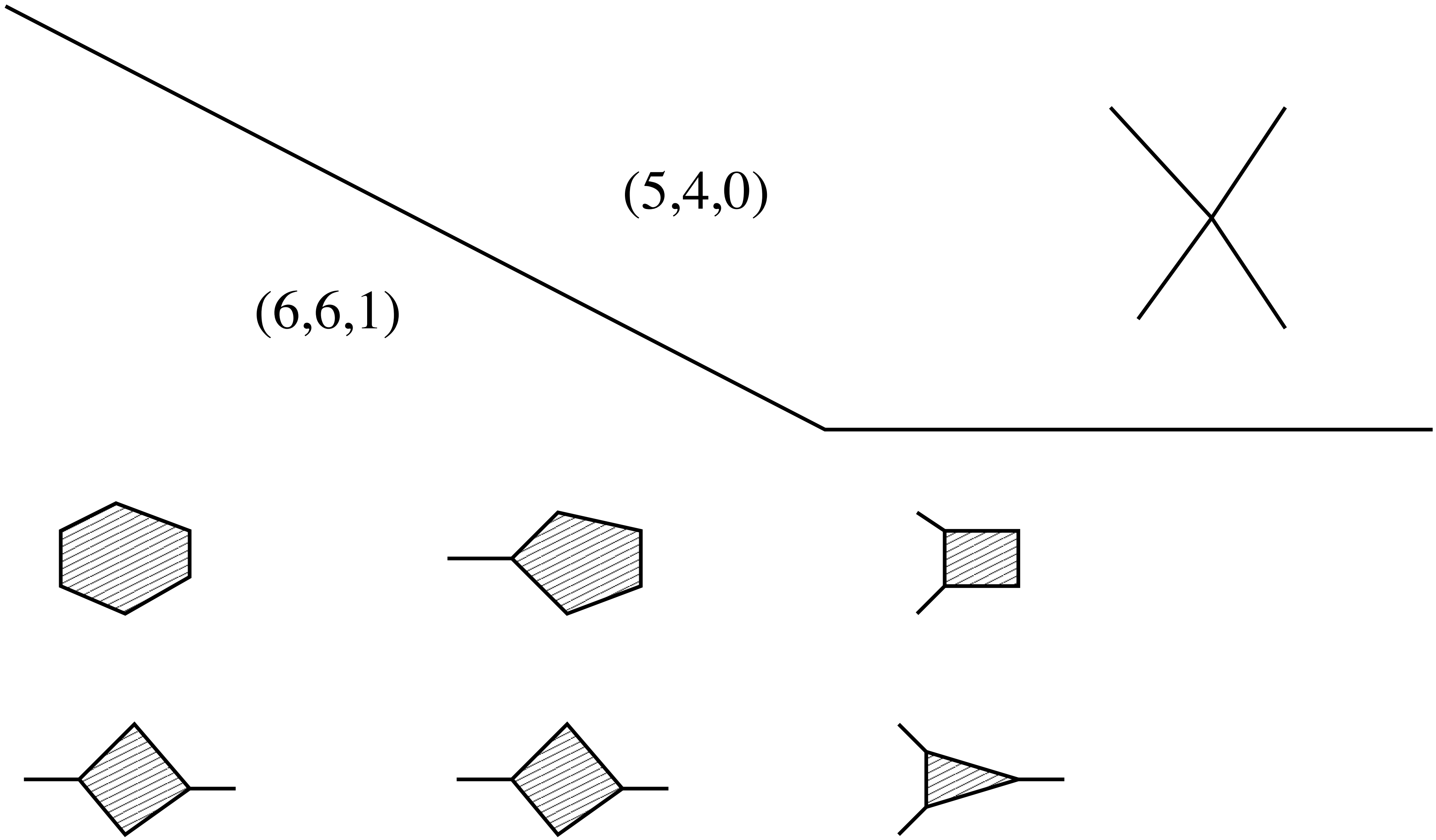
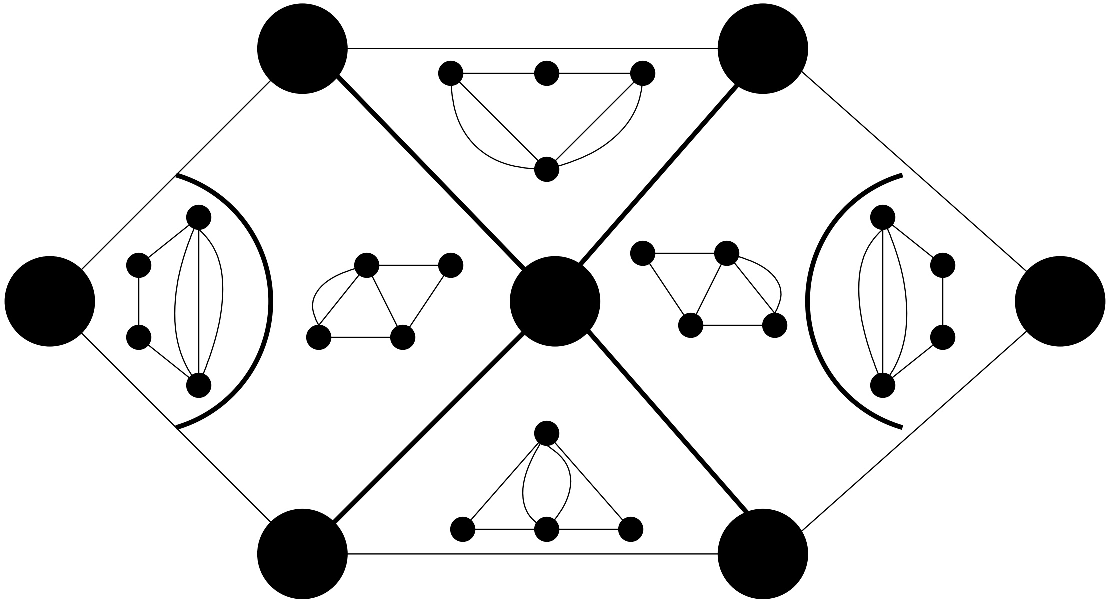
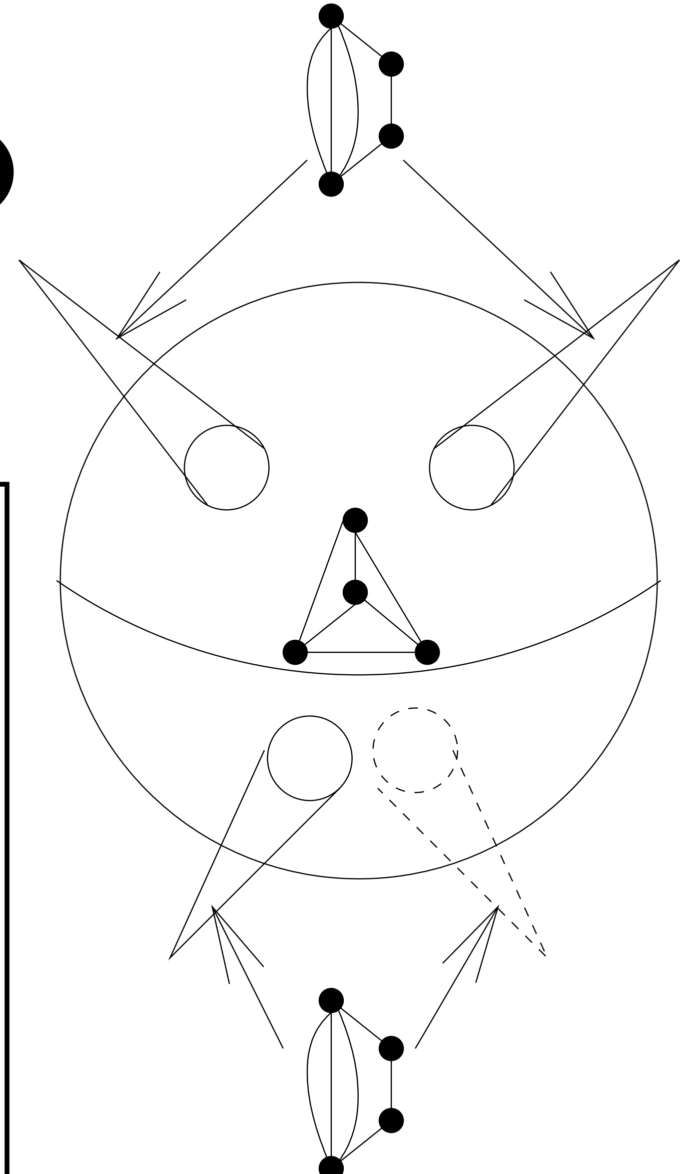
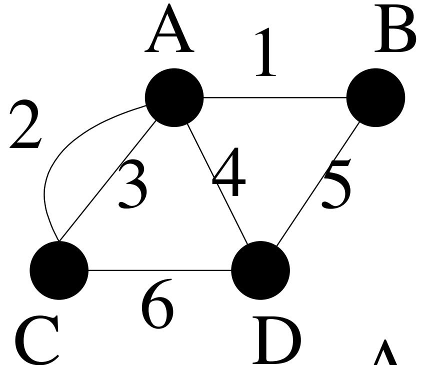


(3,6)
210
201
120
111
102
021
012
(a,b,c,d,e,f)
(a+b,c+d,e+f)
123
145
246
356
antipodal
nonantipodal


(5,4,0)
(6,6,1)







$$
\begin{array}{c}
\\ A \\ B \\ C \\ D
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1
\end{pmatrix}
\end{array}
$$

In the above diagrams, a graph represents the matroid arising as the rowspan of the adjacency matrix.