On some constrained polynomial optimization problems in nonlinear computational geometry

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(joint work with René Brandenberg)

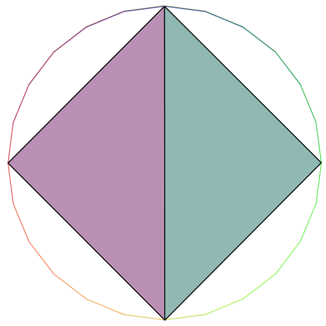
Radii of polytopes

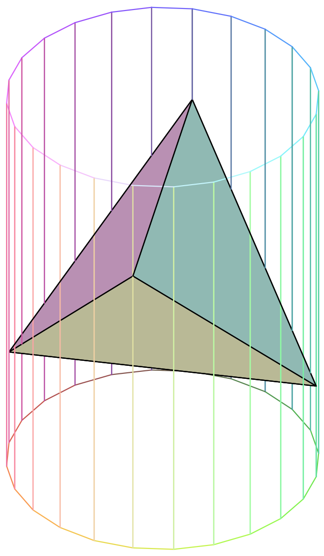
$\mathbb{E}^n$: n-dimensional Euclidean space
$\mathcal{L}_{j,n} :=$ the set of all j-dimensional subspaces in $\mathbb{E}^n$

For a polytope $P \subset \mathbb{E}^n$ and $L \in \mathcal{L}_{j,n}$:
$\pi_L(P) :=$ orthogonal projection of P on L

Outer j-radius of a polytope P:
$R_j(P) := \min_{L \in \mathcal{L}_{j,n}}$ j-dim. circumradius of $\pi_L(P)$

2-radius in $\mathbb{E}^3$:





Constrained polynomial optimization problems

Let $P = \text{conv}\,\{v^{(1)}, \ldots, v^{(m)}\}$
and $L^{\perp} = \text{lin}\,\{s^{(1)}, \ldots, s^{(n-j)}\}$ with pairwise orthogonal
$s^{(1)}, \ldots, s^{(n-j)} \in \mathbb{S}^{n-1}, \quad p \in L$.

$$
\begin{array}{rrcll}
 & \min \rho^2 & & & \\
\text{s.t.} & (p - \pi_L(v^{(i)}))^2 & \leq & \rho^2\,, & i = 1, \ldots, m \\
 & p \cdot s^{(k)} & = & 0\,, & k = 1, \ldots, n-j \\
 & s^{(1)}, \ldots, s^{(n-j)} & \in & \mathbb{S}^{n-1}, & \text{pairwise orthogonal}
\end{array}
$$

Already for the 2-radius of simplices in $\mathbb{E}^3$:

$\rightsquigarrow$ polynomial systems of degree $2 \cdot 18$ (Brandenberg, Th., AAECC 2004)

Regular simplex

T^n: regular simplex with edge length $\sqrt{2}$. Let $1 \le j \le n-1$.

$$R_1(T^n) = \begin{cases} \sqrt{\frac{1}{n+1}} & \text{if } n \text{ odd} \\ \sqrt{\frac{n+1}{n(n+2)}} & \text{if } n \text{ even} \end{cases}$$

$$R_j(T^n) = \sqrt{\frac{j}{n+1}} \qquad \text{for } 2 \le j \le n-2$$

$$R_{n-1}(T^n) = \sqrt{\frac{n-1}{n+1}} \qquad \text{if } n \text{ odd}$$

($n = 1$: classical; $n > 1$: Pukhov '80, Weißbach '83)

And the even case ...

In another paper, Weißbach reduced the determination of the outer $(n-1)$-radius for even n to

$$\min \sum_{i=1}^{n+1} u_i^4$$

$$\text{s.t.} \qquad \sum_{i=1}^{n+1} u_i^2 = 1\,, \quad \sum_{i=1}^{n+1} u_i = 0$$

Using Lagrange multipliers λ_1, λ_2 yields the equations

$$\begin{aligned} 4u_i^3 + 2\lambda_1 u_i + \lambda_2 &= 0\,, \qquad 1 \le i \le n+1 \\ \sum_{i=1}^{n+1} u_i^2 = 1\,, &\quad \sum_{i=1}^{n+1} u_i = 0 \end{aligned}$$

A SINGULAR *computation*

Errneously, it is argued that symmetry arguments imply that $\lambda_2 = 0$ in any solution (gives 50 solutions for $n = 4$).

SINGULAR computation:

```
ring R = 0, (u1,u2,u3,u4,u5,la1,la2), dp;
ideal I = 4*u1^3 + 2*la1*u1 + la2,
          4*u2^3 + 2*la1*u2 + la2,
          4*u3^3 + 2*la1*u3 + la2,
          4*u4^3 + 2*la1*u4 + la2,
          4*u5^3 + 2*la1*u5 + la2,
          u1^2 + u2^2 + u3^2 + u4^2 + u5^2 - 1,
          u1 + u2 + u3 + u4 + u5;
degree(std(I));
```

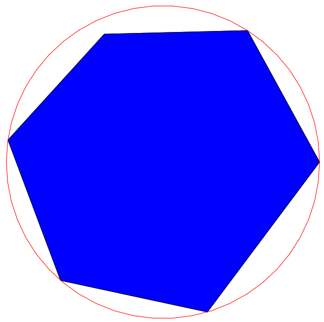
Output:

```
codimension = 7
dimension   = 0
degree      = 80
```

30 additional solutions with $\lambda_2 \neq 0$, which are also real.

Enclosing vs. circumscribing

In general, not all vertices of the polytope are projected onto the enclosing sphere in a minimal projection (not even for simplices)
$\rightsquigarrow \ \leq$ vs. $=$



Theorem. Let $1 \leq j \leq n < m$ and $P = \mathrm{conv}\{v^{(1)}, \ldots, v^{(m)}\} \subset \mathbb{E}^n$ be an n-polytope. Then one of the following is true.

a) In every R_j-minimal projection of P there exist $n+1$ affinely independent vertices of P which are projected onto the minimal enclosing j-sphere.

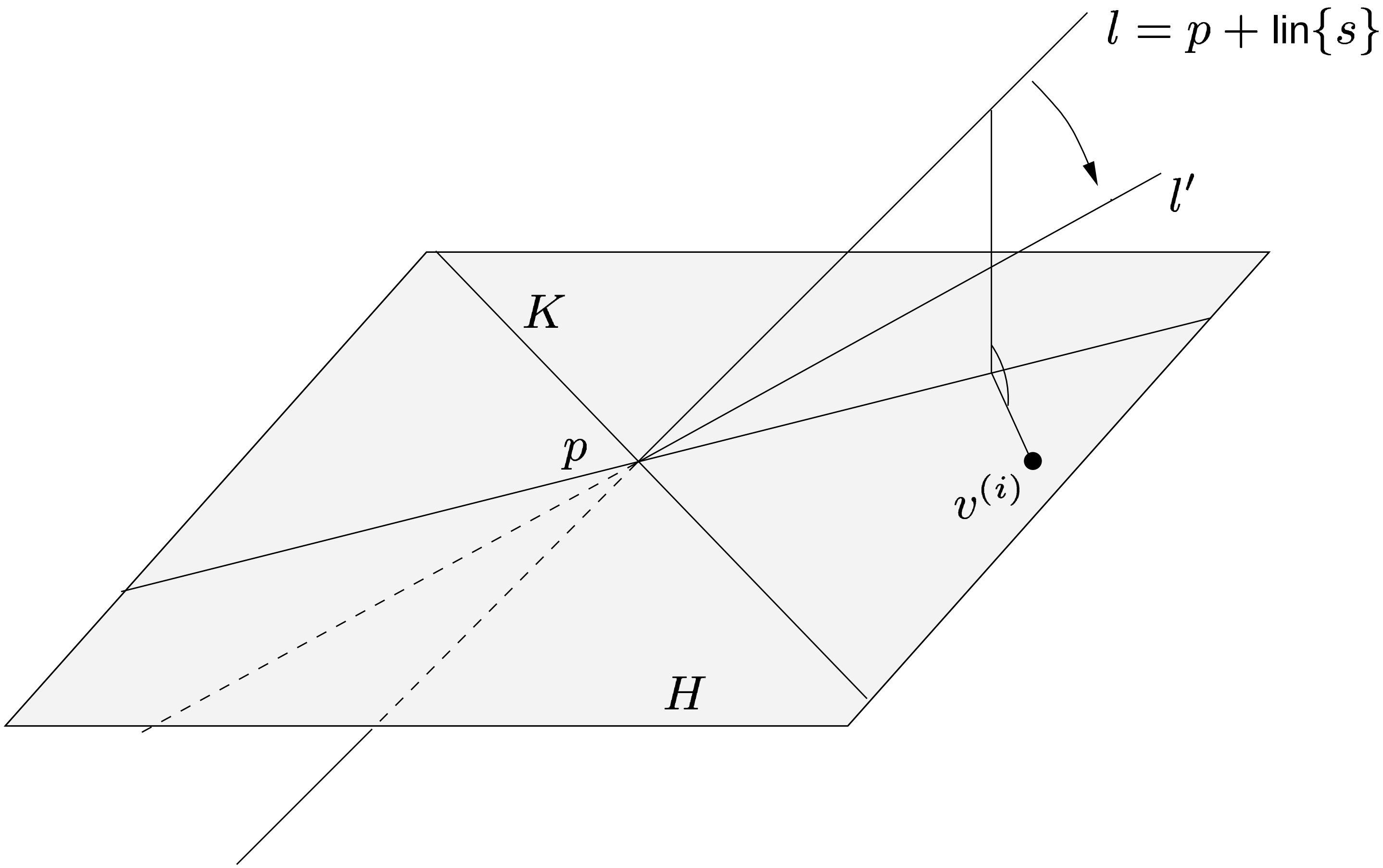
b) $R_j(P) = R_{j-1}(P \cap H)$ for $j \geq 2$ and some hyperplane $H = \mathrm{aff}\{v^{(i)} : i \in I\}$ with $I \subset \{1, \ldots, m\}$.

If $j = 1$ or if P is a regular simplex then case a) holds.

Proof idea (for the regular simplex and $j = n - 1$)

Show: Every minimal enclosing cylinder is circumscribing.

H: hyperplane underlying one of the facets



The optimization problem

$\mathbf{T}^n = \text{conv}\,\{e^{(1)}, \ldots, e^{(n+1)}\}$ ($\subset$ hyperplane $\sum_{i=1}^{n+1} x_i = 1$ in $\mathbb{E}^{n+1}$).
$s :=$ projection direction.

$$\min \sum_{i=1}^{n+1} s_i^4\,, \quad \text{s.t. } \sum_{i=1}^{n+1} s_i^3 \;=\; 0\,, \;\sum_{i=1}^{n+1} s_i^2 \;=\; 1\,, \;\sum_{i=1}^{n+1} s_i \;=\; 0$$

Lagrange multipliers yield $|\{s_1, \ldots, s_{n+1}\}| \leq 3\,.$

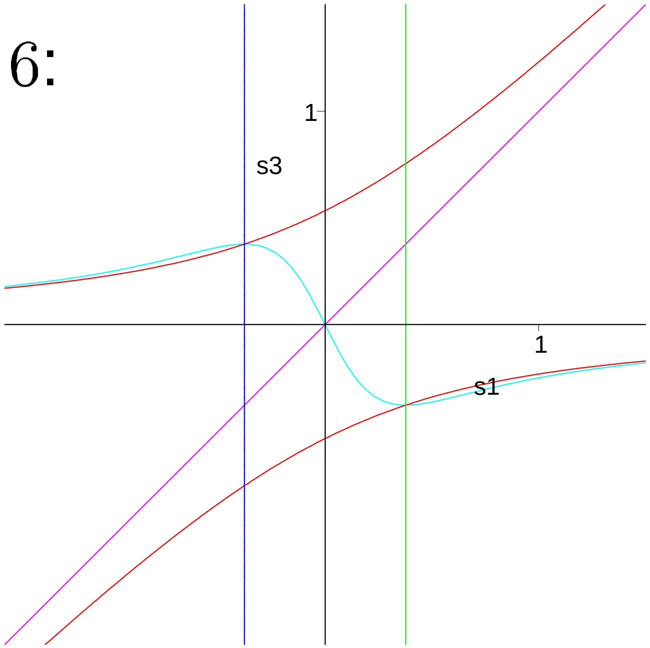
$$\begin{array}{lrcl} & \min\; k_1 s_1^4 + k_2 s_2^4 + k_3 s_3^4 & & \\ \text{s.t.} & k_1 s_1^3 + k_2 s_2^3 + k_3 s_3^3 & = & 0 \\ & k_1 s_1^2 + k_2 s_2^2 + k_3 s_3^2 & = & 1 \\ & k_1 s_1 + k_2 s_2 + k_3 s_3 & = & 0 \\ & k_1 + k_2 + k_3 & = & n+1 \\ & s_1, s_2, s_3 \in \mathbb{R}, \quad k_1, k_2, k_3 \in \mathbb{N}_0 & & \end{array}$$

Integer constraints irrelevant for n odd, but crucial for n even.

Solving the system

k_1, k_2, k_3, s_2 can be rationally expressed in s_1, s_3. Properties (such as $k_1 \leq (n+1)/2$) are regions bounded by plane algebraic curves.

E.g., $k_1 \leq (n+1)/2$ for $n = 6$:



Theorem.

$$R_{n-1}(T^n) = \begin{cases} \sqrt{\frac{n-1}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2\sqrt{n(n+1)}} & \text{if } n \text{ is even.} \end{cases} \qquad \text{(edge length } \sqrt{2}\text{)}$$

Connections to the Real Nullstellensatz

Stengle ('74): A polynomial system in m variables

$$f(x) \geq 0, \; g_1(x) = 0, \ldots, g_r(x) = 0$$

either has a solution $x \in \mathbb{R}^m$, or there exists a polynomial identity

$$\sum_{i=1}^{r} a_i g_i + (\sum_{j=1}^{u} b_j^2) f + \sum_{k=1}^{v} c_k^2 + 1 = 0$$

with $u, v \in \mathbb{N}_0$ and $a_i, b_j, c_k \in \mathbb{R}[x_1, \ldots, x_m]$.

For n odd: polynomial identity ($\rightsquigarrow$ degree bound of 4 for every n)

$$\sum_{i=1}^{n+1} s_i^4 - \frac{1}{n+1} = \frac{2}{n+1}(\sum_{i=1}^{n+1} s_i^2 - 1) + \sum_{i=1}^{n+1}(s_i^2 - \frac{1}{n+1})^2$$

For n even: already for $n = 4$ degree 8 necessary