Lower Bounds for Some Sparse Polynomial Systems

(with F. Sottile)

Consider a system of real polynomial equations:

$$f_1(t_1, \dots, t_n) = 0$$

$$\dots$$

$$f_n(t_1, \dots, t_n) = 0$$

$r :=$ # real solutions

$d :=$ # complex solutions

$$r \geq d \bmod 2$$

We try to do better.

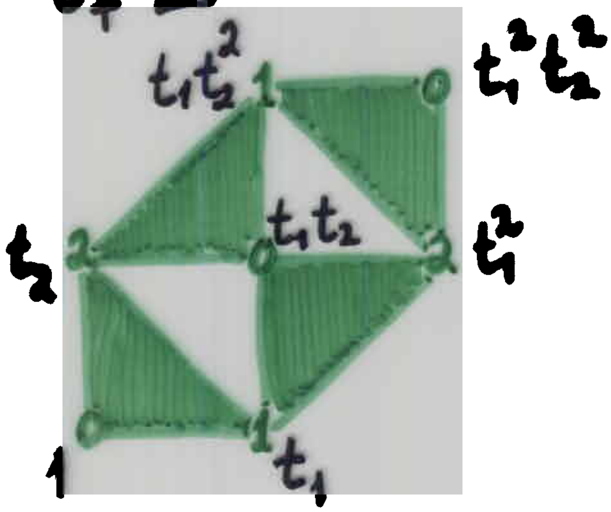
Eremenko & Gabrielov: computed the degree of the Wronski map on the real Grassmannian, which is a lower bound to certain problems from the Schubert calculus.

$\Delta \subset \mathbb{R}^n$ - lattice polytope

$m \in \Delta \cap \mathbb{Z}^n \longleftrightarrow$ monomial $t^m = t_1^{m_1} \cdots t_n^{m_n}$

$f = \sum_{m \in \Delta} a_m t^m$ - polynomial w/ support Δ

Δ_w - regular unimodular triangulation of Δ whose dual graph is bipartite.



Let t^{m_1} and t^{m_2} in f have same coefficients if m_1 and m_2 fold onto each other under the natural 'folding' of Δ_w

(*) $f(t) = a_0(1 + t_1 t_2 + t_1^2 t_2^2) + a_1(t_1 + t_1 t_2^2) + a_2(t_2 + t_2^2 t_1)$

$$f(t) = \sum_m a_{k(m)} t^m$$

(*) Consider a system of n such equations

Sometimes, # real solutions $\geq$ sign imbalance of Δ_w

For example, a system of two polynomial equations of the form (*) has at least two real solutions.

Define $\varphi_\Delta : t \in (\mathbb{C}^\times)^n \longmapsto [t^m \mid m \in \Delta] \in \mathbb{P}^\Delta$

Set $X_\Delta = \overline{\varphi_\Delta((\mathbb{C}^\times)^n)} \subset \mathbb{P}^\Delta$

- projective toric variety

Polynomial w/ support Δ

$\longleftrightarrow$ hyperplane section of X_Δ

$$\sum_{m \in \Delta} a_{\kappa(m)} t^m \longleftrightarrow \sum_{m \in \Delta} a_{\kappa(m)} x_m$$

System of n equations $\longleftrightarrow \Lambda \cap X_\Delta$

The folding gives a projection

$\pi : \mathbb{P}^\Delta \to \mathbb{P}^n$ by $\quad x_m \longmapsto x_{\kappa(m)}$

For the hexagon:

$[x_0 : \ldots : x_6] \longmapsto [x_0 + x_3 + x_6 : x_2 + x_4 : x_1 + x_5]$

$\pi : \Lambda \longrightarrow$ point p

solutions to $(*) = F^{-1}(p)$, where

$F = \pi|_{X_\Delta}$

/$\mathbb{R}$

$$Y_\Delta := X_\Delta(\mathbb{R}) \qquad f := \pi\big|_{Y_\Delta}$$

$$f : Y_\Delta \longrightarrow \mathbb{RP}^n$$

if Y_Δ, $\mathbb{RP}^n$ - orientable, $\# f^{-1}(p) \geq \deg f$

$$\begin{tikzcd} Y_\Delta^+ \arrow[d] & \subset & S^\Delta \arrow[r, dashed, "\pi^+"] \arrow[d] & S^n \arrow[d] \\ Y_\Delta & \subset & \mathbb{RP}^\Delta \arrow[r, dashed, "\pi"] & \mathbb{RP}^n \end{tikzcd} \qquad f^+ := \pi^+\big|_{Y_\Delta^+}$$

if Y_Δ^+ is orientable, define

$$\operatorname{char} f := \deg f^+$$

Proposition $\# f^{-1}(p) \geq \operatorname{char} f$

Let $\Delta = \{x \in \mathbb{R}^n : Ax \geq -b\}$

Theorem If 1) $\Delta \cap \mathbb{Z}^n$ affinely spans $\mathbb{Z}^n$

2) Integer column span of A has odd index in its saturation

3) There is an odd vector in the integer column span of $[A : b]$

then Y_Δ^+ is orientable

How to compute charf ?

a regular unimodular triangulation of Δ defines an $\mathbb{R}^{\times}$-action on S^{Δ}

$\lim_{s \to 0} s.\, Y_{\Delta}^{+} =$ union of coordinate spheres, one for each simplex Δ_ω

Theorem If

1) $s.\, Y_{\Delta}^{+}$ does not intersect center of projection
2) integer column span of A has odd index in its saturation
3) there is an odd vector in the integer column span of $[A : b]$

then charf = sign imbalance of Δ_ω

Toric varieties from posets

P - finite poset, order polytope:

$$\mathcal{O}(P) = \{ f : P \to [0,1] \mid a < b \Rightarrow f(a) < f(b) \}$$

vertices = char. fn's of upper order ideals

- $Y^{+}_{\mathcal{O}(P)}$ is orientable if all max chains in P have length of the same parity

- $\mathcal{O}(P)$ has a canonical triangulation defined by linear extensions. It is regular, unimodular, and bipartite with sign imbalance = $\sigma(P)$, sign imbalance of P.
- s. $Y^{+}_{\mathcal{O}(P)}$ does not meet center of projection

I - order ideal $t^{I} = \prod_{a \in I} t_a \in \mathbb{R}[t_a \mid a \in P]$

$c = (c_0, \dots, c_{\#P}) \in (\mathbb{R} \setminus 0)^{\#P+1}$

$|I| = \#I$,

$$f_c(t) = \sum_{I} c_{|I|} t^{I}$$

Theorem If all max chains in P have length of the same parity, a system of $\#P$ polynomials of the form ⊛ has at least $\sigma(P)$ solutions

Systems from Chain Polytopes

P - finite poset, chain polytope:

$$C(P) = \{ f : P \to [0,1] \mid f(a) + f(b) + \dots + f(c) \leq 1$$
whenever $a < b < \dots < c$ is a chain in $P \}$

vertices = char. fn's of antichains

- $Y^+_{C(P)}$ is orientable if all max chains in P have length of the same parity
- $C(P)$ has a canonical triangulation, which is regular, unimodular, bipartite, and has same sign imbalance $\sigma(P)$ as $O(P)$.
- s. $Y^+_{C(P)}$ does not meet center of projection

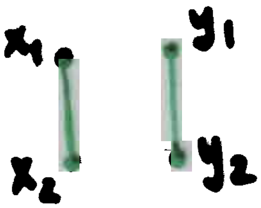
A - antichain $\longmapsto \langle A \rangle = \{ a \in P \mid a \geq b$ for some $b \in P \}$

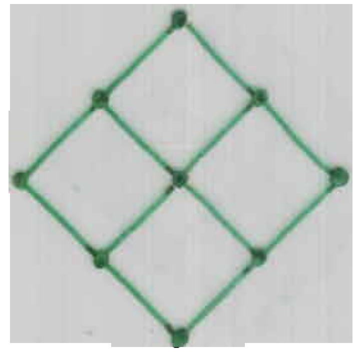
Let $|A| := \# \langle A \rangle$, $c = (c_0, \dots, c_{\# P}) \in (\mathbb{R} \setminus 0)^{\# P + 1}$

$$** \quad f_c(t) = \sum_A c_{|A|} t^A, \quad t^A = \prod_{a \in A} t_a \in \mathbb{R}[t_a \mid a \in P]$$

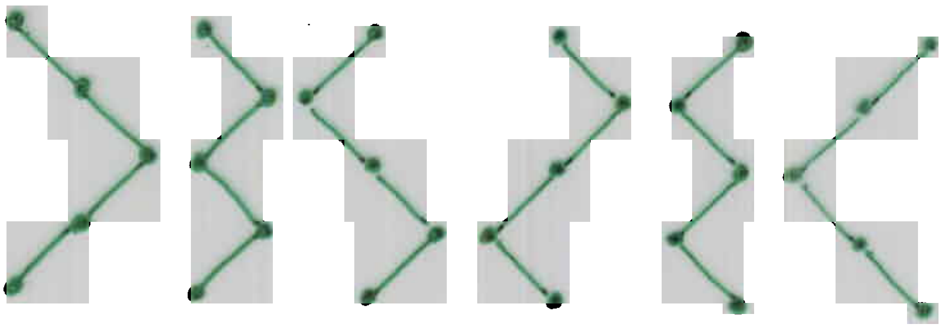
Theorem If all max chains of P have length of the same parity, a system of $\# P$ polynomials of the form ** has at least $\sigma(P)$ solutions

Example

$P =$ 

lattice of order ideals: 

$\sigma(P) = \#$ positive max chains $-$ $\#$ neg. max chains:



$\sigma(P) = 2$

$+ \quad - \quad + \quad + \quad - \quad +$

$$f(x,y) = c_0 + c_1(x_1 + y_1) + c_2(x_2 + x_1 y_1 + y_2) + c_3(x_2 y_1 + x_1 y_2) + c_4 x_2 y_2, \quad c_i \in \mathbb{R} \setminus 0$$

Theorem A system of 4 such polynomial equations has at least 2 real solutions.

Gaps

$$\mathbb{P}^{2k_1} \times \mathbb{P}^{2k_2} \times \dots \times \mathbb{P}^{2k_m}$$

$k_1,\dots,k_m$	Observed # of real solutions	# of complex solutions
1,1	2 6	6
1,2	3 7 15	15
1,3	4 8 16 28	28
1,4	5 9 17 29 45	45
1,5	6 10 18 $\boxed{30}$ $\boxed{46}$ 66	66
2,2	6 14 30 70	70
2,3	10 22 46 $\boxed{98}$ 210	210
1,1,1	6 18 90	90

$m>1$ $\quad N_{k_1,\dots,k_m} :=$ N^{th} number in the set of numbers of real solutions attained by systems on $\mathbb{P}^{2k_1} \times \dots \times \mathbb{P}^{2k_m}$

$m=1$ $\quad N_k := 1$ if $N < k$

Conjecture (1) # of attained solutions $= k_1 + \dots + k_m$

(2) $1_{k_1\dots k_m} = \binom{k_1+k_2+\dots+k_m}{k_1,\, k_2,\, \dots,\, k_m}$, $\quad (k_1+\dots+k_m)_{k_1\dots k_m} = \binom{2k_1+\dots+2k_m}{2k_1,\, \dots,\, 2k_m}$

(3) $$N_{k_1,k_2,\dots,k_m} = \sum_{j=1}^{m} (N)_{k_1,\dots,k_j-1,\dots,k_m}$$