Upper bounds for some sparse polynomial systems.

Joint with F. Bihan and F. Sottile

Let (S) be a general system of

n polynomials in n unknowns.

Let d be the total number of different

exponent vectors involved in (S)

Thm [Khovanski]

The number of real solutions of (S) in $(\mathbb{R}^*)^n$

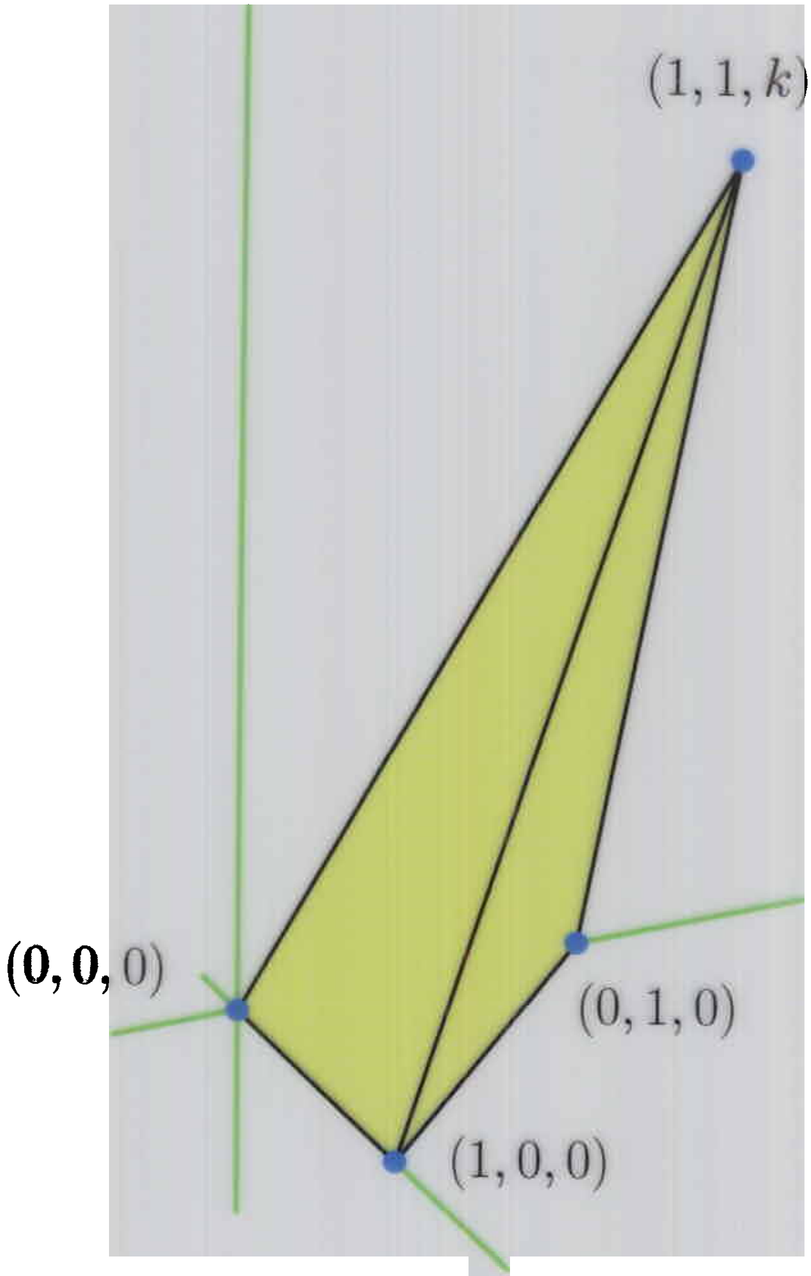
is no more than

$$X(n,d) = 2^n 2^{\binom{d}{2}} \cdot (n+1)^d$$

Li - Rojas - Wang

Thm: The number of isolated real roots of a system of two 3-nomials in the plane is less or equal to 20

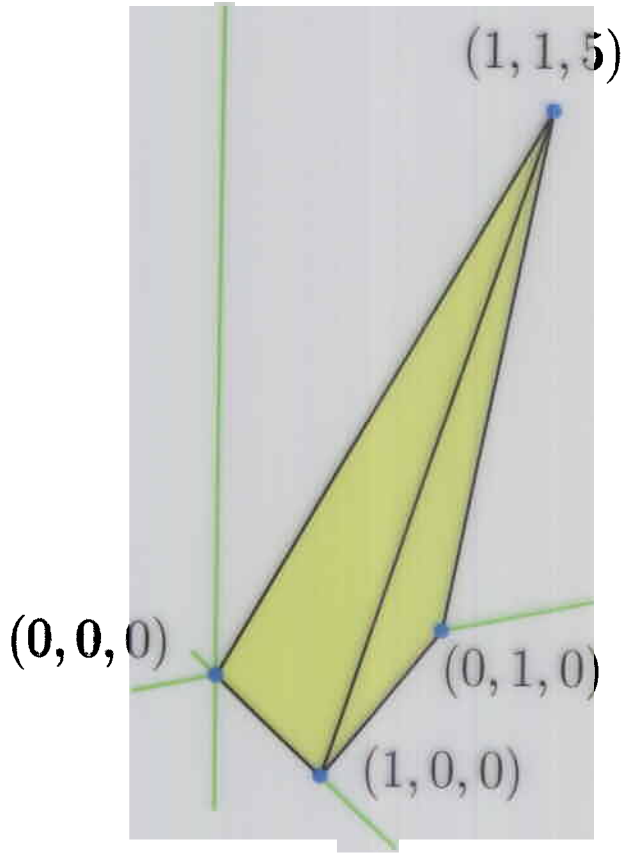
Rmk: $20 \ll 4 \cdot 2^{30} \cdot 3^{6}$


$(1,1,k)$
$(0,0,0)$
$(0,1,0)$
$(1,0,0)$

The polynomial

$$f = 1 + x - y + xyz^5$$

has Newton polytope



A system with this support

$$\begin{aligned} A_1xyz^5 + B_1x + C_1y + D_1 &= 0\,, \\ A_2xyz^5 + B_2x + C_2y + D_2 &= 0\,, \\ A_3xyz^5 + B_3x + C_3y + D_3 &= 0\,, \end{aligned}$$

is equivalent to

$$z^5 = a, \quad x = b, \quad y = c\,,$$

which has 5 complex solutions, but only 1 is real.

Let P be a polytope with integer vertices in $(\mathbb{R}^+)^n$.

Suppose that P admits a regular unimodular triangulation.

Thm [Sturmfels]

There exist polynomials $g_1, \dots, g_n$ with Newton polytope P such that all the solutions of the system $(S): g_i = 0, \ i = 1..n$ are real.

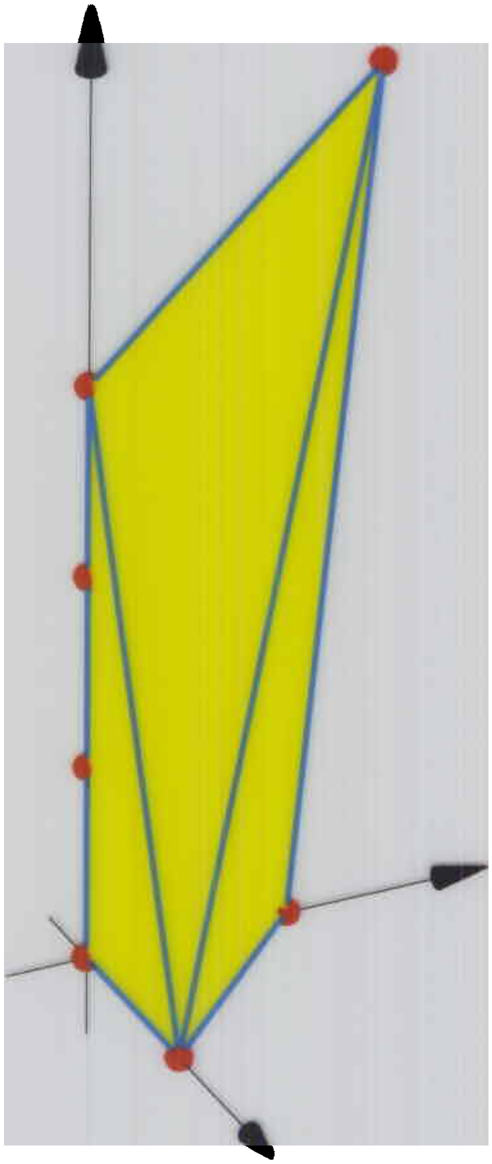
Let $\Delta_{k,\ell}$ be the convex hull of

$(0,0,0), (1,0,0), (0,1,0), (0,0,k), (1,1,\ell)$

Thm: The number r of real solutions to a general system of 3 real polynomials with support $\Delta_{k,\ell}$ satisfies

$$0 \leq r \leq \min\{2k+\ell, 3k+2\}$$

Moreover, every number in this interval, congruent to $2k+\ell$ mod 2 occurs.



$\Delta_{3,5}$

proof:

$$(S): a_{i1}x + a_{i2}y + a_{i3}xyz^{\ell} + F_i(z) = 0, \quad i = 1..3$$

$$\deg F_i = k$$

Then perturbing slightly the matrix (a_{ij}) if necessary, we can assume that it is invertible (small perturbation will not change the number of solutions).

We perform Gaussian elimination on (S):

$$(S) \Leftrightarrow \begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 xyz^{\ell} = F_1'(z) \\ \beta_2 y + \gamma_2 xyz^{\ell} = F_2'(z), \\ \gamma_3 xyz^{\ell} = F_3'(z) \end{cases}$$

$$(S) \iff \begin{cases} x = g_1(z) \\ y = g_2(z) \\ xyz^\ell = g_3(z) \end{cases} \qquad \deg g_i = k$$

(3)

Substituting in (3) we get:

$$f(z) = z^\ell g_1(z) g_2(z) - g_3(z) = 0$$

assume $\ell \geq k+1$,

$$f^{(k+1)}(z) = z^{\ell-(k+1)} \cdot Q$$

with $\deg Q = 2k$

then $f^{(k+1)}$ has at most $2k+1$ roots

and by Rolle Theorem

f has at most $3k+2$ roots.

Sharpness:

Find g_1, g_2, g_3 such that

$$z^\ell g_1 g_2 - g_3 \text{ has } 3k+\delta \text{ real roots}$$

$$\text{where } \delta = \begin{cases} 1 & \text{if } \ell-k \text{ is odd} \\ 2 & \text{if } \ell-k \text{ is even} \end{cases}$$

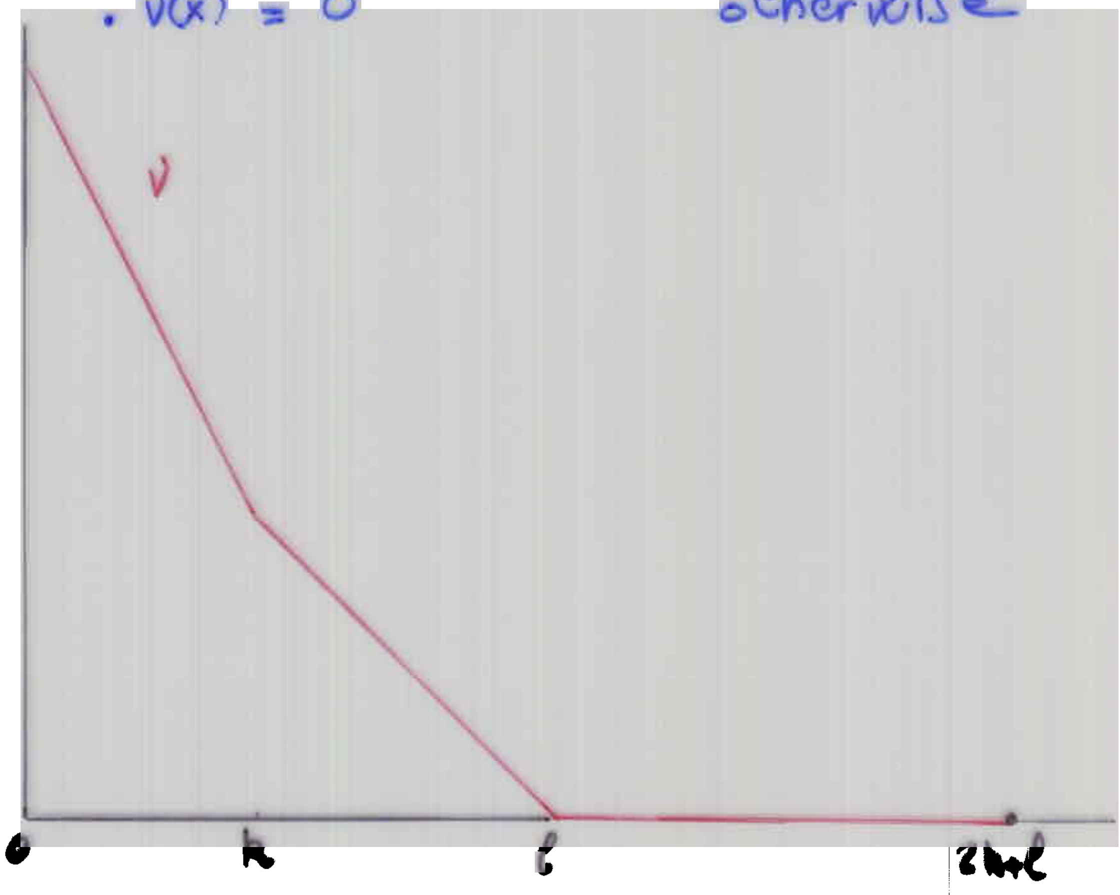
- Take $f_3(x) = c_k x^k + \text{———} + c_0$ with k real roots and $c_k > 0$
- Take f_1, f_2 such that $f_1 . f_2$ has $2k$ real roots and $f_1(0) . f_2(0) > 0$

Consider the piecewise linear convex function

$$v : [0, 2k+\ell] \longrightarrow \mathbb{R}$$

defined by:

- $v(x) = -2x + \ell + 2k$ if $x \in [0, k]$
- $v(x) = -x + \ell$ if $x \in [k, \ell]$
- $v(x) = 0$ otherwise



$$f_t(x) := \sum c_j t^{\nu(j)} x^j + x^\ell . f_1 . f_2$$

The number of real roots of the binomial

$$f_1(0) . f_2(0) \, x^\ell - c_k x^k \quad \text{in } \mathbb{R}^* \text{ is } \delta$$

Thus, by Viro Theorem,

for $t>0$ small enough,

f_t has $3k + \delta$ real roots $\blacksquare$

<u>Rmk</u>: All other admissible values are obtained by picking the f_i's with less real roots

In dimension n

Let $\Delta^n_{k,\ell} \subset \mathbb{R}^n$ be the convex hull of

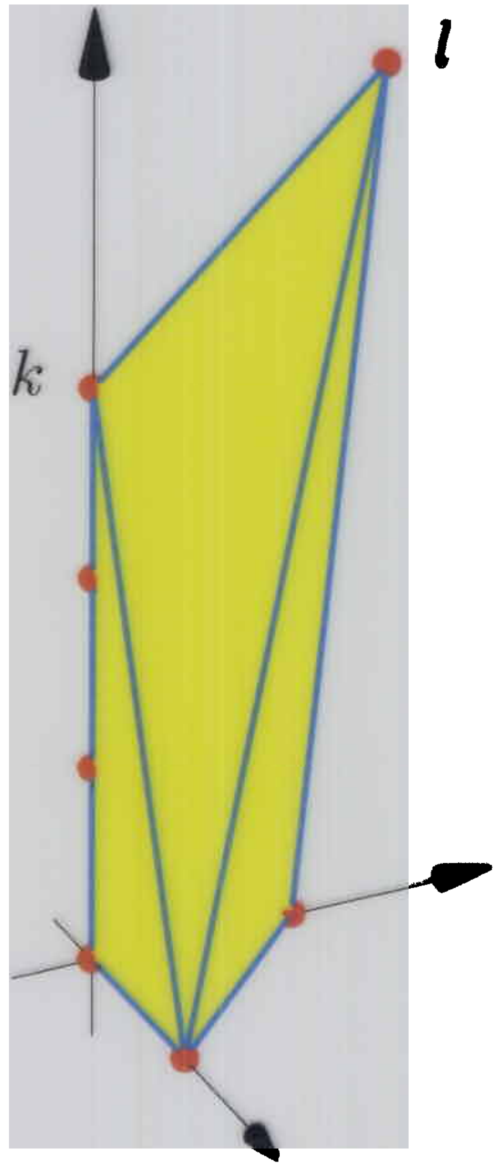
$$(0,\ldots,0),\ (1,0,\ldots,0),\ (0,1,0,\ldots,0),\ \ldots,\ (0,\ldots,0,1)$$

$$(0,\ldots,0,k),\ (1,\ldots,1,\ell)$$

Thm: The number r of real solutions to a general system of real polynomials with support $\Delta^n_{k\ell}$ satisfies

$$0 \le r \le \min\{(n-1)k+\ell,\ nk+2\}$$

and every admissible value occurs.



$\Delta^n_{k,l}$

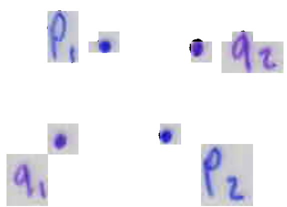
Rmk: The Khovanskii bound on the number of real solutions for these sparse systems would give

$$2^n \, 2^{\binom{n+k+1}{2}} \cdot (n+1)^{n+k+1}$$

Circuit : $n+2$ pts in $\mathbb{R}^n$

$$\sum \alpha_i p_i = \sum \beta_j q_j \qquad \alpha_i, \beta_j > 0$$

$$\sum \alpha_i + \sum \beta_i = 0$$



Any proper subset is affinely independent

but the $n+2$ pts are affinely dependent.

Near Circuits

Circuit : $n+2$ points affinely dependent in $\mathbb{R}^n$

Near circuit: add $k-1$ evenly spaced points between two of them.

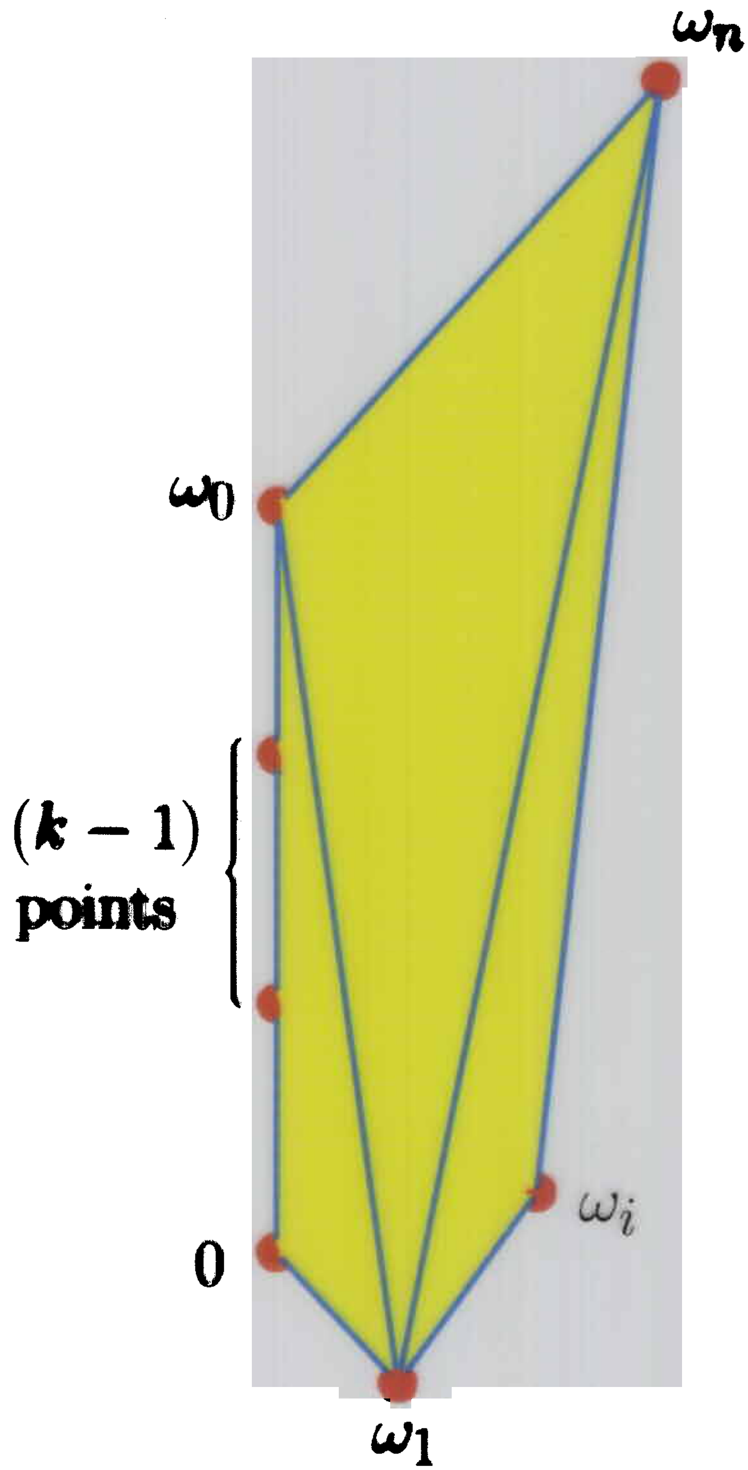
Consider the exponent vectors

$$(0, \dots, 0)\ ,\ \omega_0 = (k, 0, \dots, 0)\ ,\ \omega_1, \dots \omega_n$$

such that $\omega_n = \sum \lambda_i \omega_i \quad , \quad \lambda_i \in \mathbb{N}^*$

$$(S) : \sum a_\omega^{(i)} x^\omega = f_i(x) \quad , \quad i = 1 \dots n$$

$x = (x_1, \dots, x_n) \quad \deg f_i = k$



ω_n
ω_0
$(k-1)$ points
ω_i
0
ω_1

Using Gaussian elimination, we get

$$(S) \iff x^{\omega_i} = g_i(x_1), \quad \deg g_i = k$$

Put $y_i = x^{\omega_i}$, $i = 1 \ldots n-1$, $y_n = x_1$

$$(\tilde{S}) : \begin{cases} y_i = g_i(y_n) \\ y_n^{\lambda_0 k} \prod_i g_i^{\lambda_i}(y_n) = g_n(y_n) \end{cases}$$

$$f(z) = z^{k\lambda_0}\, g_1^{\lambda_1}(z) \ldots g_{n-1}(z)^{\lambda_{n-1}} - g_n(z)$$

Assume $\lambda_i \geq k+1$, $i = 1 \ldots n-1$, $\lambda_0 k \geq k+1$

$$f^{(k+1)}(z) = z^{k\lambda_0 - (k+1)} \cdot \prod_{i=1}^{n-1} g_i^{\lambda_i - (k+1)} \cdot Q$$

$\deg Q = (n-1)k(k+1)$

$f^{(k+1)}$ has at most

$(k+2)\,k(n-1)+1$ real roots

Thus, by Rolle Theorem,

f has at most

$(k+2)\,k(n-1)+k+2$ real roots

Let T be a n-simplex with integer vertices $(0, v_1, \dots, v_n)$

Let (T) be the system

$$(T) : x^{v_i} = c_i$$

Let V be the matrix of v_i and $\overline{V}$ its reduction mod 2

Lemma: The number of real solutions of (T) is at most $2^{\operatorname{cork}(\overline{V})}$

Rmk: If T has odd volume then $\operatorname{cork}(\overline{V}) = 0$

Thus we can bound the number of real solutions of (S) using the bound found for $(\tilde{S})$:

Let W be the matrix $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, w_1, \ldots, w_{n-1}\Big)$ and $\overline{W}$ its reduction mod 2, Then

The number of solution of (S) is at most

$$2^{\mathrm{cork}(\overline{W})} \cdot \left[(k+2)k(n-1)+k+2\right].$$