Experimentation and Conjectures in the Real Schubert Calculus

with Y. Sivan, E. Soprunova, F. Sottile

Enumerative Geometry:

Count geometric objects satisfying conditions imposed by fixed objects.

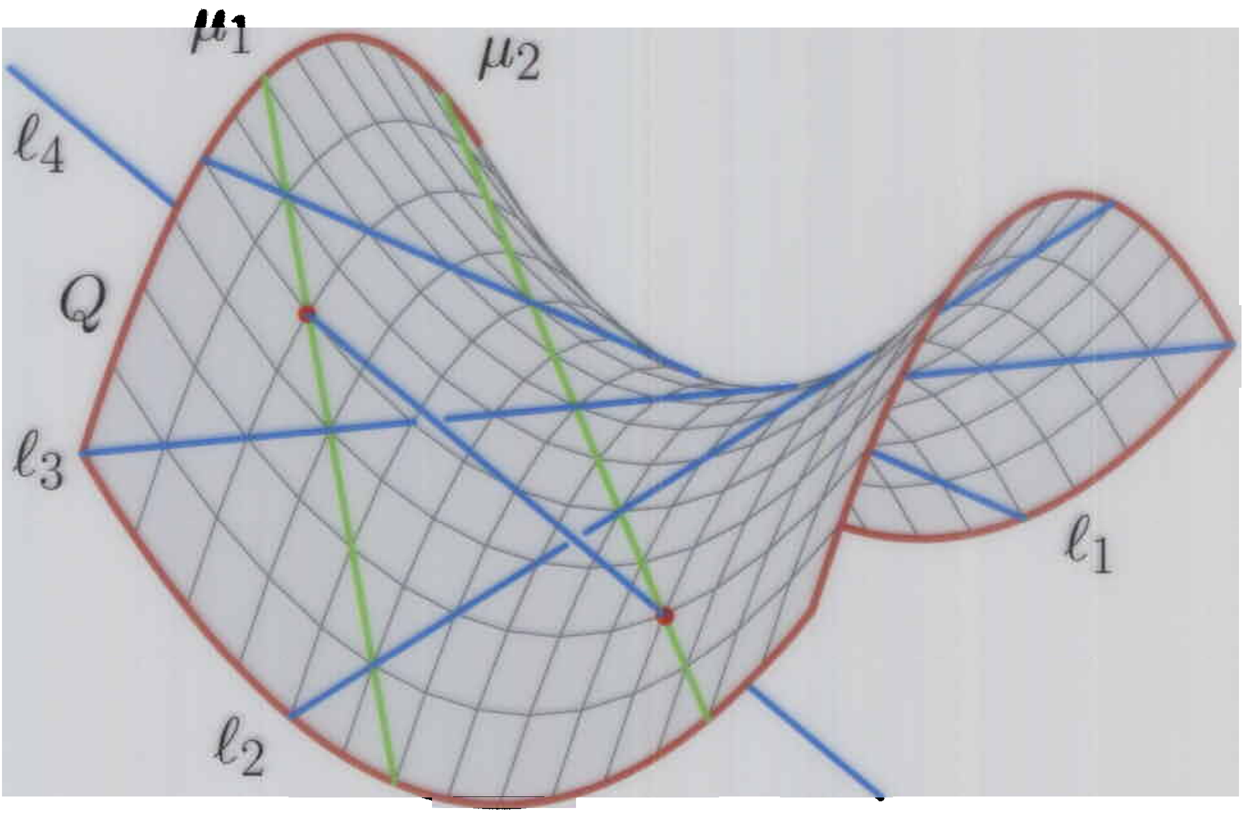
Example: Lines meeting 4 general fixed lines, $\ell_1, \ell_2, \ell_3, \ell_4$, in $\mathbb{P}^3_K$

$K = \mathbb{C}$: ℓ_1, ℓ_2, ℓ_3 are in one ruling of a quadric, Q.

ℓ_4 meets Q in $\underbrace{\text{2 points}}$.

2 lines in the 'other' ruling.

$K = \mathbb{R}$: 0 or 2 solutions.



The two lines meeting four general lines in space

Classical Schubert Calculus

Compute number of complex solutions to enumerative problems, when "geometric objects" are linear subspaces.

Theorem (Sottile '97, '99; Vakil '03):
The (classical) Schubert Calculus is real.
($\mathbb{R}$-upper bound = # $\mathbb{C}$-solutions)

- lower bounds $\neq$ upper bounds

Is there a uniform way of choosing fixed subspaces so that

$$\#_{\mathbb{R}} = \#_{\mathbb{C}} \quad ?$$

Shapiro's Conjecture

Flag: chain of linear subspaces

$$F_\bullet = (0 \subset F_{a_1} \subset \cdots \subset F_{a_s} \subset K^n)$$

$$\dim F_a = a$$

$$\gamma : \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1} \qquad C := \operatorname{im} \gamma$$

$$t \longmapsto [1, t, \ldots, t^{n-1}]$$

$F_\bullet$ **osculates** C if

$$F_a = \operatorname{span}\left\{\gamma(T), \frac{d\gamma}{dt}(T), \ldots, \frac{d^{a-1}\gamma}{dt^{a-1}}(T)\right\}$$

for some T.

Conjecture (B. Shapiro - M. Shapiro):

For enumerative problems involving conditions on subspaces imposed by fixed flags osculating C, all solutions are real.

$$\begin{tikzcd} & \mathbb{P}^{n-1} \supset E \arrow[dr, "\pi_E"] & \\ \mathbb{P}^1 \arrow[ur, "\gamma"] \arrow[rr, "f"] & & \mathbb{P}^1 \end{tikzcd}$$

$\operatorname{codim} E = 2$

$\deg f = n-1$

p is a critical point of $f \Leftrightarrow \underbrace{E \text{ meets } T_p C}_{\text{Schubert condition}}$

Theorem (Eremenko-Gabrielov '02): f has real critical points $\Rightarrow f$ is real.

Equivalently, Shapiro's Conjecture holds for codimension-2 planes (or lines) in $\mathbb{P}^{n-1}$.

Open (true, experimentally) for k-planes in $\mathbb{R}^n$.

Example: In $\mathbb{P}^3$, fix

- 2 points p_1, p_2
- 3 lines ℓ_1, ℓ_2, ℓ_3

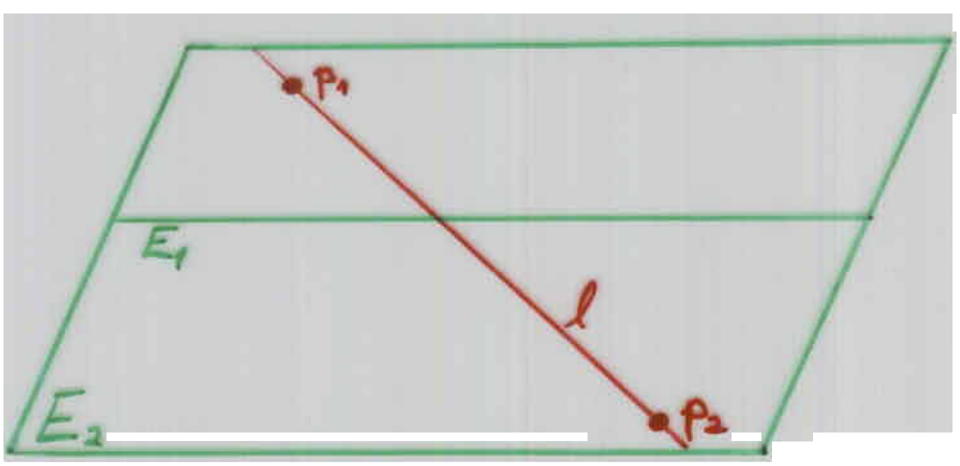
How many flags $E_1 \subset E_2$ satisfy

- E_1 meets each ℓ_i
- E_2 contains each p_i ?

$\ell :=$ line spanned by p_1 and p_2

E_2 contains each $p_i \Leftrightarrow E_2$ contains ℓ

$\Rightarrow E_1$ meets ℓ.



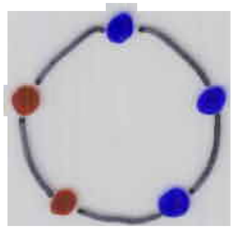
As before, ℓ_1, ℓ_2, and ℓ_3 lie in one ruling of a quadric $Q \subset \mathbb{P}^3$, and the conditions imposed by these force:

$$E_1 \in \text{the other ruling of } Q$$

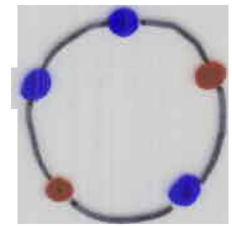
In addition, E_1 meets ℓ, so

$$\#\{\mathbb{R}\text{-solutions}\} = \#(\ell \cap Q)$$

$$E_2 = \overline{\ell, E_1}$$



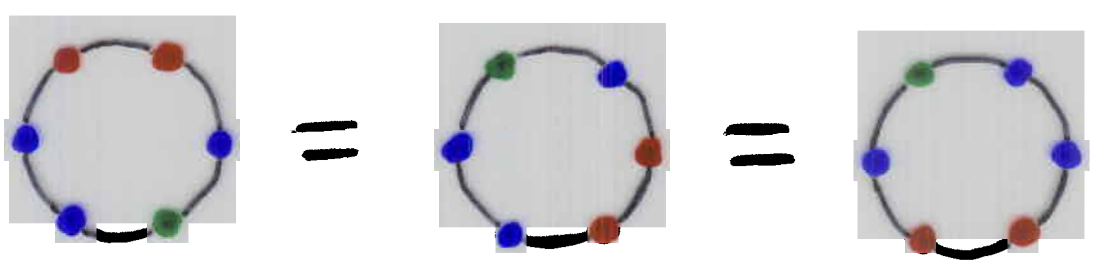
all real



not all real

Moral: Reality depends on order in which different types of conditions are arranged on C.

Encoded by necklace: Isotopy type of finite collections of colored points on $\mathbb{P}^1_{\mathbb{R}}$.



Remark: This approach applies only when conditions are Grassmanian: involve only one part of the flag.

Conjecture: For enumerative problems with only Grassmannian conditions imposed by fixed flags osculating C,

Non-crossing necklace ⇒ All solutions real

Computations:

- 7.5 GHz-years
- 934 enumerative problems on 26 flag varieties
- 400,000,000 polynomial systems

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