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Algorithmic, Combinatorial and Applicable Real Algebraic Geometry

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How Many Totally Mixed Nash Equilibria Can Graphical Games Have?

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Running Example of a Game

Players: Alice, Bob, Chris

Two pure strategies each:

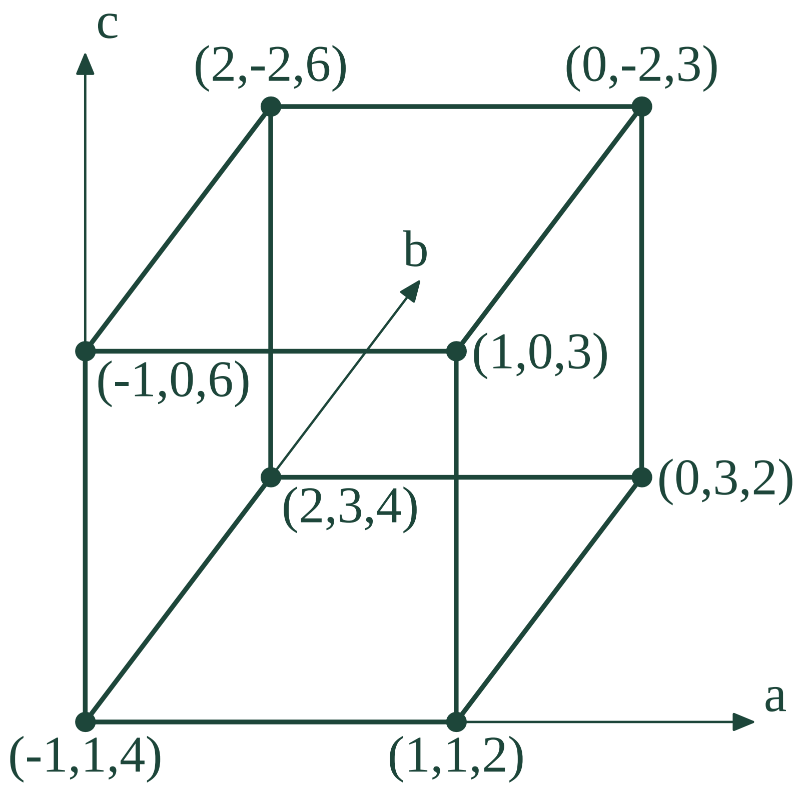
0 *and* 1

$a = \mathbf{Pr}[\textit{Alice chooses } 1]$

$b = \mathbf{Pr}[\textit{Bob chooses } 1]$

$c = \mathbf{Pr}[\textit{Chris chooses } 1]$

If Alice chooses 1,
Bob chooses 0,
and Chris chooses 1,
then Alice's payoff is 1,
Bob's payoff is 0,
and Chris's payoff is 3.



Expected Payoff Functions

Payoff to Alice of picking pure strategy 0, *is expected payoff conditioned on the event Alice chooses* 0:

$$2bc + 2b(1-c) + -1(1-b)c + (-1)(1-b)(1-c) = 2b - (1-b) = 3b - 1$$

Similarly, payoff to Alice from picking pure strategy 1:

$$0bc + 0b(1-c) + 1(1-b)c + 1(1-b)(1-c) = 1 - b$$

Payoff to Bob of picking pure strategy 0:

$$0ca + 0c(1-a) + 1(1-c)a + 1(1-c)(1-a) = 1 - c$$

Payoff to Bob of picking pure strategy 1:

$$-2ca - 2c(1-a) + 3(1-c)a + 3(1-c)(1-a) = -2c + 3(1-c) = 3 - 5c$$

More Expected Payoff Functions

Payoff to Chris of picking pure strategy 0*:*

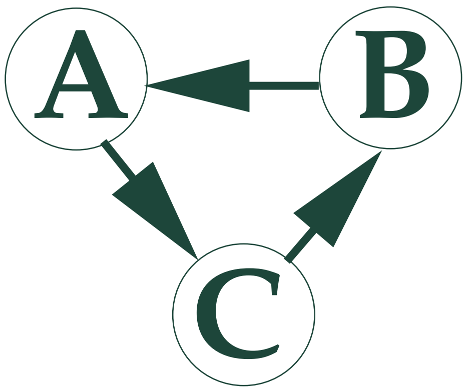
$$2ab + 2a(1-b) + 4(1-a)b + 4(1-a)(1-b) = 2a + 4(1-a) = 4 - 2a$$

Payoff to Chris of picking pure strategy 1*:*

$$ab + a(1-b) + 5(1-a)b + 5(1-a)(1-b) = a + 5(1-a) = 5 - 4a$$

Graphical Game

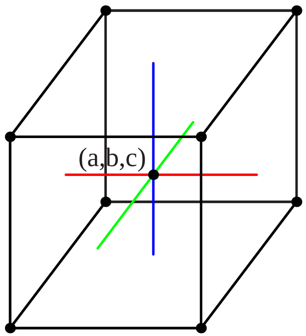
In a graphical game, the payoff to each player depends only on the actions of certain other players. We can draw the dependencies as a directed graph. Our example obeys the following graph:



Totally Mixed Nash Equilibria

Nash equilibrium: No player could unilaterally improve own payoff.

Totally mixed Nash equilibria are those in interior of cube.



Payoffs to each player of own pure strategies are equal.

(1) $3b - 1 = 1 - b$, *i.e.*, $b = 1/2$

(2) $1 - c = 3 - 5c$, *i.e.*, $c = 1/2$

(3) $4 - 2a = 5 - 4a$, *i.e.*, $a = 1/2$

Single totally mixed Nash equilibrium: $(1/2, 1/2, 1/2)$.

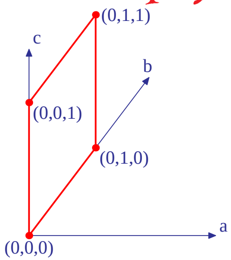
Newton Polytope

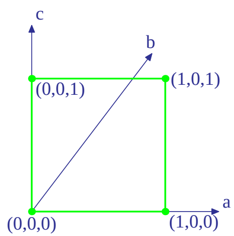
Polytope: convex hull of a finite set of points in affine space.

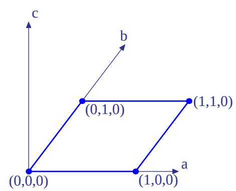
Each monomial $a^{\alpha} b^{\beta} c^{\gamma} \cdots$ in n variables is associated with a *lattice point* $(\alpha, \beta, \gamma, \ldots) \in \mathbb{N}^n$.

Support of polynomial: monomials occurring with nonzero coefficient.

Newton polytope of polynomial: convex hull of lattice points in its support.





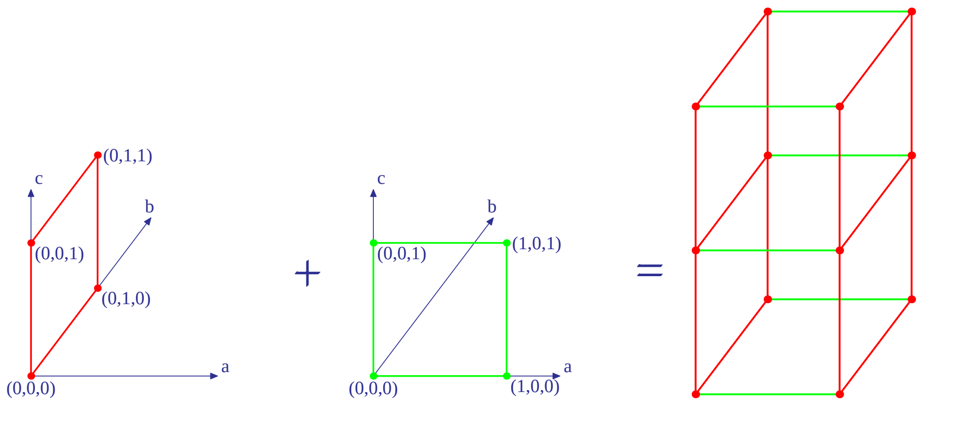


$\bullet bc + \bullet b + \bullet c + \bullet$ $\quad$ $\bullet ac + \bullet a + \bullet c + \bullet$ $\quad$ $\bullet ab + \bullet a + \bullet b + \bullet$

Minkowski Sum and Mixed Subdivision

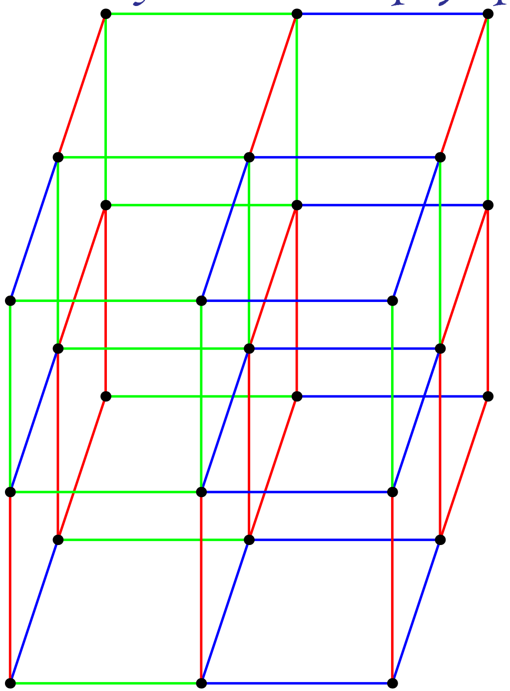
Minkowski sum of polytopes $P_1, \ldots, P_n$ is convex hull of $v_1 + \cdots + v_n$ where v_i is a vertex of P_i.

Translate faces of P_i *along edges of* P_j *to get decomposition of Minkowski sum into* *mixed subdivision* *(not unique).*



Bernstein-Kouchnirenko Theorem

The number of roots of a generic sparse system of polynomials is given by the mixed volume of their Newton polytopes.



Computing the mixed volume is not easy!

Polynomial Graph

To a system of n polynomial equations $f_1 = 0, \ldots, f_n = 0$ in n unknowns $\sigma_1, \ldots, \sigma_n$, we can associate a (non-unique) graph, the polynomial graph on n vertices, as follows:

- To each vertex i assign one of the unknowns, σ_i, and one of the equations, f_i.
- Draw an edge from vertex j to vertex k if and only if σ_j occurs in f_k.

Example of a Polynomial System

$$\bullet d_1 + \bullet d_2 + \bullet = 0;$$

$$\bullet d_1 + \bullet d_2 + \bullet = 0;$$

$$\bullet a_1 + a_2 + \bullet = 0;$$

$$\bullet = 0;$$

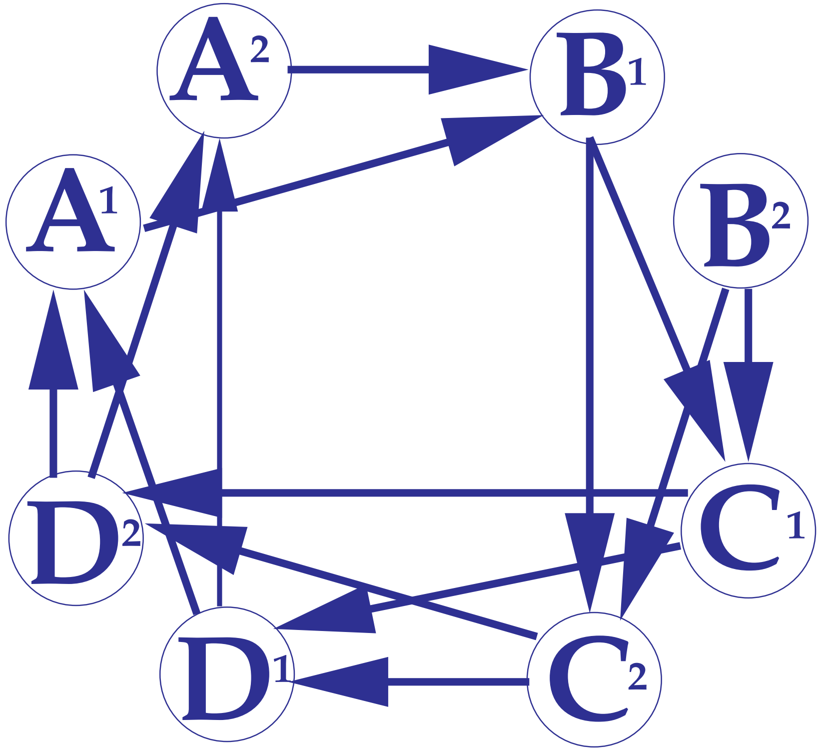
$$\bullet b_1 + \bullet b_2 + \bullet = 0;$$

$$\bullet b_1 + \bullet b_2 + \bullet = 0;$$

$$\bullet c_1 + \bullet c_2 + \bullet = 0;$$

$$\bullet c_1 + \bullet c_2 + \bullet = 0;$$

Example of a Polynomial Graph



Determinant and Permanent

Recall that the determinant is an antisymmetric sum:

$$\det\left(a_{ij}\right) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

The permanent is the corresponding symmetric sum:

$$\operatorname{per}\left(a_{ij}\right) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

Theorem (–, 2003)

Suppose the variables can be partitioned into sets $T_1, \ldots, T_N$ of cardinalities $d_1, \ldots, d_N$ such that
1) All monomials occurring in the f_i's are squarefree;
2) If $\sigma_j, \sigma_k \in T$ with $j \neq k$ then σ_j and σ_k do not both occur in any monomial of any of the f_i's;
3) If there is some $j \in T_i$ such that there is an edge from v_j to v_k in G, then for every $j \in T_i$ there is an edge from v_j to v_k in G.
Then if the polynomial system is 0-dimensional, the number of its solutions in $(\mathbb{C}^*)^d$ is the permanent of the adjacency matrix of G, divided by $\prod_{i=1}^{N}((d_i)!)$.

Example Permanental Formula

$$\mathrm{per}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} = 0$$

Real Roots?

Theorem (Mclennan, Beitrage zur Algebra und Geometrie 1999) The maximum number of real roots of a multihomogeneous system of polynomial equations is equal to the mixed volume.

Real Roots for Real People

Corollary The permanental formula gives the maximum number of real roots of a system obeying a polynomial graph, and in particular, the maximum number of totally mixed Nash equilibria of a graphical game.