**Big Question**: What makes a polynomial non-negative?

**First Guess**: A polynomial is nonnegative if and only if it is a sum of squares (SOS) of polynomials.

**Theorem**: (Hilbert) A nonnegative polynomial is necessarily SOS if it is univariate or quadratic or of degree 4 in 2 variables. In all other cases there exist non-negative polynomials that are not SOS.

**Hilbert's 17th Problem**: Is every nonnegative polynomial a sum of squares of rational functions?

**Yes** - Artin, Schreier.

However, the denominators can have very large degrees.

Big Question for this talk: What is the (quantitative) relationship between non-negative polynomials and sums of squares if we stay on the level of polynomials?

Non-negative polynomials that are not sos are hard to find explicitly. Hilbert's proof of their existence was non-constructive.

First example due to Motzkin. There are several known families due to Robinson, Reznick, Lam, Choi.

Remark: If a polynomial is non-negative then so is its homogenization. From now on we restrict our attention to homogeneous polynomials.

Let $P_{n,2k}$ be the vector space of real homogeneous polynomials in $n$ variables of degree $2k$.

Three interesting convex cones in $P_{n,2k}$:

$Pos_{n,2k}$ - cone of non-negative polynomials

$$Pos_{n,2k} = \{ f \in P_{n,2k} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \}$$

$Sq_{n,2k}$ - cone of sums of squares

$$Sq_{n,2k} = \{ f \in P_{n,2k} \mid f = \sum_i g_i^2 \text{ where } g_i \in P_{n,k} \}$$

$Lf_{n,2k}$ - cone of 2k-th powers

$$Lf_{n,2k} = \{ f \in P_{n,2k} \mid f = \sum_i \ell_i^{2k} \text{ where } \ell_i \in P_{n,1} \}$$

To compare sizes of these cones take compact sections with a hyperplane.

$M_{n,2k}$ - hyperplane of polynomials of average on $S^{n-1}$, unit sphere in $\mathbb{R}^n$:

$$M_{n,2k} = \{ f \in P_{n,2k} \mid \int_{S^{n-1}} f \, d\sigma = 1 \}$$

$\sigma$ - rotation invariant probability measure on $S^{n-1}$.

Define

$$\overline{Pos}_{n,2k} = Pos_{n,2k} \cap M, \quad \overline{Sq}_{n,2k} = Sq_{n,2k} \cap M$$

$$\overline{Lf}_{n,2k} = Lf_{n,2k} \cap M.$$

Why these sections? $M_{n,2k}$ is the only hyperplane in $P_{n,2k}$ fixed by the natural action of $SO(n)$.

$\overline{Pos}_{n,2k}$, $\overline{Sq}_{n,2k}$, $\overline{Lf}_{n,2k}$ are full-dimensional compact, convex sets in $M$.

Aside: How to measure the size of a convex set. $K \subset \mathbb{R}^n$ compact convex.

If take $\operatorname{Vol} K$ then $\operatorname{Vol} \alpha K = \alpha^n \operatorname{Vol} K$ so if $n$ is large then $\operatorname{Vol}(1+\varepsilon)K \gg \operatorname{Vol} K$.

Therefore we should take

$$(\operatorname{Vol} K)^{1/n}.$$

Let $D = \dim M_{n,2k} = \dim P_{n,2k} - 1 =$

$$= \binom{n+2k-1}{2k} - 1.$$

$B$ - unit ball of $L^2$ metric in $P_{n,2k}$.

$$\langle f, g\rangle = \int_{S^{n-1}} f g \, d\sigma.$$

**Theorem 1** (B) There are the following bounds on volumes of $\overline{Pos}_{n,2k}$ and $\overline{Sq}_{n,2k}$:

$$\left(\frac{\mathrm{Vol}\,\overline{Pos}_{n,2k}}{\mathrm{Vol}\,B}\right)^{1/D} \geq \frac{\gamma_1}{\sqrt{n \log 2k}}$$

$$\left(\frac{\mathrm{Vol}\,\overline{Sq}_{n,2k}}{\mathrm{Vol}\,B}\right)^{1/D} \leq \gamma_2 \binom{n+k-1}{k}^{1/2} \binom{n+2k-1}{2k}^{-1/2}$$

$\gamma_1, \gamma_2$ - absolute constants. Can take

$\gamma_1 \approx .21$, $\gamma_2 \approx 13$.

**Corollary**: Let the degree and number of variables vary. If the degree is not at least doubly exponential in the number of variables then

$$\lim_{n,k\to\infty} \left(\frac{\mathrm{Vol}\,\overline{Sq}}{\mathrm{Vol}\,\overline{Pos}}\right)^{1/D} = 0.$$

$$\left(\frac{\text{Vol Pos}}{\text{Vol B}}\right)^{1/D} \geq \frac{\gamma_1}{\sqrt{n \log 2k}}$$

$$\left(\frac{\text{Vol Sq}}{\text{Vol B}}\right)^{1/D} \leq \sqrt{\frac{(2k)(2k-1)\dots(k+1)}{(n+2k-1)\dots(n+k)}} \cdot \gamma_2$$

Say $2k = 6$, then need the number of variables $n$ to be about $2500$ to have

$$\left(\frac{\text{Vol Pos}}{\text{Vol Sq}}\right)^{1/D} \geq 4.$$

We also have full asymptotic understanding of volumes for the case that the degree is fixed.

Theorem 2: (B) There exist constants $c_1, c_2, c_3, c_4, c_5, c_6$ depending on $k$ only such that

$$c_1 n^{-\frac{1}{2}} \leq \left(\frac{\mathrm{Vol}\,\overline{P_{os}}}{\mathrm{Vol}\,B}\right)^{1/D} \leq c_2 n^{-\frac{1}{2}}$$

$$c_3 n^{-k/2} \leq \left(\frac{\mathrm{Vol}\,\overline{\Sigma_q}}{\mathrm{Vol}\,B}\right)^{1/D} \leq c_4 n^{-k/2}$$

$$c_5 n^{-k+\frac{1}{2}} \leq \left(\frac{\mathrm{Vol}\,\overline{L_q}}{\mathrm{Vol}\,B}\right)^{1/D} \leq c_6 n^{-k+\frac{1}{2}}$$

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$G$ - compact group, $V$ real vector space

$G$ acts on $V$, let $v \in V$ and let $K$ be the convex hull of the orbit of $v$.

$$K = \mathrm{conv}\{gv \mid g \in G\}.$$

Let $K^\circ$ be the polar of $K$.

$$K^\circ = \{ x \in V \mid \langle x, y \rangle \leq 1 \text{ for all } y \in K \}.$$

$K^\circ$ is cut out of $V$ by an orbit of one linear inequality.

Translate $\overline{Pos}$ to the hyperplane of polynomials of average $0$ on $S^{n-1}$ by subtracting $(x_1^2 + \dots + x_n^2)^k$. Call translation $Pos'_{n,2k}$. Then

$$Pos'_{n,2k} = \left\{ f \in P_{n,2k} \;\middle|\; \int_{S^{n-1}} f \, d\sigma = 0 \text{ and } f(v) \geq -1 \text{ for all } v \in S^{n-1} \right\}.$$

Orbit of one inequality under the action of $SO(n)$.

$\overline{Sq}_{n,2k}$ can be viewed as a projection of a convex hull of the orbit of a point which allows us to derive bounds for Theorem 1.

**Key Ingredient**: Maximal volume ellipsoid of $K^\circ$. Suppose $K^\circ$ is defined by the orbit of inequality $\ell(x) \le 1$. The Max Volume Ellipsoid of $K^\circ = v \in \mathbb{R}^n$ such that average of $\ell^2$ over the orbit of $v$ is at most $\frac{1}{n}$.

$$\left\{ v \in \mathbb{R}^n \;\middle|\; \int_G \ell^2(gv)\, d\mu \le \frac{1}{n} \right\}.$$

**Intuitive Idea**: Max Volume ellipsoids give "quadratic information" about $K^\circ$ and $K$. But if $K$ is an orbit of $v$ then can take $v \otimes v \otimes v \dots \otimes v$ and by considering Max Volume Ellipsoids of $v^{\otimes k}$ we can extract $2k$-th order information about $K$ and $K^\circ$.

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