April 16, 2004

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Barrier functions and cones of positive semidefinite forms.

(Work of E. Becker, A. Bemig and A. Diaz-Cano)

Let F be a vector space of finite dimension.

Let $K \subseteq F$ be a full dimensional closed convex cone (e.g., the cone of functions which are positive on some set).

<u>Problem:</u> Minimize $L(f)$ over $f \in K \cap A$, where $L : F \to \mathbb{R}$ is a linear functional and $A \subseteq F$ is a linear subspace.

<u>Definition.</u> $B : \mathring{K} \to \mathbb{R}$ is called a regular barrier function for K, if

1) B is convex
2) $\forall f \in K \setminus \mathring{K}$ "$B(f) = \infty$"
3) $B \in C^3(\mathring{K})$
4) $\exists\, c > 0 : \forall f \in \mathring{K}, \forall h \in F$

$$|B'(f)(h)| \leq c \sqrt{B''(f)(h,h)}$$

5) $\exists\, d > 0 : \forall f \in \mathring{K}, \forall h \in F$ $\sqrt[3]{|B'''(f)(h,h,h)|} \leq d \sqrt{B''(f)(h,h)}$.

Let $\langle \cdot, \cdot \rangle$ denote a scalar product on F.

Definition. $K^* := \{ g \in F : \forall f \in K \ \ \langle f, g \rangle \geq 0 \}$

Observe that $K^{**} = K$.

Theorem (Nesterov, Nemirovski, Güler)

Let us define the functional

$$\varphi_K : \mathring{K} \to \mathbb{R} \quad \text{by} \quad \varphi_K : f \mapsto \int_{K^*} e^{-\langle f, g \rangle} \, dg .$$

Then $B_K := \log \circ \varphi_K$ is a regular barrier function for K.

Examples

1) $F = \mathbb{R}^n$, $K = [0, \infty)^n = K^*$. This is an LP problem.

$$\forall a \in \mathring{K} \qquad B_K(a) = -\sum_{i=1}^{n} \log a_i .$$

2) $F = S\mathbb{R}^{n \times n}$, $K = S\mathbb{R}_+^{n \times n} = K^*$.

This is a problem in semidefinite programming.

$$\forall M \in \mathring{K} \qquad B_K(M) = \text{const} - \frac{n+1}{2} \log \det M$$

(This is a result due to Güler).

3

Proposition For all "nice cross sections" S of K^* $\exists$ $c = c(S) > 0$ such that $\forall f \in \overset{\circ}{K}$ $\varphi_K(f) = c(S) \int_S \frac{dg}{\langle f, g\rangle^n}$.

Definition. Let $F_{n,k}$ be the space of real forms in $x_1, \dots, x_n$ of degree k, equipped with the scalar product

$$\left\langle \sum_\alpha a_\alpha x^\alpha, \sum_\alpha b_\alpha x^\alpha \right\rangle = \sum_\alpha \frac{a_\alpha b_\alpha}{\binom{k}{\alpha}}.$$

$$\begin{array}{ccccc} Q_{n,2k} & \subseteq & P_{n,2k} & \subseteq & F_{n,2k} \\ \shortparallel & & \shortparallel & & \\ \sum F_{n,1}^{2k} & & \text{psd} & & \end{array}$$

Remark. $\forall f \in F_{n,2k}$, $\forall a \in \mathbb{R}^n$

$$\langle f, \underbrace{(a_1 x_1 + \dots + a_n x_n)^{2k}}_{= \sum_\alpha \binom{k}{\alpha} x^\alpha} \rangle = f(a)$$

Hence $Q_{n,2k}^* = P_{n,2k}$ and $P_{n,2k}^* = Q_{n,2k}$.

Let S be "nice cross sections of $Q_{n,2k}$".

$S = S^{n-1} \subseteq \mathbb{K}^n$.

$f \in \mathring{P}_{n,2k}$ and $h \in F_{n,2k}$

Let $f = \langle f, \cdot \rangle$, $h = \langle h, \cdot \rangle$ be linear forms.

$f, h : \mathbb{R}^n \supset S^{n-1} \longrightarrow \mathbb{R}$ - 2k-forms.

For $\alpha \in \mathbb{R}$ $\quad \varphi'(f)(h) = -\alpha \int_S \frac{h}{f} \frac{1}{f^\alpha}$,

$\varphi(f) = \int_S \frac{1}{f^\alpha}$, $\quad \varphi''(f)(h,h) = \alpha(\alpha+1) \int_S \left(\frac{h}{f}\right)^2 \frac{1}{f^\alpha}$,

$$\varphi'''(f)(h,h,h) = -\alpha(\alpha+1)(\alpha+2) \int_S \left(\frac{h}{f}\right)^3 \frac{1}{f^\alpha}.$$

For $n > (2k-1)^2$ and $\alpha = \frac{n+1}{2k} - 1$ all the conditions in the definition on page 1 are satisfied.