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On G-Invariant MOMENT PROBLEMS

— JOINT WORK WITH J. CIMPRIČ —

Talk by

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1/2 hour talk

Plan of the talk:

1) The multidimensional moment problem (background and overview of recent results).

2) Two G-Invariant moment problems (discussion & problematic)

3) INVARIANT s.a. representations of G-invariant b.c.s.a sets (discussion and examples)

4) G-Saturatedness of G-invariant preorderings (discussion and examples)

5) Examples of G-Invariant b.c.s.a sets for which the MP is NOT finitely solvable but the IMP is (orbit space)

§1 The multidimensional KMP:

$K \subset \mathbb{R}^n$ closed. $L \neq 0$ linear functional on $\mathbb{R}[X_1, \dots, X_n] := \mathbb{R}[\underline{X}]$.

Question: when is there a positive Borel measure on $\mathbb{R}^n$ (supported on K) s.t. $L(f) = \int f \, d\mu \quad \forall f \in \mathbb{R}[\underline{X}]$?

ANSWER: (HAVILAND): such a μ exists iff $L(\mathrm{Psd}(K)) \geq 0$

where $\mathrm{Psd}(K) :=$ the (preordering) of positive semidefinite polynomials:

$= \{ f \in \mathbb{R}[\underline{X}] \mid f \geq 0 \text{ on } K \}$.

DRAWBACK:

$\mathrm{Psd}(K)$ is huge (not finitely generated in general)! we want to make the "checking list" smaller...

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Let $S \subseteq \mathbb{R}[\underline{X}]$ finite and set $S = \{f_1, \dots, f_s\}$ and $K = K_S := \{\underline{x} \mid f_1(\underline{x}) \geq 0, \dots, f_s(\underline{x}) \geq 0\}$ consider the f.g. preordering

$$T_S = \left\{ \sum_{e \in \{0,1\}^s} \sigma_e f^e \;\middle|\; \sigma_e \in \sum \mathbb{R}[\underline{X}]^2 \right\}$$

multi-index   sums of squares

Say S solves the KMP if for any $L \neq 0$ we have:

$$\text{(1)} \quad L(T_S) \geq 0 \Rightarrow L(\mathrm{Psd}(K_S)) \geq 0.$$

Given b.c.s.a. set K say KMP is finitely solvable if there exists $S \subseteq \mathbb{R}[\underline{X}]$ finite s.t $K = K_S$ and (1) holds.

Define

$$T_S^{\text{lin}} = \left\{ f \in \mathbb{R}[\underline{X}] \;\middle|\; L(f) \geq 0 \text{ whenever } L(T_S) \geq 0 \right\}$$

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Fact 1: $T_S \subseteq T_S^{lin} \subseteq Psd(K_S)$.

Fact 2: S solves the KMP iff

$$T_S^{lin} = Psd(K_S).$$

- Say T_S is saturated if $T_S = Psd(K_S)$.
- Define $T_S^{(\dagger)} = \left\{ f \in \mathbb{R}[\underline{x}] \,\middle/\, \forall \varepsilon > 0 \text{ real} : f + \varepsilon \in T_S \right\}$

Fact 3: T_S saturated $\Rightarrow$ S solves KMP.

Fact 4: $T_S \subseteq T_S^{(\dagger)} \subseteq T_S^{lin} \subseteq Psd(K_S)$

Schmüdgen 1991: K compact b.c.s.a

S any finite description, then $T_S^{(\dagger)} = Psd(K_S)$. Therefore

S solves KMP.

Say T_S is closed if $T_S = T_S^{lin}$

Fact 5: if T_S closed then: T_S sat. $\Longleftrightarrow$ S solves KMP.

NON-COMPACT CASE? Works by Powers-Scheider Schmüdgen-K-Marshall, K-M-Schwartz

I'll only quote the results that I will need today from:

[K-M] Positivity, sums of squares and the multi-dimensional moment problem, TAMS Vol 354 2002

[K-M-S] Positivity, Sums of Squares and the multi-dimensional moment problem II, to appear in Advances in Geometry.

~~~ END of Commercial break...
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PART II : $n = 1$

Semi-algebraic subsets

– of the real line –

K = finite union of intervals.

The natural description:

$K = K_{\mathcal{N}}$ where $\mathcal{N}$ is defined as follows:

- if $a \in K$ and $(-\infty, a) \cap K = \phi$ then $(x-a) \in \mathcal{N}$
- if $a \in K$ and $(a, +\infty) \cap K = \phi$ then $(a-x) \in \mathcal{N}$
- if $a, b \in K$, $a < b$, $(a,b) \cap K = \phi$ then $(x-a)(x-b) \in \mathcal{N}$.
- nothing else is in $\mathcal{N}$.

$\mathcal{N}$ is the natural set of generators of K.

Example: $K = (-\infty, -1] \cup [1, \infty)$ get

$$\mathcal{N} = \{(x-1)(x+1)\} = \{(x^2-1)\}.$$

$n=1$

- moment problem for non compact case -

Theorem ([K-M], generalizes all known results for $n=1$)

If K_S is not compact then T_S is closed (so all conditions are equivalent) (ie T_S saturated iff S solves.)

Moreover the following are equiv:

(i) T_S is saturated.

(ii) T_S contains the natural set of generators $\mathcal{N}$.

(iii) S contains the natural set of generators $\mathcal{N}$ (up to scaling by positive reals).

Example $K = (-\infty, -1] \cup [1, \infty)$

$\mathcal{N} = \{(x^2-1)\}$ solves the K-moment problem but

$S = \{(x^2-1)x^2\}$ does not!

Theorem (Cylinders) [K-M]

$S = \{f_1, \dots, f_s\} \subseteq \mathbb{R}[x_1, \dots, x_n, Y]$

(only involving variables $\underline{x}$).

So $K_S = K \times \mathbb{R}$.

Assume K is compact.

(so K_S cylinder with compact cross section).

Then S solves KMP.

[K-M-S] extend this to subsets of cylinders, in particular:

Theorem (Polyhedra)

K_S b.c.s.a set defined by

$S = \{l_1, \dots, l_s\}$ linear polys.

Then: S solves KMP iff

K_S does NOT contain a 2-D cone.

§2 TWO G-Invariant KMP

- G finite group; $\varphi : G \longrightarrow GL_n(\mathbb{R})$ (linear representation)
  $\begin{pmatrix} G \text{ acting} \\ \text{on } \mathbb{R}^n \end{pmatrix}$.
- $K \subseteq \mathbb{R}^n$ is (setwise) G-invariant if: $\forall g \in G,\ \forall x \in K : x^g \in K$.
- $f(\underline{x}) \in \mathbb{R}[\underline{X}]$ define

$$f^g(\underline{x}) := f(\underline{x}^g).$$

f G-invariant if

$$f^g = f \quad \forall g \in G$$

$\mathbb{R}[\underline{X}]^G :=$ ring of invariant polys.

(I'll omit the reference to G and say invariant instead of G-invariant whenever clear)

G-Invariant moment problem :

$K \subseteq \mathbb{R}^n$ closed invariant

$L \neq 0$ invariant lin. funct. on $\mathbb{R}[\underline{X}]$

When is there a positive Borel meas. μ on $\mathbb{R}^n$ (supported on K) st

$$L(f) = \int f \, d\mu \quad \forall f \in \mathbb{R}[\underline{X}] \; ?$$

Note 1 :

There is a 1-1 corres. between:

Invariant lin. funct on $\mathbb{R}[\underline{X}]$ $\longleftrightarrow$ Linear funct. on $\mathbb{R}[\underline{X}]^G$

$$L \longmapsto L|_{\mathbb{R}[\underline{X}]^G}$$

$$F^* \longleftarrow F$$

where $F^*(f) := F\left(\frac{1}{|G|} \sum_{g \in G} f^g\right)$

Note 2: via this corresp. it is clear that Haviland's theorem holds:

Theorem (Haviland): Let $L \neq 0$ lin. funct. on $\mathbb{R}[\underline{X}]^G$, let $K \subseteq \mathbb{R}^n$ closed and invariant. There exists a positive Borel meas. s.t

$$L(f) = \int f \, d\mu \text{ for all } f \in \mathbb{R}[\underline{X}]^G$$

if and only if

$$L(\mathrm{Psd}(K)^G) \geq 0.$$

where $\mathrm{Psd}(K)^G := \mathrm{Psd}(K) \cap \mathbb{R}[\underline{X}]^G$

So now we want to consider lin. funct on $\mathbb{R}[\underline{X}]^G$ and formulate the G-invariant moment problems.

Let $S \subseteq \mathbb{R}[\underline{x}]$ finite and $K = K_S$ invariant.

Invariant KMPI:

Say S solves Inv KMPI if for all lin funct $L \neq 0$ defined on $\mathbb{R}[\underline{x}]^G$ we have:

$$(2) \quad L(T_S^G) \geq 0 \Rightarrow L(Psd(K)^G) \geq 0$$

where

$$T_S^G := T_S \cap \mathbb{R}[\underline{x}]^G.$$

Say Inv KMPI is finitely solvable if there exists finite $S \subseteq \mathbb{R}[\underline{x}]$ s.t $K = K_S$ and (2) holds.

motivation: exploit the symmetry of the set to get an integral representation of all invariant funct even if such does not exist for all functionals ...

Note 1 It may be "wiser" to take

$S \subseteq \mathbb{R}[\underline{X}]^G$ a (pointwise) invariant description of K.

Such invariant descriptions always exist (e.g see [Bröcker]):

given K_S invariant there is $S' \subseteq \mathbb{R}[\underline{X}]^G$ s.t $K_S = K_{S'}$

Unfortunately not so for the associated preorderings:

Note 2 If T_S (setwise) invariant then $T_{S'} \subseteq T_S$. We often have

$T_{S'} \subsetneq T_S$ (examples next section)

[Bröcker] on symmetric semi algebraic sets and orbit spaces

Singularities Symposium, Banach Centre Publ. Vol 44, 1998.

Note 3 if $S \subseteq \mathbb{R}[X]^G$ then one can describe T_S^G as follows:

T_S^G = the preordering gener. by S over $\left(\sum \mathbb{R}[\underline{X}]^2\right)^G := \sum \mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]^G$

This is seen by the following argument:

Pf: $S = \{f_1, \ldots, f_s\}$ $\quad h \in T_S^G$

$h \in T_S \Rightarrow h = \sum_{e \in \{0,1\}^s} \sigma_e \underline{f}^e \; ; \; \sigma_e \in \sum \mathbb{R}[\underline{X}]^2$

$h \in \mathbb{R}[\underline{X}]^G \Rightarrow \forall g \in G:$

$$h = h^g = \sum_{e \in \{0,1\}^s} \sigma_e^g (\underline{f}^e)^g = \sum_{e \in \{0,1\}^s} \sigma_e^g \underline{f}^e$$

(Since f_i's $\in \mathbb{R}[\underline{X}]^G$)

so $|G|\, h = \sum_{g \in G} h^g = \sum_{e \in \{0,1\}^s} \left(\sum_g \sigma_e^g\right) \underline{f}^e$

And of course $\sum_g \sigma_e^g \in \mathbb{R}[\underline{X}]^G$ ∎

Invariant KMP II:

S solves inv. KMP II if $\forall L \neq 0$ lin funct. on $\mathbb{R}[\underline{x}]^G$

(3) $L(T_S^{\mathbb{R}[\underline{x}]^G}) \geq 0 \Rightarrow L(\mathrm{Psd}(K)^G) \geq 0$

where $T_S^{\mathbb{R}[\underline{x}]^G}$:= preord. gen. by S in $\mathbb{R}[\underline{x}]^G$

say Inv KMP II finitely solvable if there is finite $S \subseteq \mathbb{R}[\underline{x}]^G$ s.t. $K = K_S$ and (3) holds.

Note 4: advantage: one can exploit orbit spaces as we shall see.

Disadvantage: it is asking for more - may be seldom solvable.

Note 5: Inv KMP I $\not\Leftrightarrow$ Inv KMP II

because in general

$$T_S^G \supsetneq T_S^{\mathbb{R}[\underline{x}]^G}$$

The reason is (as observed in [G - P]) that in general

$$\left(\sum \mathbb{R}[\underline{X}]^2\right)^G \supsetneq \sum \left(\mathbb{R}[\underline{X}]^G\right)^2$$

Question 1: Is $\left(\sum \mathbb{R}[\underline{X}]^2\right)^G$ a finitely generated preordering in $\mathbb{R}[\underline{X}]^G$?

Question 2: Is $\left(\sum \mathbb{R}[\underline{X}]^2\right)^G$ contained in some $\left(T_S^{\mathbb{R}[\underline{X}]^G}\right)^{lin}$ for some appropriately chosen finite $S \subseteq \mathbb{R}[\underline{X}]^G$?

Note 5 S solves KMP $\Rightarrow$ S solves Inv KMP I (but not necessarily Inv KMP II).

K. Gatermann & P. Parillo: Symmetry groups, semi definite programs and sums of squares, to appear

§3 Invariants & representations

<u>General comments</u>: Let $\underset{\text{finite}}{S} \subseteq \mathbb{R}[x]$ then S (setwise) G-inv $\Rightarrow$ T_S (setwise) G-inv $\Rightarrow$ $\underset{\text{(setwise)}}{K_S}$ G-inv $\Rightarrow$ $\underset{\text{(setwise)}}{Psd(K)}$ G-inv.

One cannot reverse arrows. For examples we need:

Theorem ([K-M-S]) $\overset{n=1}{\underset{\text{finite}}{S}} \subseteq \mathbb{R}[x]$, K_S compact. Then T_S is saturated iff T_S contains $\mathcal{N}$ (nat. gen). If K_S has no isolated points then T_S is sat. iff for each endpt $a \in K_S$ there is $f \in S$ st $(x-a) \mid f$ but $(x-a)^2$ does not.

<u>Example 1:</u> (K_S $\underset{\text{setwise}}{\text{inv}}$ but not T_S setwise)

$G = \{-1, 1\}$ acting on $\mathbb{R}$ by $x \longmapsto -x$

$S = \{1+x, (1-x)^3\}$ $K_S = [-1, 1]$ G-inv.

But $1-x \notin T_S$ (otherwise T_S sat, contr. the Theorem)

Pointwise invariant presentations:

$\boxed{\text{Example 2}}$ $G = \{-1, 1\}$ acting on $\mathbb{R}$

$S = \{f_1, \dots, f_s\}$, K_S (setwise) G-invariant.

Then $S' = \{f_i(x) + f_i(-x) ; f_i(x) f_i(-x) \mid i = 1, \dots\}$

is the pointwise inv present.

Theorem: (Showing $T_{S'} \subsetneq T_S$ happens!)

Let K non-compact inv. (no isolated points)

$S = \{f_i(x), f_i(-x) \mid i = 1, \dots, s\}$ a setwise inv. presentation. Assume T_S saturated. Then $T_{S'}$ is saturated iff all natural generators are even.

(iff K is the complement of a possibly empty open interval).

Example 3 $G = D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ acting on $\mathbb{R}^2$ by

$$(x,y)^a = (-y, x) \qquad (x,y)^b = (y,x)$$

$$S = \{1+x, 1+y, 1-x, 1-y\}$$

$$K_S = [-1,1]^2 \quad G\text{-inv (setwise)}$$

$$S' = \{(1-x^2)(1-y^2),\ 2-x^2-y^2\}$$

$T_{S'} \subsetneq T_S$ because $1-x \notin T_{S'}$.

§4 G-Saturatedness: $S \subseteq \mathbb{R}[\underline{x}]$ finite

K_S G-inv. T_S is **G-saturated** if

$$T_S^G = \mathrm{psd}(K)^G \qquad \text{Sat} \Rightarrow G\text{-Sat}, \ \not\Leftarrow$$

Example 4 $G = \{-1, 1\}$ acting on $\mathbb{R}$, $S \subseteq \mathbb{R}[\underline{x}]^G$, K_S non compact G-inv:

Theorem TFAE:

(1) T_S is G-saturated

(2) (i) (a,b) connected component of $\mathbb{R} \setminus K_S$, $0<a<b \Rightarrow (x^2-a^2)(x^2-b^2) \in S$ (up to scalar)

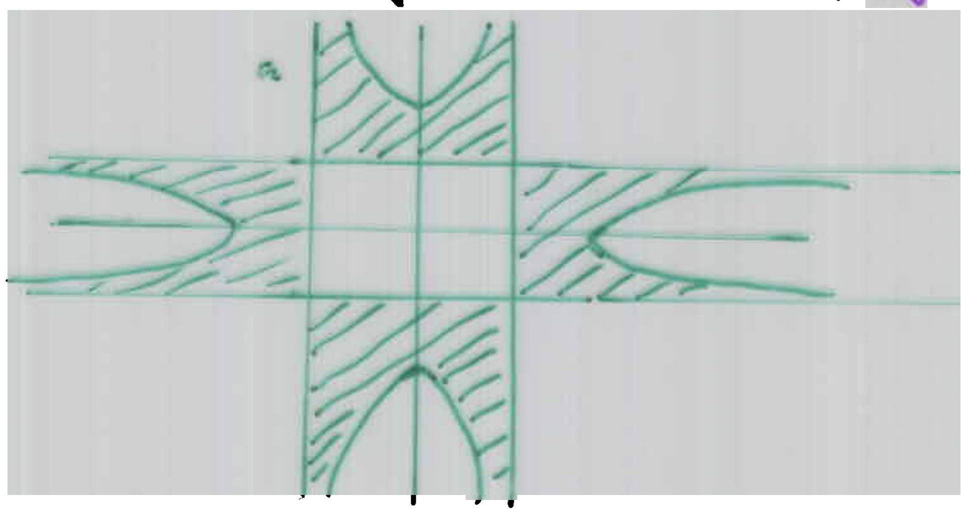
(ii) $(-a,a)$ connected component of $\mathbb{R}\setminus K_S \Rightarrow (x^2-a^2) \in S$ (up to scalar)

Last theorem provides examples of S solving Inv KMP but not KMP.

But we want more: an example where KMP is not finitely solvable but Inv KMP is:

§5 $\boxed{\text{Example 5}}$ $G = D_4$ acting on $\mathbb{R}^2$ as before

K_S defined by $-1 \le (x^2-1)(y^2-1) \le 0$



Claim

~~We~~ claim that KMP is not finitely solvable (email corresp. with Scheiderer in Feb 2004 - Hopefully more examples!)

However the Inv KMP is solvable:

$\mathbb{R}[x,y]^G$ is f.g by the polynomials

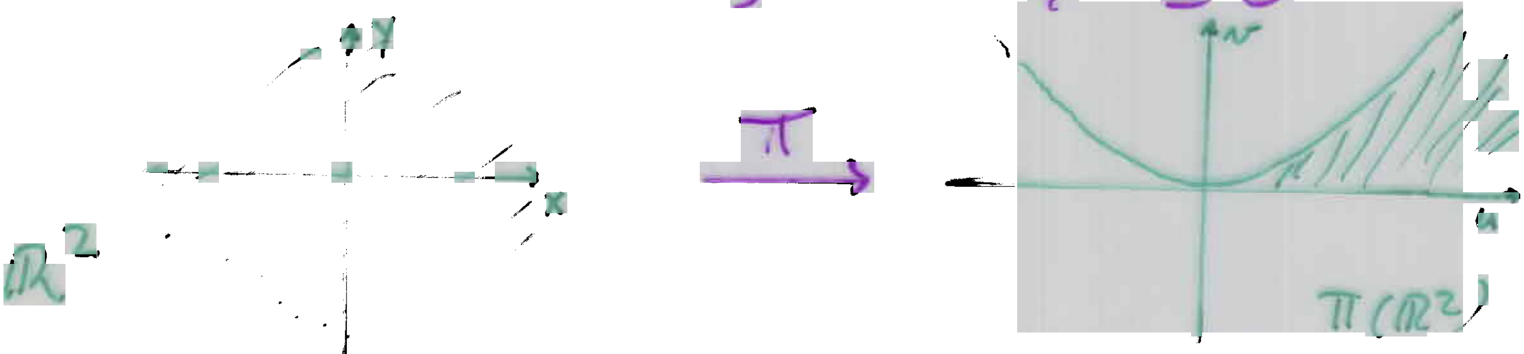
$$u = x^2+y^2 \qquad v = x^2y^2$$

$\mathbb{R}[x,y]^G = \mathbb{R}[u,v]$ a polynomial ring

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x^2+y^2,\ x^2y^2)$$

$\pi(\mathbb{R}^2)$ b.c.s.a defined by the inequalities:

$$u \geq 0, \quad v \geq 0, \quad u^2 - 4v \geq 0$$

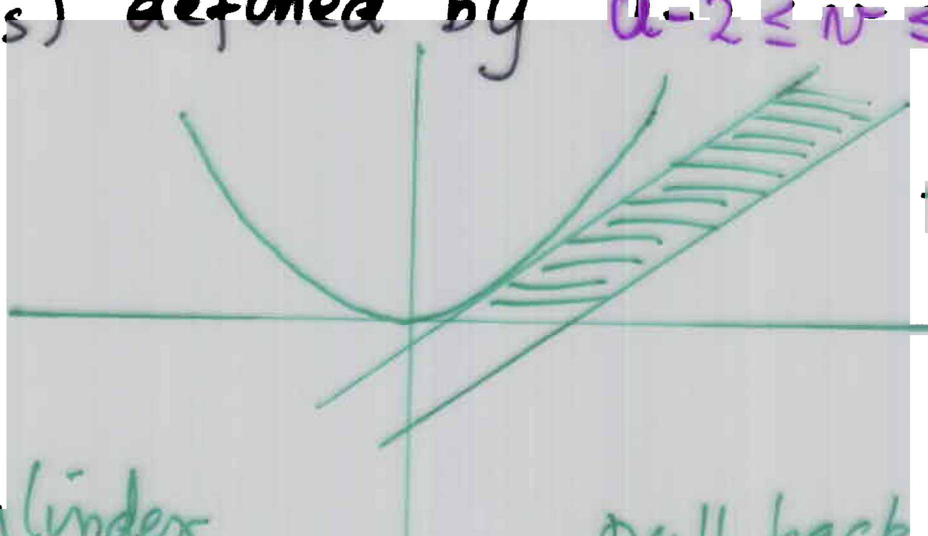


$f \in \mathbb{R}[\underline{x}] \rightsquigarrow \tilde{f}(u,v)$ s.t

$$\tilde{f}(x^2+y^2,\ x^2y^2) = f(x,y)$$

So $\overbrace{(x^2-1)(y^2-1)} = v - u + 1$

So $\pi(K_s)$ defined by $u-2 \leq v \leq u-1$



$\pi(K_s)$ cylinder pull back: get