Patterns In A Non-Local Model

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Outline

- Part I. Scalar Model.

  (i) PDE derivation.

  (ii) 3-bump formation.

- Part II. Extension to Systems.

  (i) Spirals.

  (ii) Rings.

  (iii) Breaking waves to produce spirals.

Part I. Scalar Model

Goal: Analyze pattern formation in the equation

$$\frac{\partial u(x,y,t)}{\partial t} = -u + \int\int_{\mathrm{R}^2} w(x-s, y-q) f(u(s,q,t) - th) ds dq$$

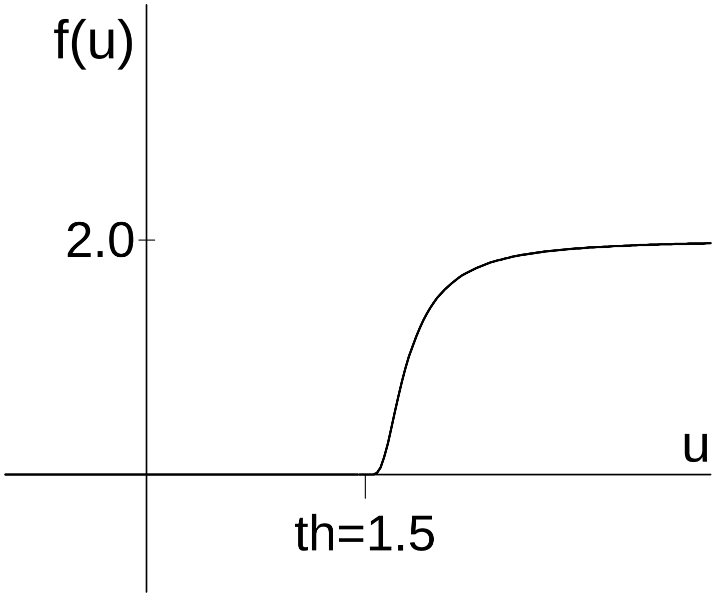
Wilson and Cowan (1972, 1973), Amari (1977)

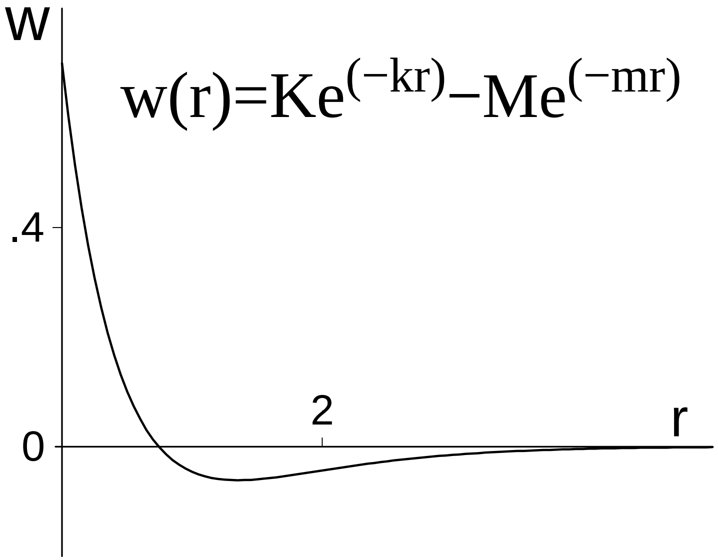
- $u(x,y,t)$ is the activity level (voltage) at position (x,y) at time t.
- $w(z)$ is the coupling weight.
- f is the firing rate function.
- $th > 0$ is the threshold.

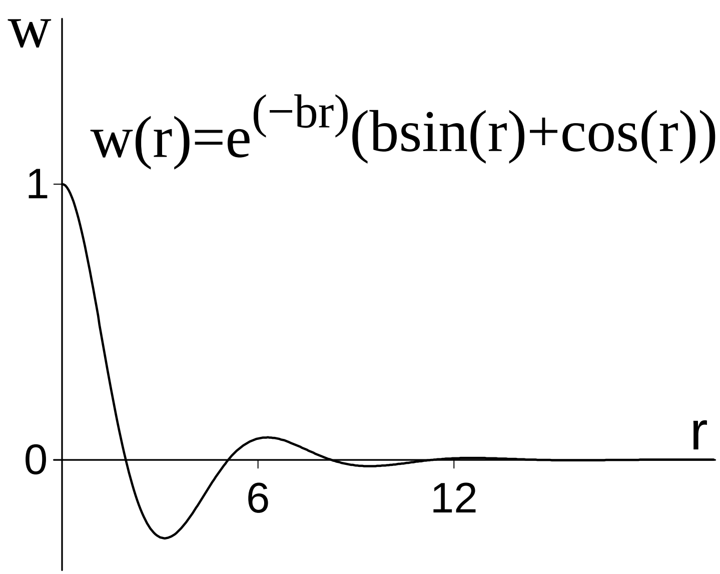
The Firing Rate

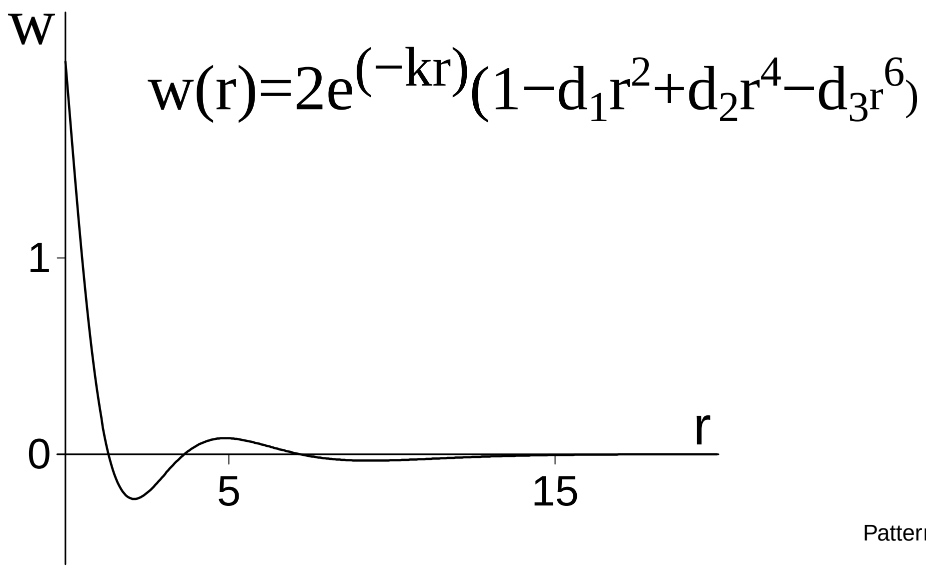
$$f(u-th) = Qexp\left(\frac{-\rho}{(u-th)^2}\right)H(u-th)$$

H is the Heaviside function. Below: $Q = 2,\ \rho = .1,\ th = 1.5$




w
$w(r)=Ke^{(-kr)}-Me^{(-mr)}$
.4
0
2
r


w
$w(r)=e^{(-br)}(b\sin(r)+\cos(r))$
1
0
6
12
r


w
$w(r)=2e^{(-kr)}(1-d_1r^2+d_2r^4-d_3r^6)$
1
0
5
15
r

PDE Derivation

$$u_t + u = \int\int_{\mathrm{R}^2} w\left(\sqrt{(x-s)^2+(y-q)^2}\right) f(u(s,q,t)-th)dsdq$$

Apply the two–dimensional Fourier transform defined by

$$\widehat{F}(g) \equiv (2\pi)^{-1}\int\int_{R^2} exp(-i(\alpha x+\beta y))g(x,y)dx\,dy$$

$$\widehat{F}(u+u_t) = \widehat{F}(w)\widehat{F}\left(f(u-th)\right)$$

If $w = w(r)$ then $\widehat{F}(w) = \widehat{F}\left(\sqrt{\alpha^2+\beta^2}\right)$. To obtain the PDE we approximate $\widehat{F}(w)$ by a rational function of $\sqrt{\alpha^2+\beta^2}$.

Example

$$\widehat{F}(u+u_t) = \widehat{F}(w)\widehat{F}(f(u-th))$$

$$\widehat{F}(w) = \frac{A}{B+(\alpha^2+\beta^2-M)^2}$$

$$\left((\alpha^2+\beta^2)^2 - 2M(\alpha^2+\beta^2) + B + M^2\right)\widehat{F}(u+u_t) = A\widehat{F}(f(u-th))$$

Identities:

$$(\alpha^2+\beta^2)^2\widehat{F}(g) = \widehat{F}(\nabla^4 g) \quad \text{and} \quad (\alpha^2+\beta^2)\widehat{F}(g) = -\widehat{F}(\nabla^2 g)$$

Resultant PDE:

$$\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u_t+u) = Af(u-th)$$

N-bump solutions.

(I) Change to polar coordinates and find symmetric solns.

$$L \equiv \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + 2M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + B + M^2$$

$$L(u_t + u) = Af(u - th),$$

(II) Find stationary solutions of the ODE problem

$$\begin{cases} Lu = Af(u - th), \\ u'(0) = u'''(0) = 0,\ (u, u', u'', u''') \to (0,0,0,0) \text{ as } r \to \infty. \end{cases}$$

(III) Linearize the PDE around the ODE solution

Linearization

$$u(r,\theta,t) = \widetilde{u}(r) + \mu\nu(r,t)\cos(m\theta), \qquad 0 < \mu << 1$$

To first order ν satisfies

$$[\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \left(\frac{2Mr^2 - 2m^2 - 1}{r^2}\right)\frac{\partial^2}{\partial r^2} + \left(\frac{2m^2 + 1 + 2Mr^2}{r^3}\right)\frac{\partial}{\partial r}$$

$$+ \frac{m^4 - 4m^2 + (B + M^2)r^4 - 2Mm^2r^2}{r^4}](\nu + \frac{\partial \nu}{\partial t}) = Af'(\widetilde{u} - th)\nu$$

Let $\nu(r,0) = e^{-r^2}$. We expect that $\nu(r,t) \sim \overline{\nu}(r)e^{\lambda t}$ as $t \rightarrow \infty$,

$$\nu(r,t) \sim \overline{\nu}(r)e^{\lambda t} \text{ as } t \rightarrow \infty,$$

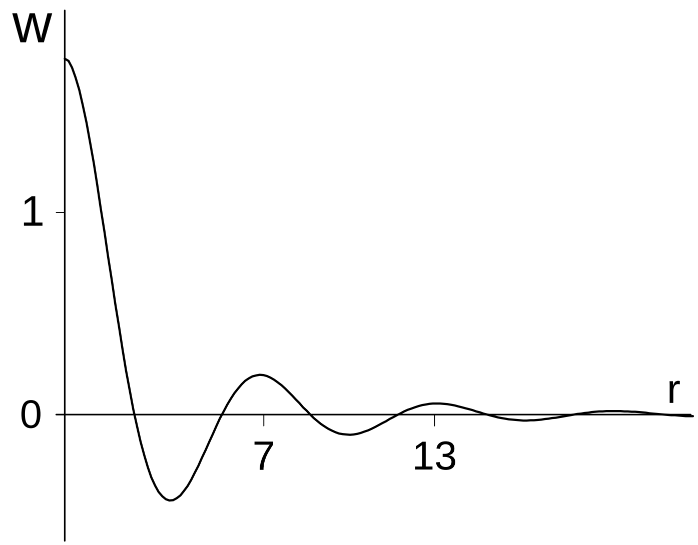
where λ is the largest eigenvalue, $\overline{\nu}(r)$ is the eigenfunction.

Example: M=1, A=.4, B=.1

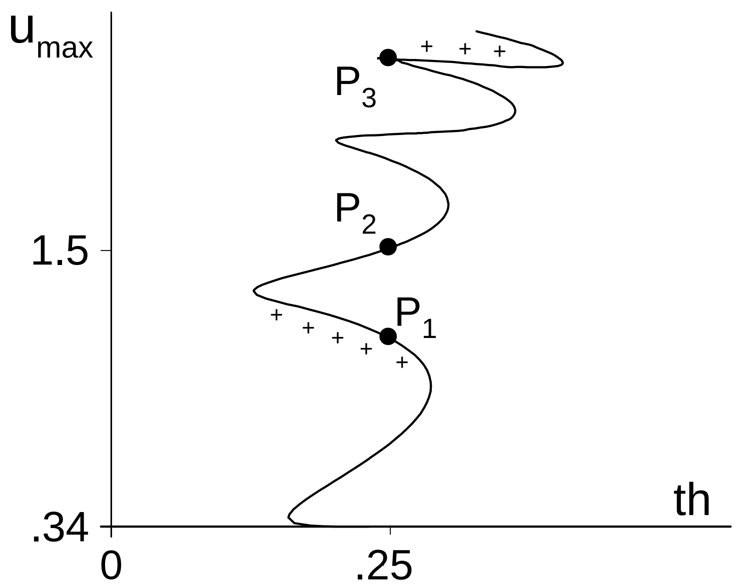
$$\widehat{F}\left(\sqrt{\alpha^2+\beta^2}\right) = \frac{A}{B+(\alpha^2+\beta^2-M)^2}$$

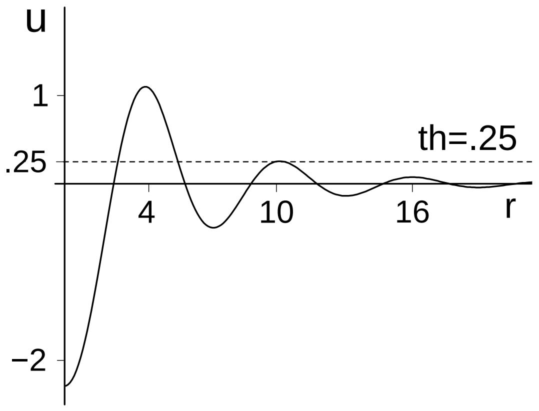
The inverse is given by

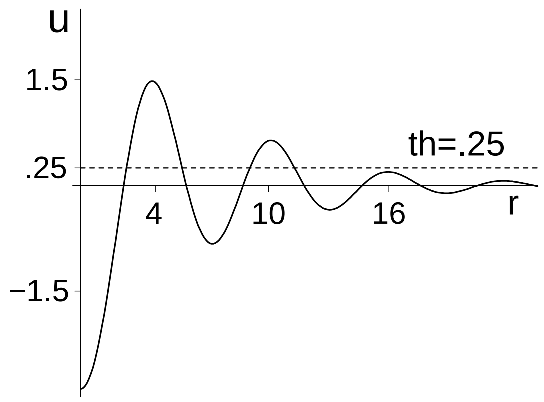
$$w(r) = \int_0^\infty s\widehat{F}(s)J_0(rs)ds$$

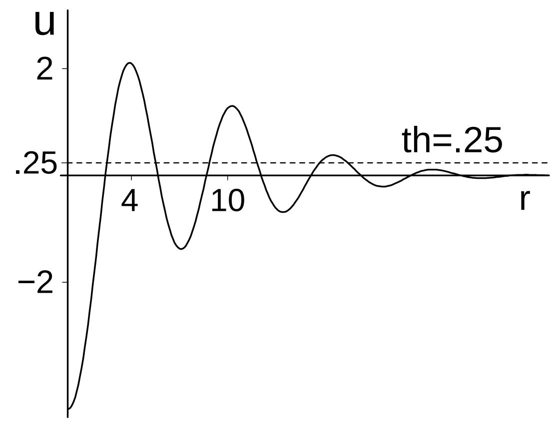


ODE Bifurcation diagram.

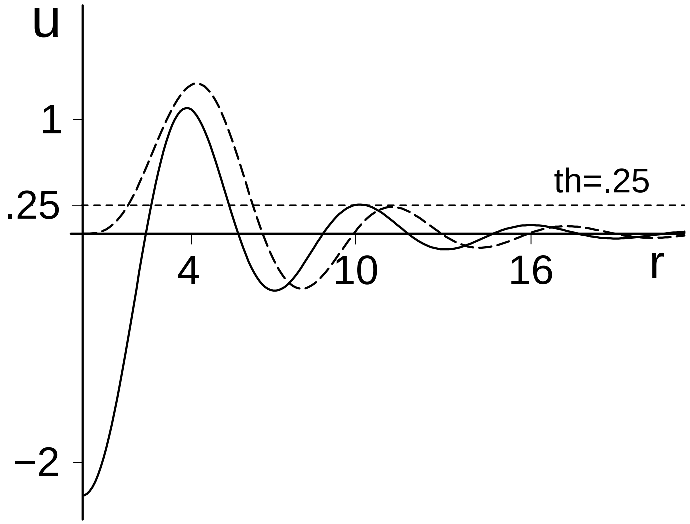


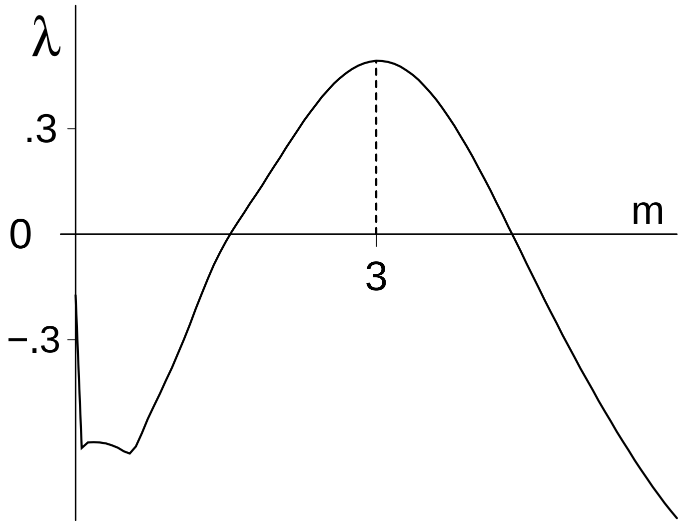


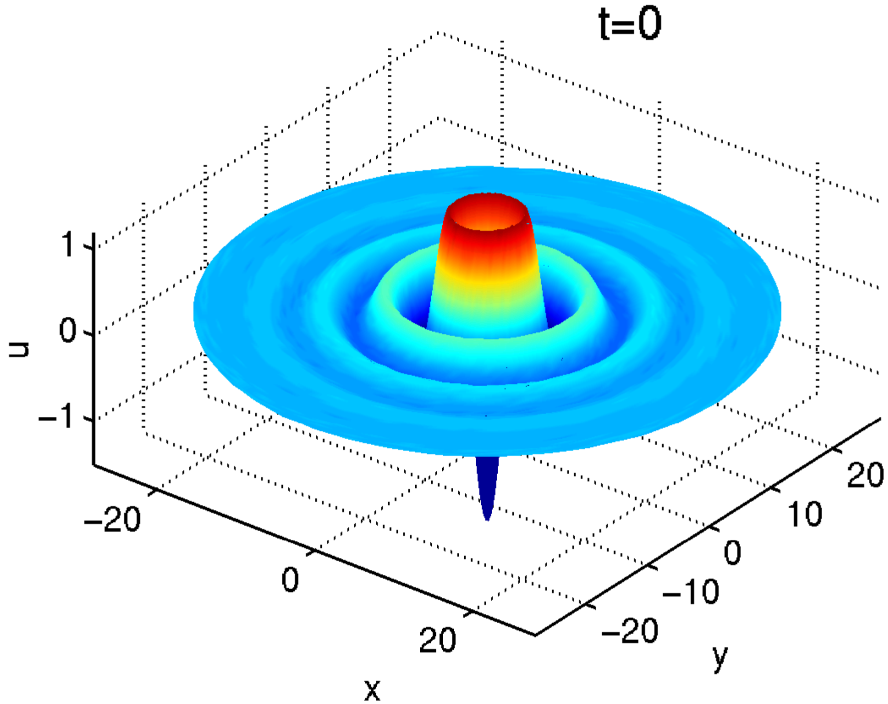




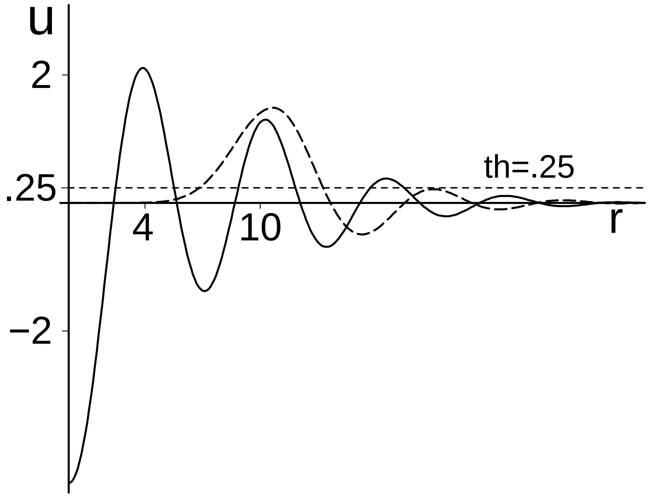
3-Bump Formation

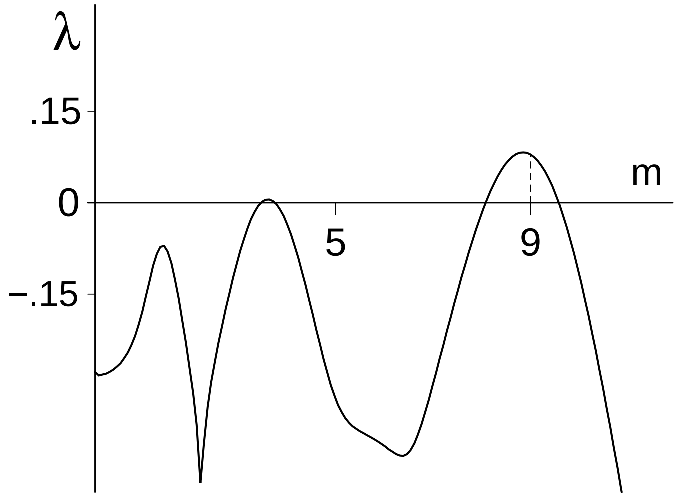


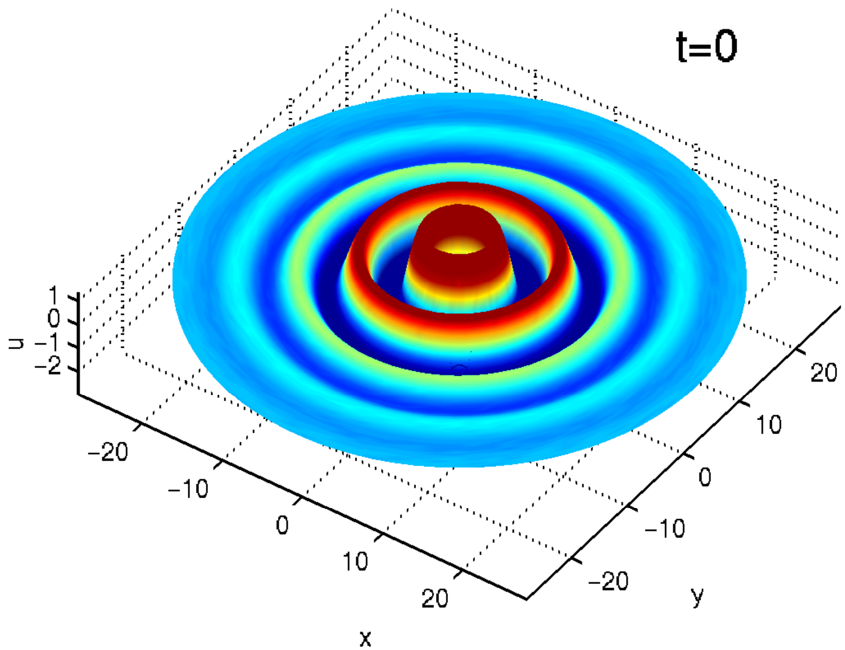




12-Bump Formation







Part II. Extension To Systems

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y,q,s) f(u-th)\, dq\, ds - a + I \\
\tau \frac{\partial a}{\partial t} &= \beta u - a
\end{aligned}
$$

a is a recovery variable. $I = I(x,y,t)$ is an external input.

Pinto and Ermentrout: I and II (2001); 1D waves, $I \equiv 0$.

Bressloff, Folias, Pratt, Li (2003); 1D waves, $I = I(x,y) \neq 0$.

Folias and Bressloff (2004); 2D waves, $I = I(x,y) \neq 0$.

Equivalent PDE's

$$w(x,y,s,q) = w\left(\sqrt{(x-q)^2+(y-s)^2}\right) g(q,s)$$

The PDE method: transform

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -u + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y,q,s) f(u-th)\, dq\, ds - a + I \\
\tau \frac{\partial a}{\partial t} &= \beta u - a
\end{aligned}$$

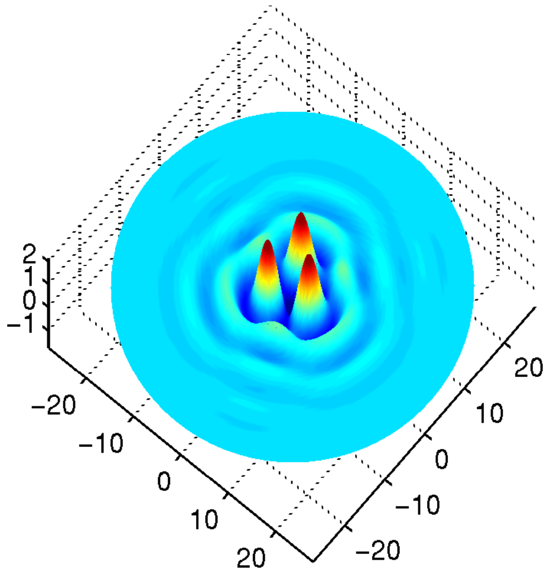
into

$$\begin{aligned}
\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u + u_t + a - I) &= Ag(x,y) f(u-th) \\
\tau a_t &= \beta u - a.
\end{aligned}$$

Let $I(x,y) \equiv 0$ and $g(x,y) \equiv 1$. For each $\beta \geq 0$ solve

$$\begin{aligned}\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u_t + u + a) &= Af(u - th)\\ \tau a_t &= \beta u - a,\end{aligned}$$

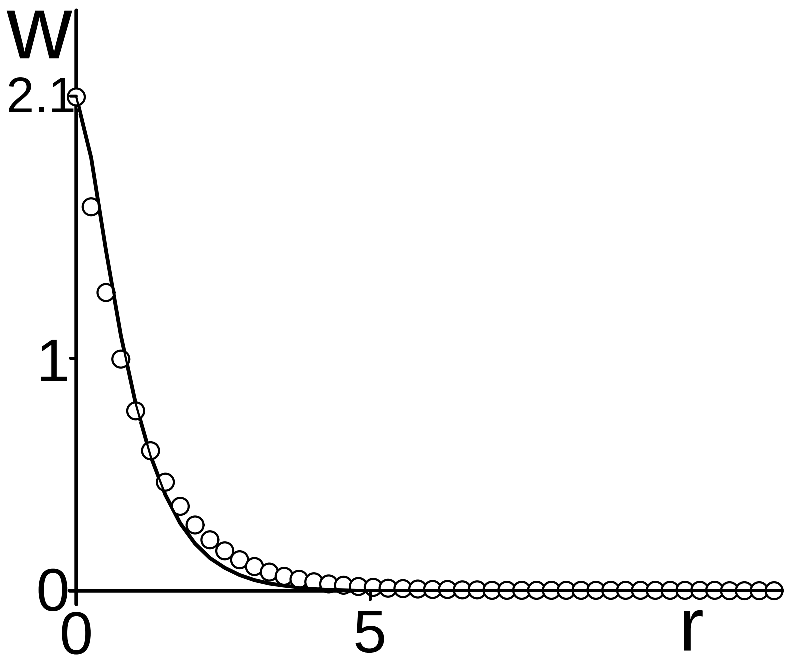
$$(u(r,0), a(r,o)) = \left(u_\beta(r), a_\beta(r)\right) + \text{ small perturbation.}$$



Positive Coupling

Approximate $w = 2.1e^{-r}$ by the Fourier transform method.

M=-2.5, A=7, B=.52



Circles: $w = 2.1e^{-r}$ Solid curve: approximation

Rotating Waves

Rabbit cortex: Petsche et al (1974)

Chic retina: Gorelova and Bures (1983)

Turtle: Prechtel et al (1997)

Mouse Hippocampus: Harris-White et al (1998)

EEG Patterns: www.ccs.fau.edu/∼jirsa/Imaging.html

<u>Models</u>

BZ Reaction Diffusion Model: Winfree (1974)

Discrete Systems: Greenberg and Hastings (1978)

Integrate and Fire Models: Chu, Milton and Cowan (1994)

RD Equations: Golubitsky, Knobloch and Stewart (2000)

Theta Neuron Model: Osan and Ermentrout (2001)

Rigid Rotating Waves

Substitute $(u, a) = (u(r, \phi), a(r, \phi))\,,\ \ \phi = \theta - \omega t$ into

$$\begin{aligned}\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u + a + u_t) &= Af(u - th) \\ \tau a_t &= \beta u - a\end{aligned}$$

and obtain

$$\begin{aligned}\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)\left(u + a - \omega\frac{\partial u}{\partial \phi}\right) &= Af(u - th) \\ -\omega\tau\frac{\partial a}{\partial \phi} &= \beta u - a.\end{aligned}$$

Limiting Case: $\omega = -1,\ \beta = 0,\ a \equiv 0,\ f = H(u - th).$

$$\begin{aligned} \left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u + a - \omega\frac{\partial u}{\partial\phi}) &= Af(u - th) \\ -\omega\tau\frac{\partial a}{\partial\phi} &= \beta u - a. \end{aligned}$$

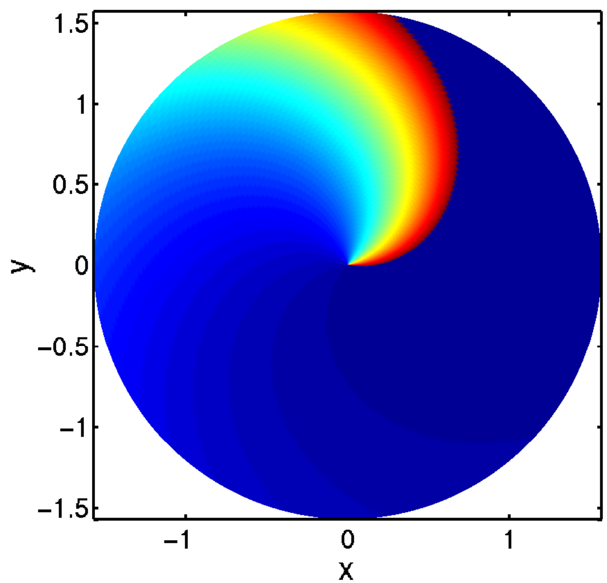
reduces to

$$\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u + \frac{\partial u}{\partial\phi}) = AH(u - th)$$

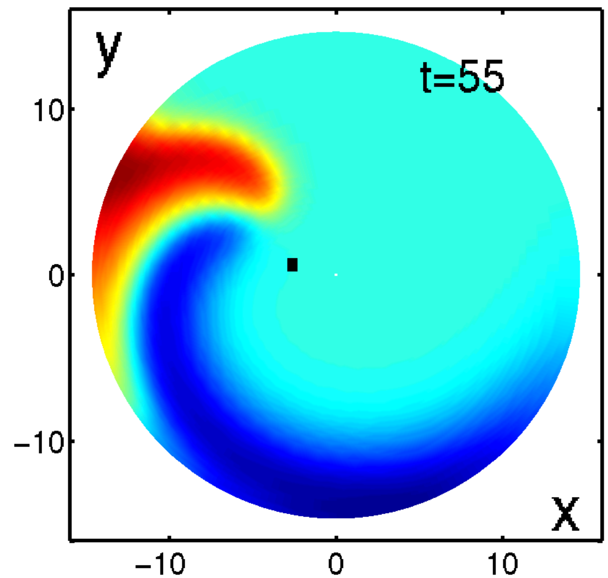
$$u(r,\phi) = \left(\frac{A}{B + M^2} + (th - \frac{A}{B + M^2})e^{r-\phi})\right) H(\phi - r)$$

$$\beta = 0,\ a \equiv 0,\ 0 < th < \frac{A}{B + M^2}$$

$$u(r, \phi) = \left(\frac{A}{B + M^2} + (th - \frac{A}{B + M^2}) e^{r - \phi} \right) H(\phi - r)$$

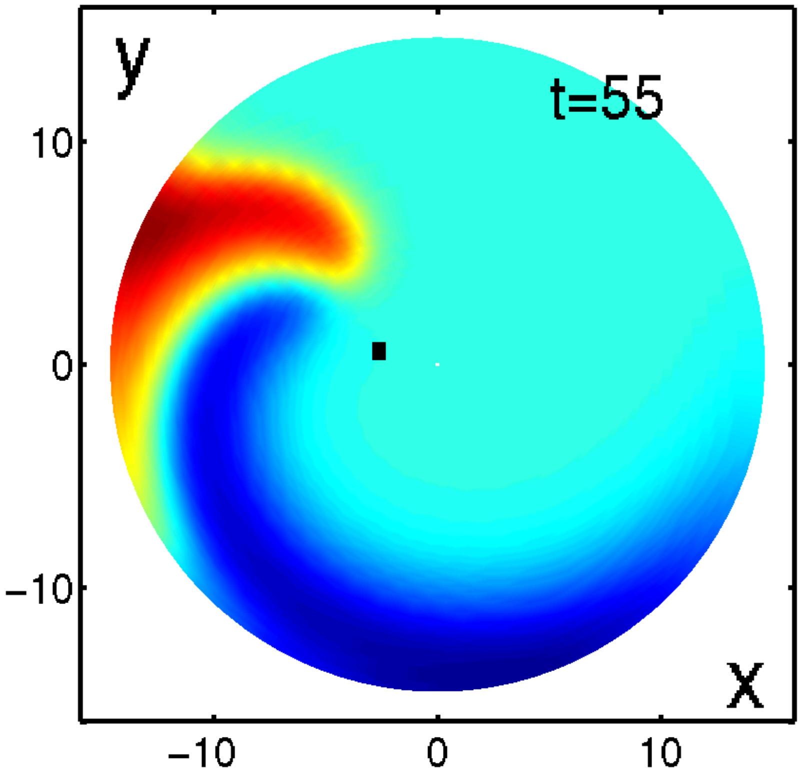


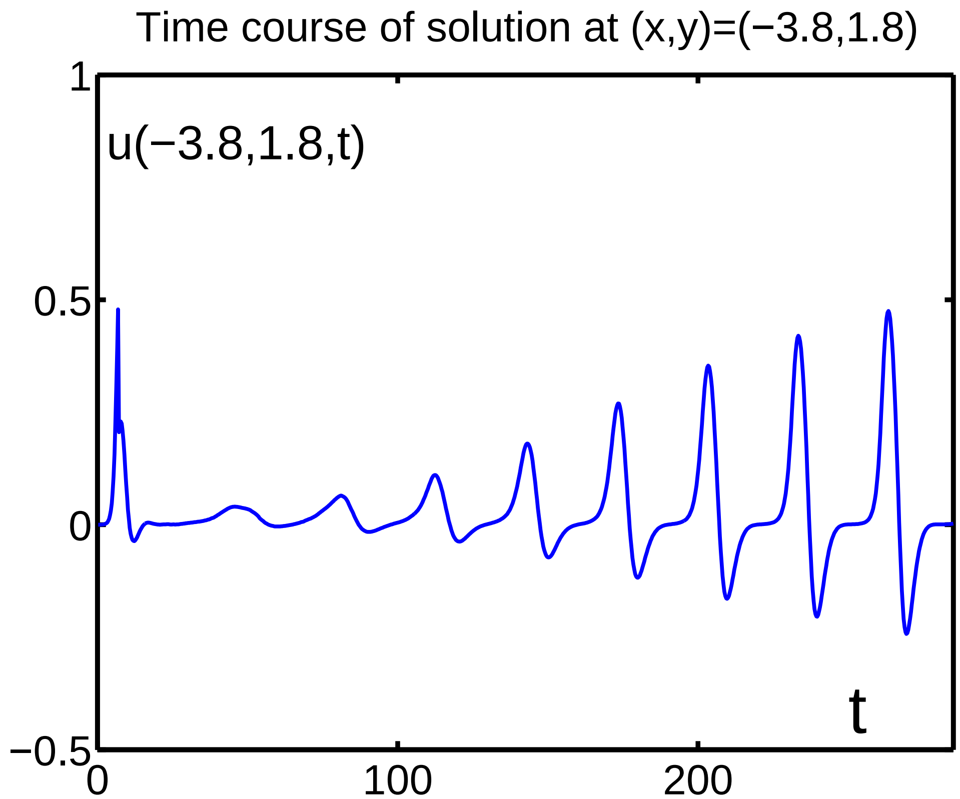
Spiral for $\beta = 0$.



Spiral for $\beta > 0$.

Spiral Drift

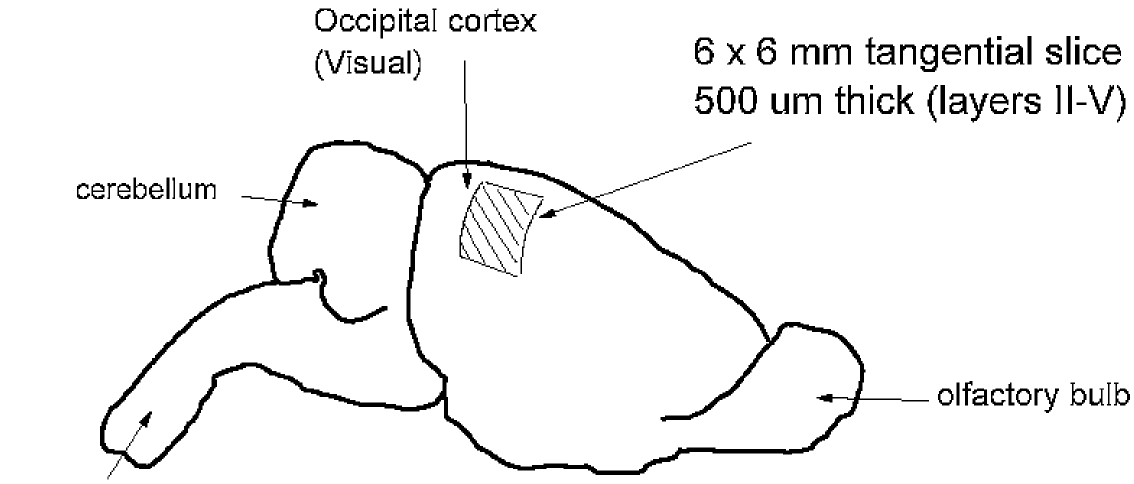




$\beta > 0$ Spiral Drift

Wu et al (August 2003)

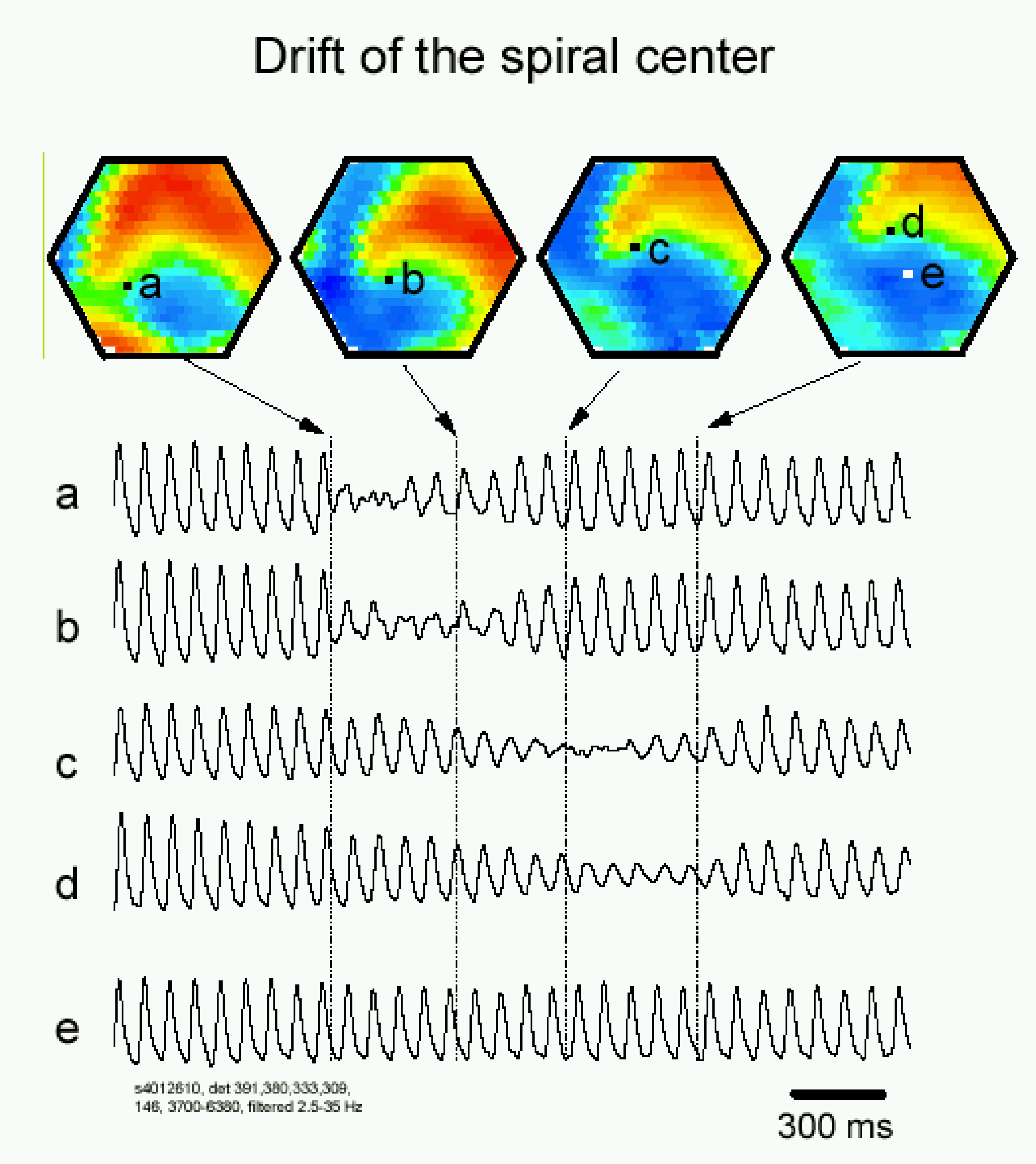
Tangential slice from rat cortex



The brain of a 3 - 5 week old rat

tail ←——→ Head

Wu et al (Feb 2004)



Non-rotating Waves

Waves can propagate with different speeds and shapes.

(1.) R. Chervin, P. Pierce & B. Connors. *Periodicity and directionality in the propagation of epileptform discharges across neocortex.* Jour. of Neurophysiology (1988)

(2.) J. Y. Wu, L. Guan, L. Bat & Q. Yang, *Spatio-temporal properties of an evoked population activity in rat sensory cortical slices.* Jour. of Neurophysiology (2001)

(3.) D. Pinto & B. Connors. *The fine structure of waves in neocortex.* (2004), in preparation

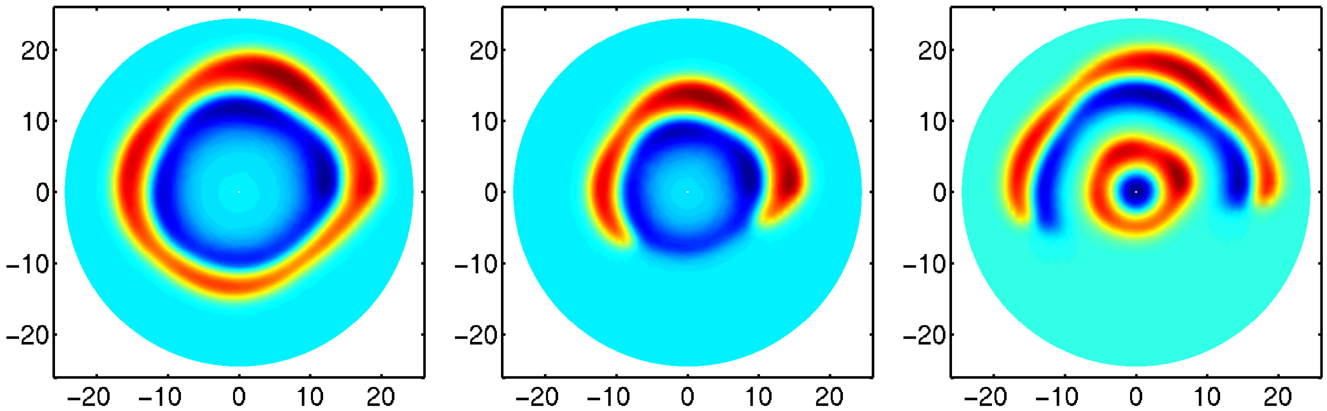
(4.) D. Glaser & D. Barch. *Bow Waves.* Neurocomp. (1999) http://foresight.berkeley.edu

Inhomogeneous Coupling

$$\left(\nabla^4 + 2M\nabla^2 + B + M^2\right)(u + u_t + a) = Ag(r,\theta)f(u - th)$$
$$\tau a_t = \beta u - a.$$

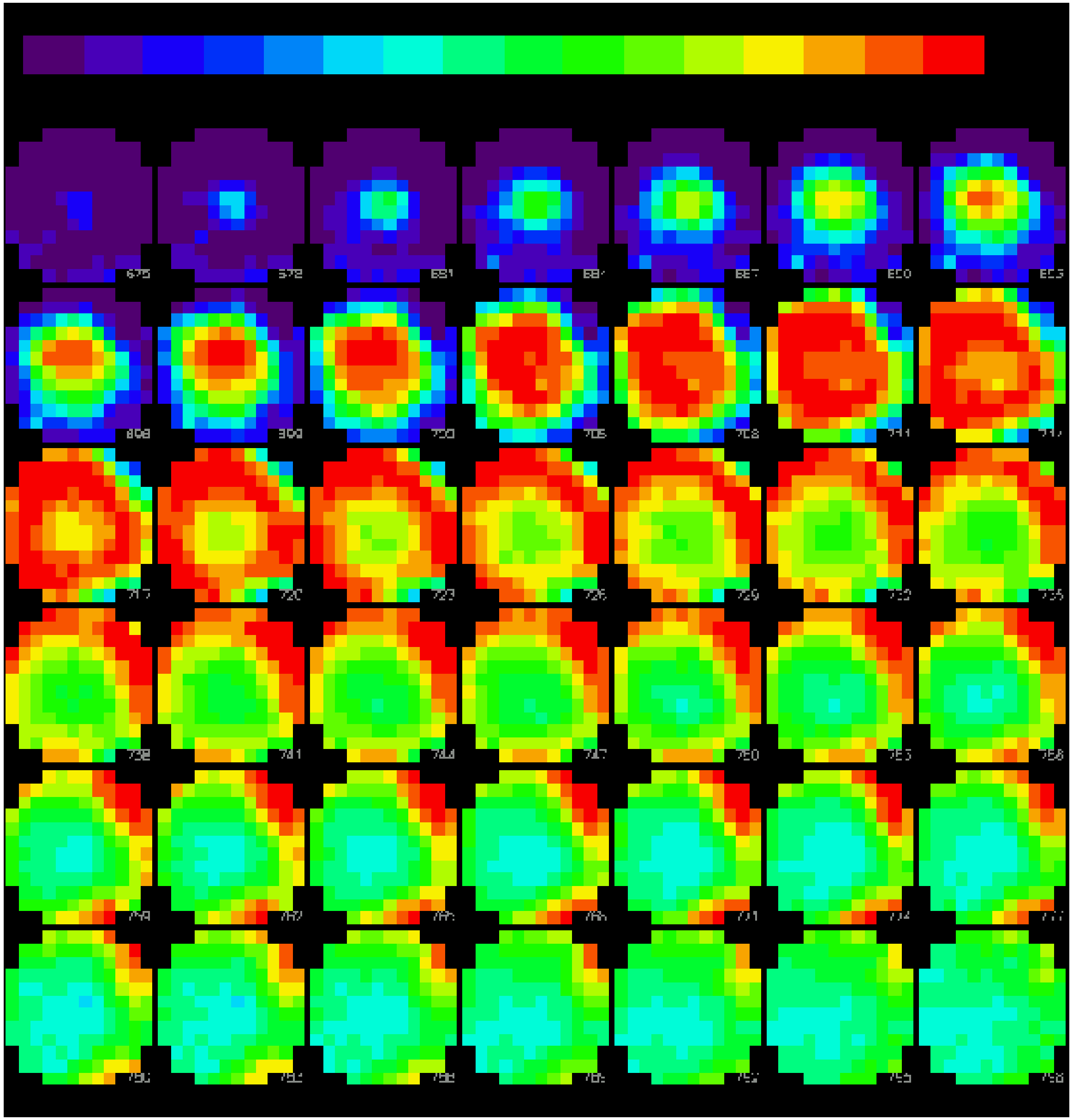
$$\tau = 10,\ \beta = 4.5, \quad g = .5\left(1 + \exp\left(.02\mathrm{rsin}(.5\theta)\right)\right)$$

$$u(r,0) = \mathrm{Kexp}(-.5r^2),\ a(r,0) \equiv 0$$

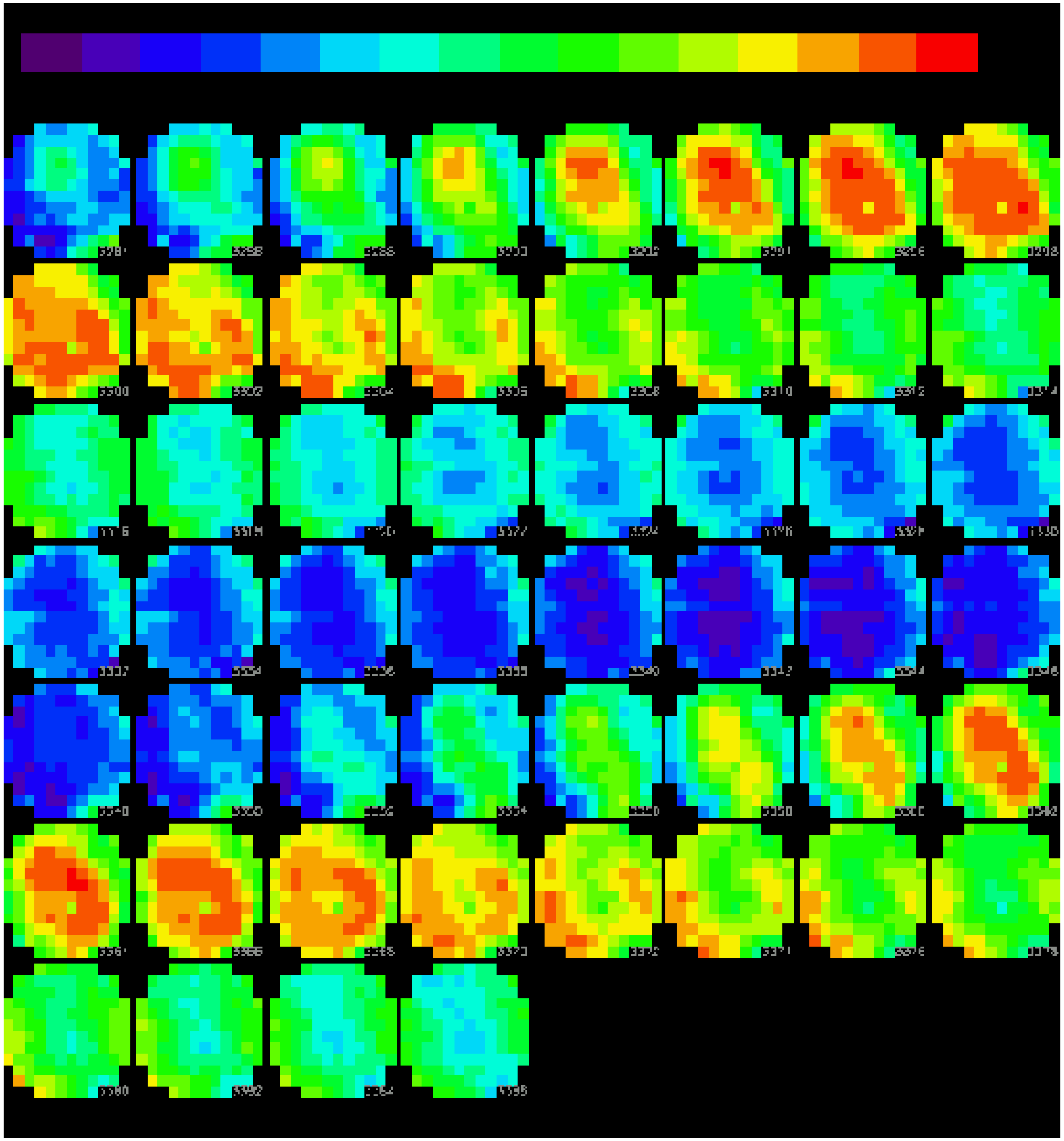


$th = .17$ $\quad th = .19$ $\quad th = .19$

Wu et al (Dec. 2003)

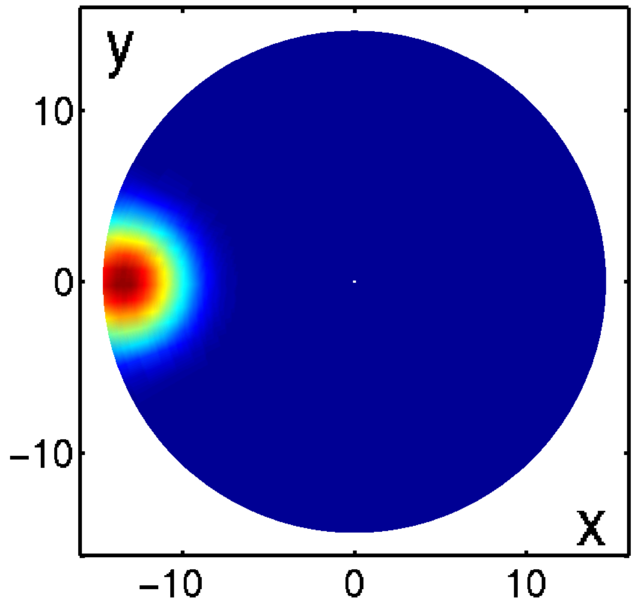


Wu et al (Dec 2003)



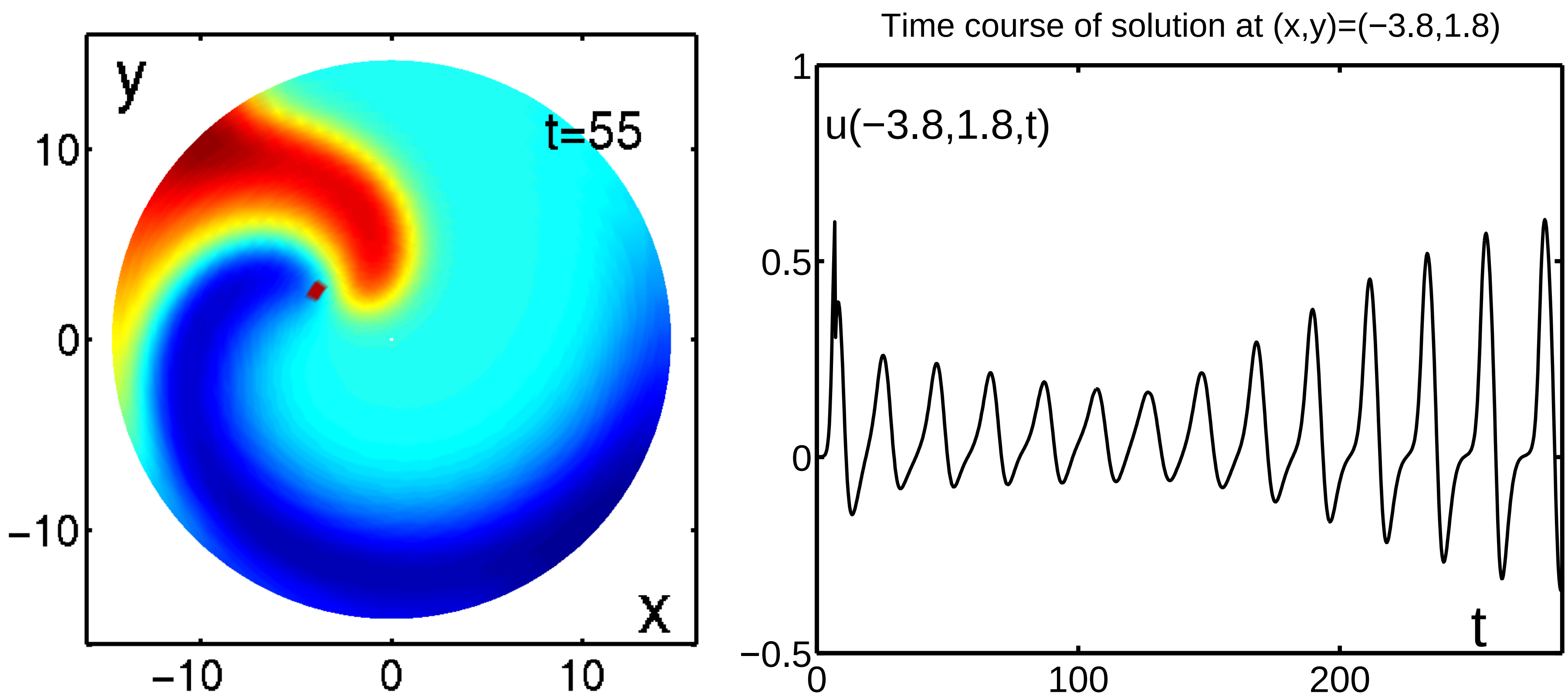
Broken Waves

$$th = .15,\ \tau = 10,\ \beta = 3$$



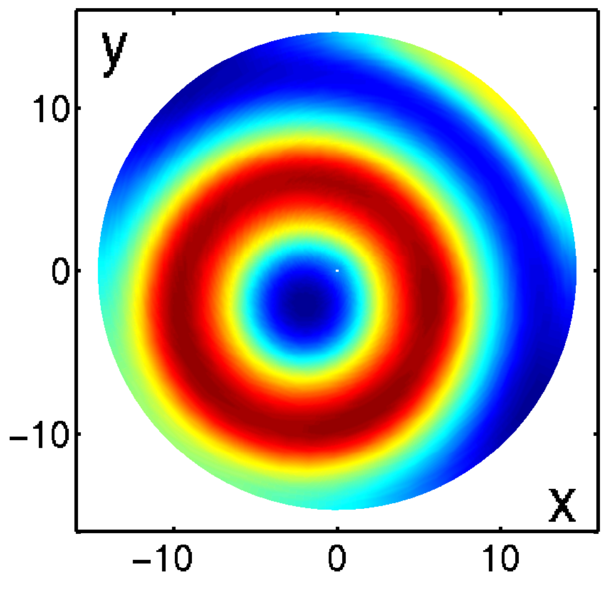
Break early   Later.   Later still. $th = .2$.

Spiral Drift

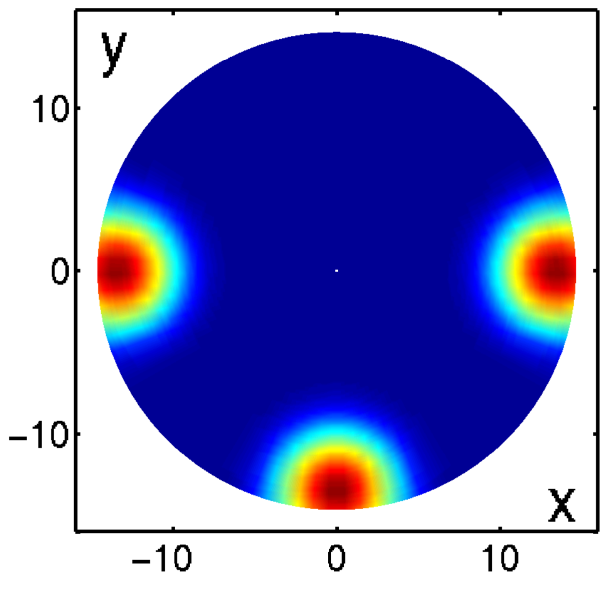


Inhomogeneous Coupling

Periodic Waves

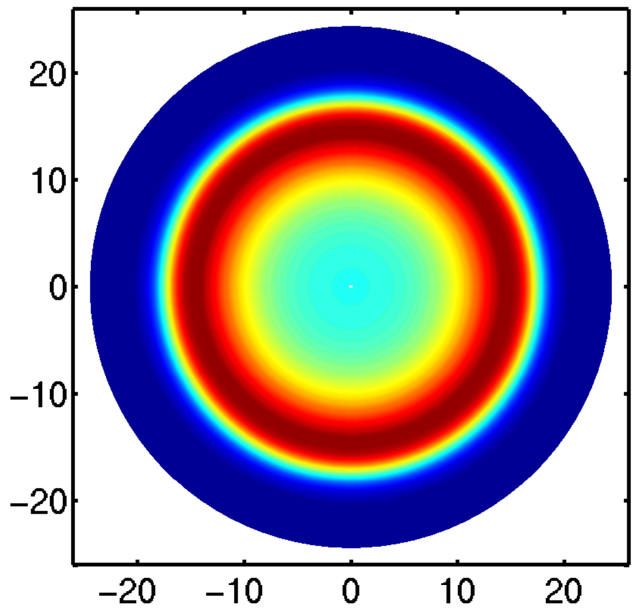


Experiment 3: Jan. 2004



Experiment 4: Jan. 2004

Broken Rings



Symmetric Coupling   Inhomogeneous Coupling