# Neural networks with space-dependent delays

*Steve Coombes*

School of Mathematical Sciences

The University of Nottingham

In collaboration with:
Markus Owen

# Evans functions for integral neural field equations with Heaviside firing rate functions

*Steve Coombes*

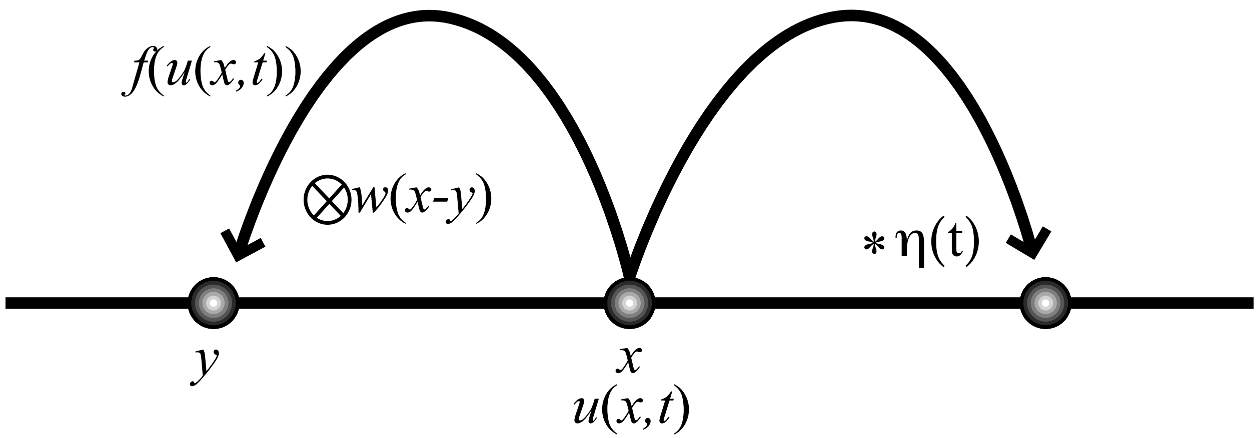
School of Mathematical Sciences

The University of Nottingham

In collaboration with:

Markus Owen

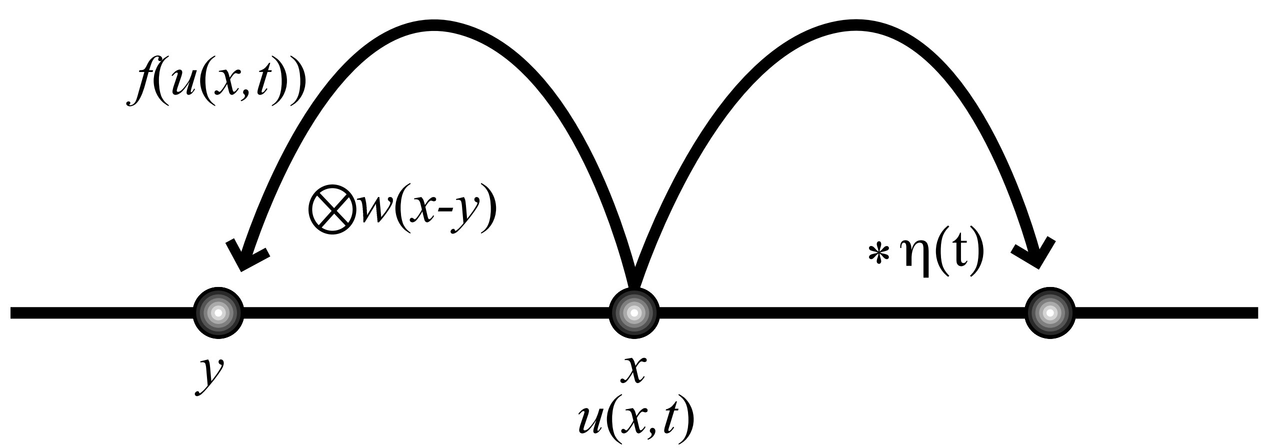
# Firing Rate Model




- Slow synapses (spike train → firing rate)

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \mathrm{d}y w(y) f \circ u(x-y,t) - ga(x,t)$$
$$\frac{1}{\epsilon}\frac{\partial a(x,t)}{\partial t} = -a(x,t) + u(x,t).$$

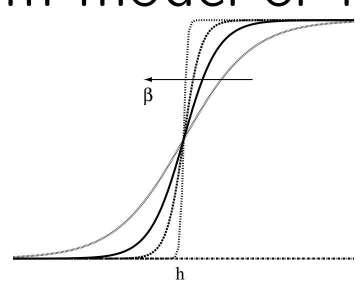
# Firing Rate Model




- Slow synapses (spike train → firing rate)

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \mathrm{d}y w(y) f \circ u(x-y,t) - ga(x,t)$$
$$\frac{1}{\epsilon}\frac{\partial a(x,t)}{\partial t} = -a(x,t) + u(x,t).$$

- Firing rate derived from model or fit to real data



# A scalar integral equation

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ u(x-y, t-s-|y|/v)$$

# A scalar integral equation

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ u(x-y, t-s-|y|/v)$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, recovers the standard model.

# A scalar integral equation

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ u(x-y, t-s-|y|/v)$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, recovers the standard model.
- TW co-ord $\xi = x - ct$, $U(\xi, t) = u(x-ct, t)$:

$$U(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ U(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

# A scalar integral equation

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ u(x-y, t-s-|y|/v)$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, recovers the standard model.
- TW co-ord $\xi = x - ct$, $U(\xi, t) = u(x - ct, t)$:

$$U(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ U(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

- TW solu $U(\xi, t) = q(\xi)$:

$$q(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ q(\xi - y + cs + c|y|/v)$$

# A scalar integral equation

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ u(x-y, t-s-|y|/v)$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, recovers the standard model.
- TW co-ord $\xi = x - ct$, $U(\xi,t) = u(x-ct,t)$:

$$U(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ U(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

- TW solu $U(\xi,t) = q(\xi)$:

$$q(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f \circ q(\xi - y + cs + c|y|/v)$$

- Heaviside firing rate $f(u) = \Theta(u-h)$

$$q(\xi) = \int_0^{\infty} \eta(s) \psi(\xi + cs) \mathrm{d}s,$$

where

$$\psi(\xi) = \int_{-\infty}^{\infty} \mathrm{d}w(y) \Theta(q(\xi - y + c|y|/v) - h)$$

# Linear stability

- Linearise about the steady state: $U(\xi,t) = q(\xi) + u(\xi,t)$, and Taylor expand

$$u(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f'(q(\xi - y + cs + c|y|/v)) \\ \times u(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

# Linear stability

- Linearise about the steady state: $U(\xi,t) = q(\xi) + u(\xi,t)$, and Taylor expand

$$u(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{0}^{\infty} \mathrm{d}s \eta(s) f'(q(\xi - y + cs + c|y|/v)) \times u(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

- Bounded smooth solutions $u(\xi,t) = u(\xi)\mathrm{e}^{\lambda t}$ generates the eigenvalue equation $u = \mathcal{L}u$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{\xi - y + c|y|/v}^{\infty} \frac{\mathrm{d}s}{c} \eta(-\xi/c + y/c - |y|/v + s/c) \times \mathrm{e}^{-\lambda(-\xi/c + y/c + s/c)} f'(q(s))u(s)$$

# Linear stability

- Linearise about the steady state: $U(\xi,t) = q(\xi) + u(\xi,t)$, and Taylor expand

$$u(\xi,t) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) f'(q(\xi - y + cs + c|y|/v)) \\ \times u(\xi - y + cs + c|y|/v, t - s - |y|/v)$$

- Bounded smooth solutions $u(\xi,t) = u(\xi)\mathrm{e}^{\lambda t}$ generates the eigenvalue equation $u = \mathcal{L}u$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{\xi - y + c|y|/v}^{\infty} \frac{\mathrm{d}s}{c} \eta(-\xi/c + y/c - |y|/v + s/c) \\ \times \mathrm{e}^{-\lambda(-\xi/c + y/c + s/c)} f'(q(s)) u(s)$$

- Let $\sigma(\mathcal{L})$ be the spectrum of $\mathcal{L}$. TW is linearly stable if

$$\max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathcal{L}),\ \lambda \neq 0\} \leq -K,$$

for some $K > 0$, and $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}$.
We shall take it that linear stability implies nonlinear stability.

# Travelling fronts

- $q(\xi) > h$ for $\xi > 0$

$$\psi(\xi) = \begin{cases} \int_{-\infty}^{\xi/(1-c/v)} w(y)\mathrm{d}y & \xi > 0 \\ \int_{-\infty}^{\xi/(1+c/v)} w(y)\mathrm{d}y & \xi < 0 \end{cases}$$

# Travelling fronts

- $q(\xi) > h$ for $\xi > 0$

$$\psi(\xi) = \begin{cases} \int_{-\infty}^{\xi/(1-c/v)} w(y) \mathrm{d}y & \xi > 0 \\ \int_{-\infty}^{\xi/(1+c/v)} w(y) \mathrm{d}y & \xi < 0 \end{cases}$$

- Self-consistent speed

$$q(0) = h$$

(choice of origin for one-parameter family of waves)

# Travelling fronts

- $q(\xi) > h$ for $\xi > 0$

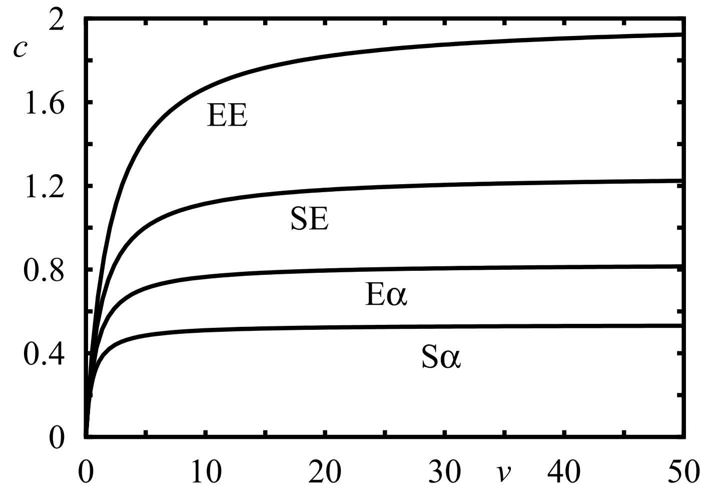
$$\psi(\xi) = \begin{cases} \int_{-\infty}^{\xi/(1-c/v)} w(y)\mathrm{d}y & \xi > 0 \\ \int_{-\infty}^{\xi/(1+c/v)} w(y)\mathrm{d}y & \xi < 0 \end{cases}$$

- Self-consistent speed

$$q(0) = h$$

(choice of origin for one-parameter family of waves)

- eg $w(x) = \mathrm{e}^{-|x|}/2$, $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$:




$c \sim v$ for small $v$.

# Evans function for travelling front

- $f(u) = \Theta(u-h)$, so $f'(q) = \delta(q-h)$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{q(\xi - y + c|y|/v)}^{q(\infty)} \frac{\mathrm{d}z}{c} \eta(q^{-1}(z)/c - \xi/c + y/c - |y|/v)$$
$$\times \, \mathrm{e}^{-\lambda(q^{-1}(z)/c - \xi/c + y/c)} \frac{\delta(z-h)}{|q'(q^{-1}(z))|} u(q^{-1}(z))$$

# Evans function for travelling front

- $f(u) = \Theta(u-h)$, so $f'(q) = \delta(q-h)$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{q(\xi - y + c|y|/v)}^{q(\infty)} \frac{\mathrm{d}z}{c} \eta(q^{-1}(z)/c - \xi/c + y/c - |y|/v)$$
$$\times\, \mathrm{e}^{-\lambda(q^{-1}(z)/c - \xi/c + y/c)} \frac{\delta(z-h)}{|q'(q^{-1}(z))|} u(q^{-1}(z))$$

- When $z = h$, $q^{-1}(h) = 0$.

$$u(\xi) = \frac{u(0)}{c|q'(0)|} \int_{-\infty}^{\infty} \mathrm{d}y w(y) \eta(-\xi/c + y/c - |y|/v) \mathrm{e}^{-\lambda(y-\xi)/c}$$

# Evans function for travelling front

- $f(u) = \Theta(u-h)$, so $f'(q) = \delta(q-h)$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{q(\xi - y + c|y|/v)}^{q(\infty)} \frac{\mathrm{d}z}{c} \eta(q^{-1}(z)/c - \xi/c + y/c - |y|/v)$$
$$\times\, \mathrm{e}^{-\lambda(q^{-1}(z)/c - \xi/c + y/c)} \frac{\delta(z-h)}{|q'(q^{-1}(z))|} u(q^{-1}(z))$$

- When $z = h$, $q^{-1}(h) = 0$.

$$u(\xi) = \frac{u(0)}{c|q'(0)|} \int_{-\infty}^{\infty} \mathrm{d}y w(y) \eta(-\xi/c + y/c - |y|/v) \mathrm{e}^{-\lambda(y-\xi)/c}$$

- Self-consistent condition (at $\xi = 0$) has a non-trivial solution if $\mathcal{E}(\lambda) = 0$, where

$$\mathcal{E}(\lambda) = 1 - \frac{1}{c|q'(0)|} \int_{-\infty}^{\infty} \mathrm{d}y w(y) \eta(y/c - |y|/v) \mathrm{e}^{-\lambda y/c}$$

# Evans function for travelling front

- $f(u) = \Theta(u-h)$, so $f'(q) = \delta(q-h)$:

$$u(\xi) = \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_{q(\xi - y + c|y|/v)}^{q(\infty)} \frac{\mathrm{d}z}{c} \eta(q^{-1}(z)/c - \xi/c + y/c - |y|/v) \times \mathrm{e}^{-\lambda(q^{-1}(z)/c - \xi/c + y/c)} \frac{\delta(z-h)}{|q'(q^{-1}(z))|} u(q^{-1}(z))$$

- When $z = h$, $q^{-1}(h) = 0$.

$$u(\xi) = \frac{u(0)}{c|q'(0)|} \int_{-\infty}^{\infty} \mathrm{d}y w(y) \eta(-\xi/c + y/c - |y|/v) \mathrm{e}^{-\lambda(y-\xi)/c}$$

- Self-consistent condition (at $\xi = 0$) has a non-trivial solution if $\mathcal{E}(\lambda) = 0$, where

$$\mathcal{E}(\lambda) = 1 - \frac{1}{c|q'(0)|} \int_{-\infty}^{\infty} \mathrm{d}y w(y) \eta(y/c - |y|/v) \mathrm{e}^{-\lambda y/c}$$

- For our example front is stable (ES: $\lambda = -\alpha + ipc$) since

$$\mathcal{E}(\lambda) = \frac{\lambda}{\frac{c}{\sigma} + \alpha\left(1 - \frac{c}{v}\right) + \lambda}$$

$v \to \infty$ to recover the result of Linghai Zhang [DIE, 16, (2003), 513-536]

# A model with linear recovery

- Spike frequency adaptation

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - ga(x,t), \qquad Q_a a(x,t) = u(x,t)$$

Notation

$$(w \otimes f)(x,t) = \int_{-\infty}^{\infty} w(y) f(x-y, t-|y|/v) \mathrm{d}y$$

$$Q\eta(t) = \delta(t), \qquad Q_a \eta_a(t) = \delta(t)$$

# A model with linear recovery

- Spike frequency adaptation

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - ga(x,t), \qquad Q_a a(x,t) = u(x,t)$$

Notation

$$(w \otimes f)(x,t) = \int_{-\infty}^{\infty} w(y) f(x-y, t-|y|/v) \mathrm{d}y$$

$$Q\eta(t) = \delta(t), \qquad Q_a \eta_a(t) = \delta(t)$$

- Integrated form $\boxed{u = \eta * w \otimes f \circ u - g\eta_b * u}$ where

$$(\eta * f)(x,t) = \int_0^t \eta(s) f(x, t-s) \mathrm{d}s, \qquad \eta_b = \eta * \eta_a$$

# A model with linear recovery

- Spike frequency adaptation

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - ga(x,t), \qquad Q_a a(x,t) = u(x,t)$$

Notation

$$(w \otimes f)(x,t) = \int_{-\infty}^{\infty} w(y) f(x-y, t-|y|/v) \mathrm{d}y$$

$$Q\eta(t) = \delta(t), \qquad Q_a \eta_a(t) = \delta(t)$$

- Integrated form $u = \eta * w \otimes f \circ u - g\eta_b * u$ where

$$(\eta * f)(x,t) = \int_0^t \eta(s) f(x, t-s) \mathrm{d}s, \qquad \eta_b = \eta * \eta_a$$

- Re-arrange: $[1 + g\eta_b *]u = \eta * w \otimes f \circ u$ giving TW:

$$q(\xi) = \int_0^{\infty} \eta_c(z) \psi(\xi + cz) \mathrm{d}z$$

$$\eta_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\eta_c}(k) \mathrm{e}^{ikt} \mathrm{d}k, \qquad \widehat{\eta_c}(k) = \frac{\widehat{\eta}(k)}{1 + g\widehat{\eta_b}(k)}$$

# Fronts

- Evans function $\mathcal{E}(\lambda) = 1 - \mathcal{H}(\lambda)/\mathcal{H}(0)$:

$$\mathcal{H}(\lambda) = \int_0^\infty \mathrm{d}y w(y) \eta_c(y/c - y/v) \mathrm{e}^{-\lambda y/c}$$

# Fronts

- Evans function $\mathcal{E}(\lambda) = 1 - \mathcal{H}(\lambda)/\mathcal{H}(0)$:

$$\mathcal{H}(\lambda) = \int_0^\infty \mathrm{d}y w(y) \eta_c(y/c - y/v) \mathrm{e}^{-\lambda y/c}$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, $\eta_a(t) = \mathrm{e}^{-t}$ and $w(x) = \mathrm{e}^{-|x|}/2$

$$\eta_c(t) = \frac{\alpha}{k_- - k_+} \left\{ (1 - k_+) \mathrm{e}^{-k_+ t} - (1 - k_-) \mathrm{e}^{-k_- t} \right\}$$

$$k_\pm = \frac{1 + \alpha \pm \sqrt{(1+\alpha)^2 - 4\alpha(1+g)}}{2}$$

# Fronts

- Evans function $\mathcal{E}(\lambda) = 1 - \mathcal{H}(\lambda)/\mathcal{H}(0)$:

$$\mathcal{H}(\lambda) = \int_0^\infty \mathrm{d}y w(y) \eta_c(y/c - y/v) \mathrm{e}^{-\lambda y/c}$$

- $\eta(t) = \alpha \mathrm{e}^{-\alpha t}$, $\eta_a(t) = \mathrm{e}^{-t}$ and $w(x) = \mathrm{e}^{-|x|}/2$

$$\eta_c(t) = \frac{\alpha}{k_- - k_+} \left\{ (1 - k_+)\mathrm{e}^{-k_+ t} - (1 - k_-)\mathrm{e}^{-k_- t} \right\}$$

$$k_\pm = \frac{1 + \alpha \pm \sqrt{(1+\alpha)^2 - 4\alpha(1+g)}}{2}$$

$q(0) = h$ gives an implicit expression for the front speed as

$$cm_- = \frac{1}{2}\left[1 + \alpha - \frac{\alpha}{2h} \pm \sqrt{\left(1 + \alpha - \frac{\alpha}{2h}\right)^2 - 4\alpha\left(1 + g - \frac{1}{2h}\right)}\right] \qquad c > 0$$

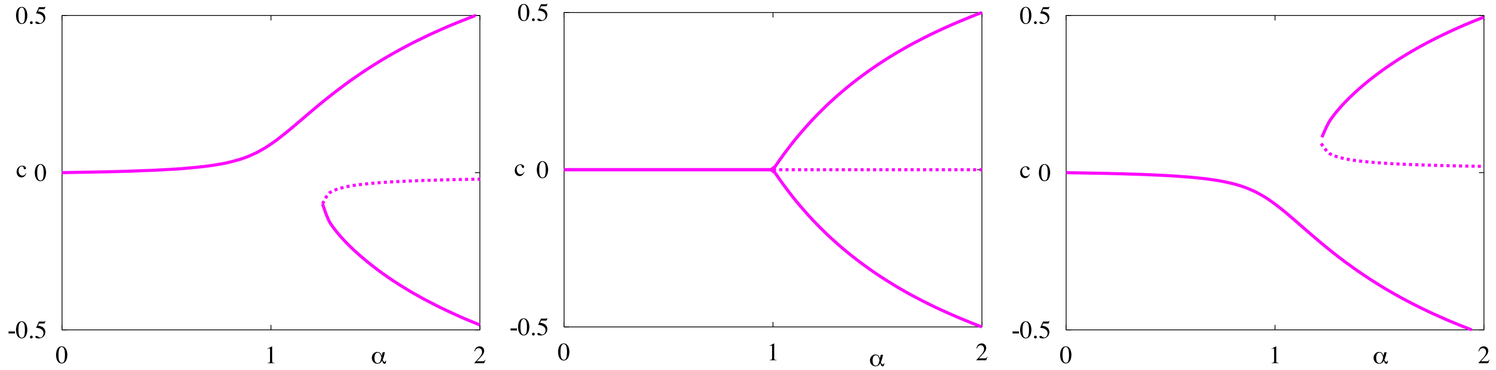
$$cm_+ = \frac{1}{2}\left[1 + \alpha - \frac{\alpha}{2h^*} \pm \sqrt{\left(1 + \alpha - \frac{\alpha}{2h^*}\right)^2 - 4\alpha\left(1 + g - \frac{1}{2h^*}\right)}\right] \qquad c < 0$$

where $h^* = 1/(1+g) - h$ and $m_\pm = v/(c \pm v)$.

$v \to \infty$ to recover the result of Bressloff and Folias [preprint (2004)]

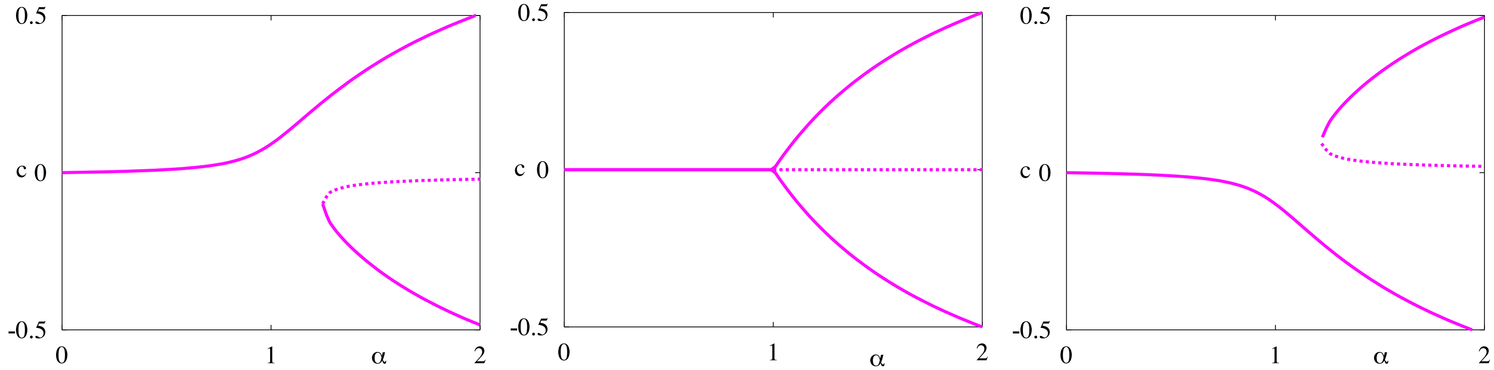
# Front bifurcations

If $g = g_c$, where $2h(1+g_c) = 1$, there is a front for all $\alpha$ with speed $c = 0$. At a critical value of $\alpha$ this stationary front undergoes a pitchfork bifurcation leading to a pair of fronts traveling in opposite directions. If this critical condition is not met then the pitchfork bifurcation is broken:



# Front bifurcations

If $g = g_c$, where $2h(1+g_c) = 1$, there is a front for all $\alpha$ with speed $c = 0$. At a critical value of $\alpha$ this stationary front undergoes a pitchfork bifurcation leading to a pair of fronts traveling in opposite directions. If this critical condition is not met then the pitchfork bifurcation is broken:

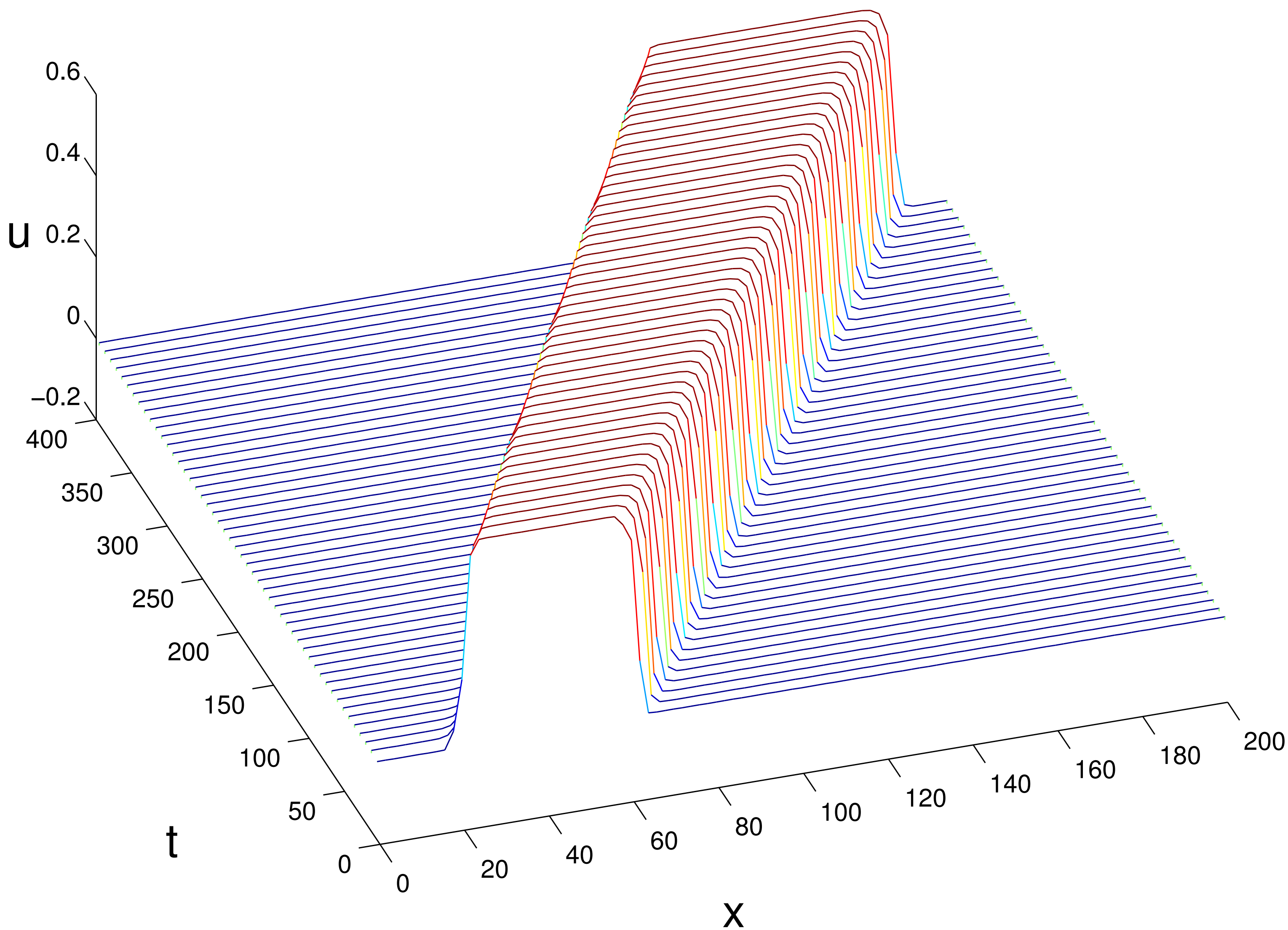



On the branch with $c = 0$ and $g = g_c$ (defining a stationary front) we find that ES: $\lambda = -k_\pm + ipc$ and

$$\mathcal{E}(\lambda) = \lambda \frac{(\lambda + k_+ + k_- - k_+ k_-)}{(\lambda + k_+)(\lambda + k_-)},$$

which has zeros when $\lambda = 0$ and $\lambda = \alpha g_c - 1$. Hence, the stationary front changes from stable to unstable as $\alpha$ is increased through $1/g_c$.

# Front bifurcations: simulations



# Pulses

- $q(\xi) > h$ for $\xi \in (0, \Delta)$ and $q(\xi) < h$ otherwise.

$$\psi(\xi) = \begin{cases} \mathcal{F}\left(\frac{-\xi}{1+c/v}, \frac{\Delta-\xi}{1+c/v}\right) & \xi \leq 0 \\ \mathcal{F}\left(0, \frac{\xi}{1-c/v}\right) + \mathcal{F}\left(0, \frac{\Delta-\xi}{1+c/v}\right) & 0 < \xi < \Delta \\ \mathcal{F}\left(\frac{\xi-\Delta}{1-c/v}, \frac{\xi}{1-c/v}\right) & \xi \geq \Delta \end{cases}$$

where

$$\mathcal{F}(a, b) = \int_a^b w(y) \mathrm{d}y.$$

The dispersion relation $c = c(\Delta)$ is then implicitly defined by the simultaneous solution of $q(0) = h$ and $q(\Delta) = h$ $(\Delta > 0)$.

# Pulses

- $q(\xi) > h$ for $\xi \in (0, \Delta)$ and $q(\xi) < h$ otherwise.

$$\psi(\xi) = \begin{cases} \mathcal{F}\left(\frac{-\xi}{1+c/v}, \frac{\Delta-\xi}{1+c/v}\right) & \xi \leq 0 \\ \mathcal{F}\left(0, \frac{\xi}{1-c/v}\right) + \mathcal{F}\left(0, \frac{\Delta-\xi}{1+c/v}\right) & 0 < \xi < \Delta \\ \mathcal{F}\left(\frac{\xi-\Delta}{1-c/v}, \frac{\xi}{1-c/v}\right) & \xi \geq \Delta \end{cases}$$

where

$$\mathcal{F}(a, b) = \int_a^b w(y) \mathrm{d}y.$$

The dispersion relation $c = c(\Delta)$ is then implicitly defined by the simultaneous solution of $q(0) = h$ and $q(\Delta) = h$ $(\Delta > 0)$. Linearising around the traveling pulse solution and proceeding as before we obtain an eigenvalue equation of the form $u = \mathcal{J}_c u$, where

$$\boxed{\mathcal{J}_c u(\xi) = A_c(\xi, \lambda) u(0) + B_c(\xi, \lambda) u(\Delta)}$$

$$A_c(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w(y)\eta_c(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$
$$B_c(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w(y)\eta_c((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

Demanding that the eigenvalue problem $u = \mathcal{J}_c u$ be self-consistent at $\xi = 0$ and $\xi = \Delta$ gives the system of equations

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}_c(\lambda)\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \qquad \mathcal{A}_c(\lambda) = \begin{bmatrix} A_c(0,\lambda) & B_c(0,\lambda) \\ A_c(\Delta,\lambda) & B_c(\Delta,\lambda) \end{bmatrix}$$

$$A_c(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w(y)\eta_c(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$B_c(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w(y)\eta_c((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

Demanding that the eigenvalue problem $u = \mathcal{J}_c u$ be self-consistent at $\xi = 0$ and $\xi = \Delta$ gives the system of equations

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}_c(\lambda)\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \qquad \mathcal{A}_c(\lambda) = \begin{bmatrix} A_c(0,\lambda) & B_c(0,\lambda) \\ A_c(\Delta,\lambda) & B_c(\Delta,\lambda) \end{bmatrix}$$

Nontrivial solution if $\mathcal{E}(\lambda) = 0$, where

$$\boxed{\mathcal{E}(\lambda) = \det(\mathcal{A}_c(\lambda) - I)}$$

which we recognise as the Evans function for a pulse.

# A pair of travelling pulses

- $Q_a = \partial_t$

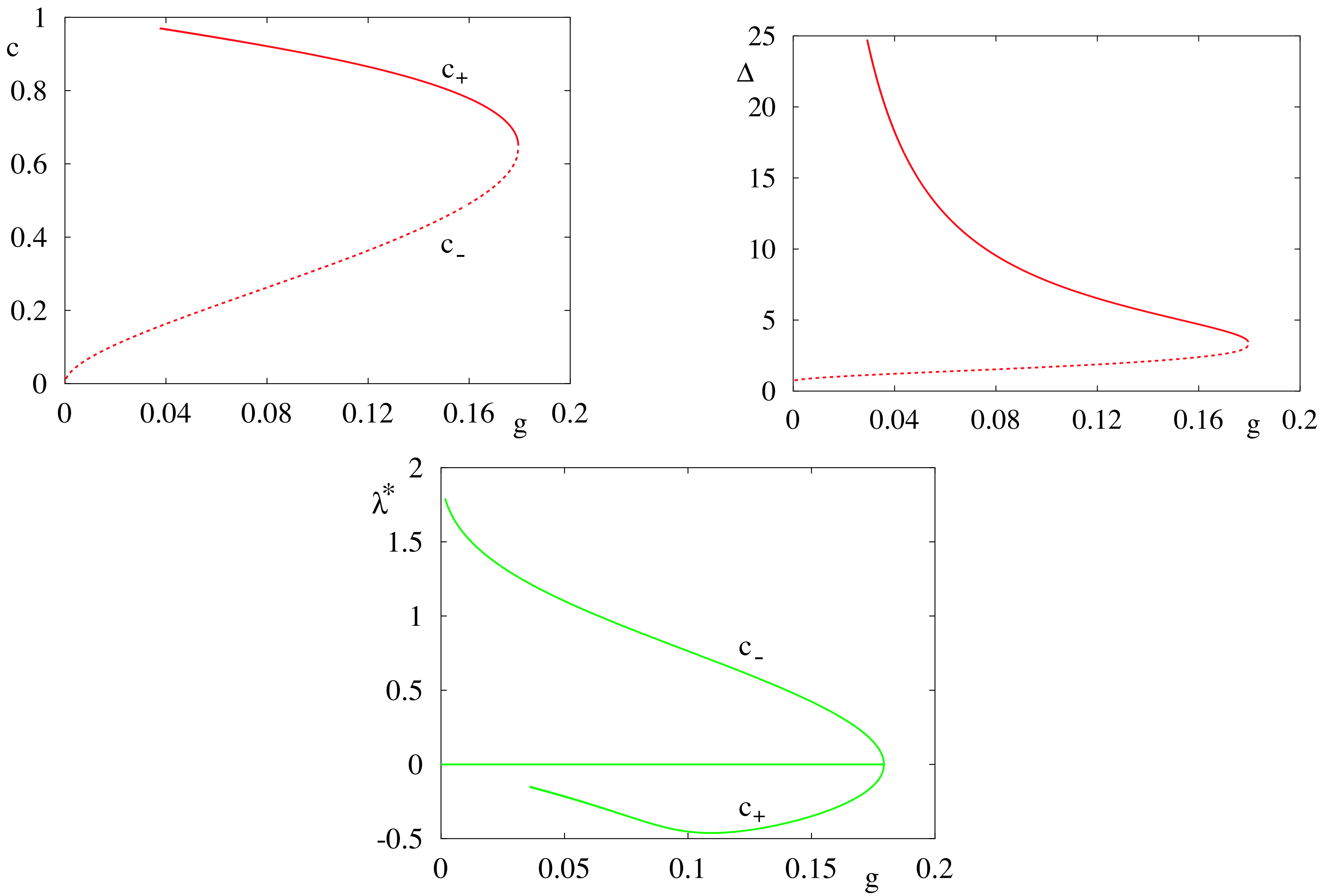
$$\eta_c(t) = \frac{\alpha}{k_+ - k_-}\left\{k_+ \mathrm{e}^{-k_+ t} - k_- \mathrm{e}^{-k_- t}\right\}$$

$$k_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4\alpha g}}{2}$$

- $q(0) = h = q(\Delta)$ gives

$$h = \frac{\alpha c(1 - \mathrm{e}^{-\Delta})}{2(c^2 + \alpha c + \alpha g)}$$

$$h = \frac{\alpha}{k_+ - k_-}\left\{\mathrm{e}^{-k_-\Delta/c} + \frac{k_-}{2}\left(\frac{\mathrm{e}^{-\Delta} - \mathrm{e}^{-k_-\Delta/c}}{k_- - c} + \frac{1 - \mathrm{e}^{-k_-\Delta/c}}{k_- + c}\right)\right.$$
$$\left. - \mathrm{e}^{-k_+\Delta/c} - \frac{k_+}{2}\left(\frac{\mathrm{e}^{-\Delta} - \mathrm{e}^{-k_+\Delta/c}}{k_+ - c} + \frac{1 - \mathrm{e}^{-k_+\Delta/c}}{k_+ + c}\right)\right\}$$




$v \to \infty$ to confirm a conjecture of Pinto and Ermentrout [SIAM J Appl Math, 62 (2001), 206-225]

# A model with nonlinear recovery/lateral inhibition

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - g(w_a \otimes a)(x,t) \qquad Q_a a(x,t) = f \circ u(x,t)$$

$$u = [\eta * w \otimes -g\eta_b * w_a \otimes] f \circ u$$

# A model with nonlinear recovery/lateral inhibition

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - g(w_a \otimes a)(x,t) \qquad Q_a a(x,t) = f \circ u(x,t)$$

$$\boxed{u = [\eta * w \otimes - g\eta_b * w_a \otimes] f \circ u}$$

- TW

$$q(\xi) = \left( \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) - g \int_{-\infty}^{\infty} \mathrm{d}y w_a(y) \int_0^{\infty} \mathrm{d}s \eta_b(s) \right) \times \Theta(q(\xi - y + cs + c|y|/v) - h)$$

# A model with nonlinear recovery/lateral inhibition

$$Qu(x,t) = (w \otimes f \circ u)(x,t) - g(w_a \otimes a)(x,t) \qquad Q_a a(x,t) = f \circ u(x,t)$$

$$\boxed{u = [\eta * w \otimes - g\eta_b * w_a \otimes] f \circ u}$$

- TW

$$q(\xi) = \left( \int_{-\infty}^{\infty} \mathrm{d}y w(y) \int_0^{\infty} \mathrm{d}s \eta(s) - g \int_{-\infty}^{\infty} \mathrm{d}y w_a(y) \int_0^{\infty} \mathrm{d}s \eta_b(s) \right) \times \Theta(q(\xi - y + cs + c|y|/v) - h)$$

- Stability: eigenvalue equation of the form $u = \mathcal{L}u - g\mathcal{J}u$, where

$$\mathcal{L}u(\xi) = A(\xi,\lambda)u(0) + B(\xi,\lambda)u(\Delta)$$
$$\mathcal{J}u(\xi) = C(\xi,\lambda)u(0) + D(\xi,\lambda)u(\Delta)$$

where

$$A(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y\, w(y)\eta(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$B(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y\, w(y)\eta((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

$$C(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y\, w_a(y)\eta_b(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$D(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y\, w_a(y)\eta_b((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

where

$$A(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w(y)\eta(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$B(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w(y)\eta((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

$$C(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w_a(y)\eta_b(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$D(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w_a(y)\eta_b((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

Demanding that perturbations be determined self consistently at $\xi = 0$ and $\xi = \Delta$ gives the system of equations

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}(\lambda)\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \quad \mathcal{A}(\lambda) = \begin{bmatrix} A(0,\lambda) - gC(0,\lambda) & B(0,\lambda) - gD(0,\lambda) \\ A(\Delta,\lambda) - gC(\Delta,\lambda) & B(\Delta,\lambda) - gD(\Delta,\lambda) \end{bmatrix}$$

where

$$A(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w(y)\eta(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$B(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w(y)\eta((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

$$C(\xi,\lambda) = \frac{1}{c|q'(0)|}\int_{\frac{\xi}{1-c/v}}^{\infty} \mathrm{d}y w_a(y)\eta_b(-\xi/c + y/c - y/v)\mathrm{e}^{-\lambda(y-\xi)/c}$$

$$D(\xi,\lambda) = \frac{1}{c|q'(\Delta)|}\int_{\frac{\xi-\Delta}{1+c/v}}^{\infty} \mathrm{d}y w_a(y)\eta_b((\Delta-\xi)/c + y/c - |y|/v)\mathrm{e}^{-\lambda(y-(\xi-\Delta))/c}$$

Demanding that perturbations be determined self consistently at $\xi = 0$ and $\xi = \Delta$ gives the system of equations

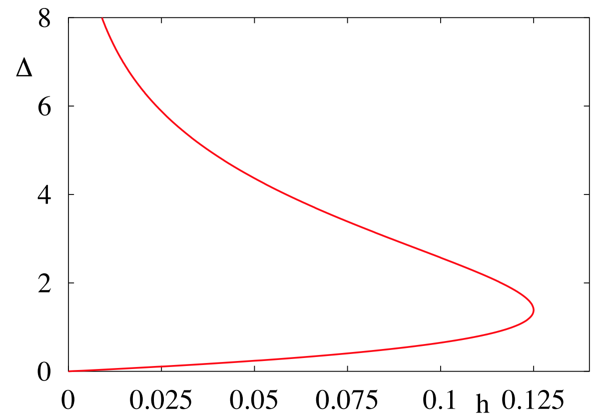
$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}(\lambda)\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \quad \mathcal{A}(\lambda) = \begin{bmatrix} A(0,\lambda) - gC(0,\lambda) & B(0,\lambda) - gD(0,\lambda) \\ A(\Delta,\lambda) - gC(\Delta,\lambda) & B(\Delta,\lambda) - gD(\Delta,\lambda) \end{bmatrix}$$

Nontrivial solution if $\mathcal{E}(\lambda) = 0$, where the Evans function is

$$\boxed{\mathcal{E}(\lambda) = \det(\mathcal{A}(\lambda) - I)}$$

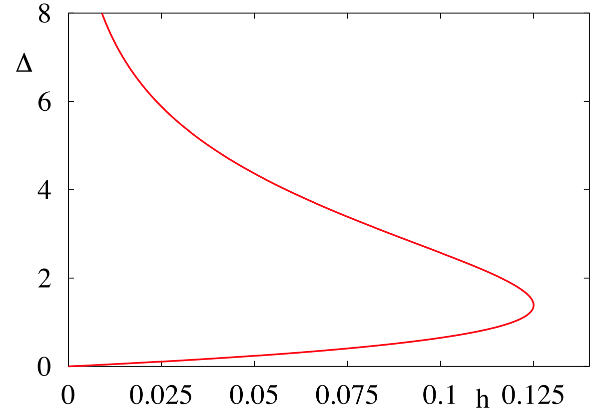
# Dynamic instability of a standing pulse

$$\eta(t) = \alpha e^{-\alpha t},\ \eta_a(t) = e^{-t},\ w(x) = e^{-|x|}/2,\ w_a(x) = e^{-|x|/\sigma_a}/(2\sigma_a)$$

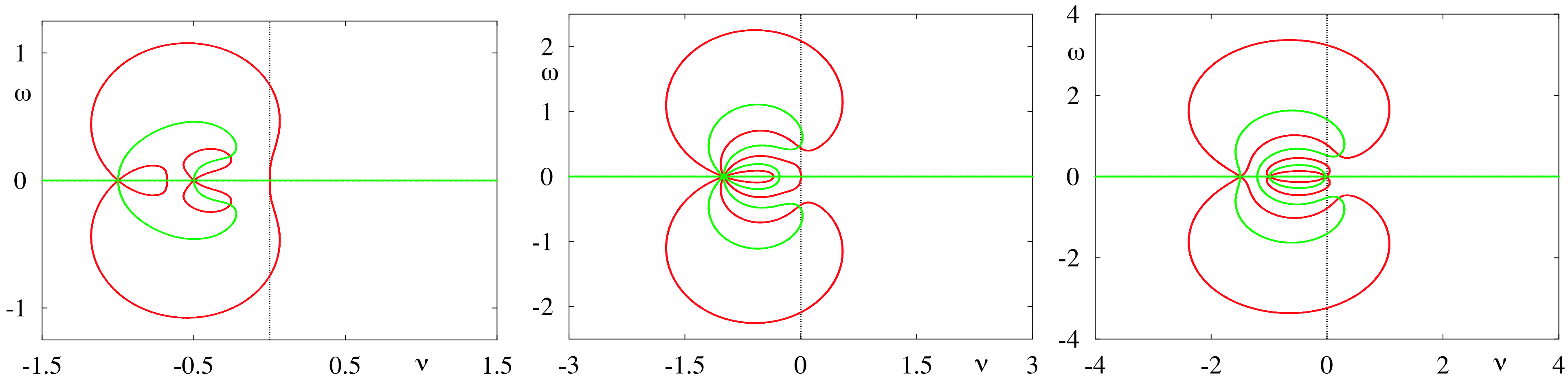


# Dynamic instability of a standing pulse

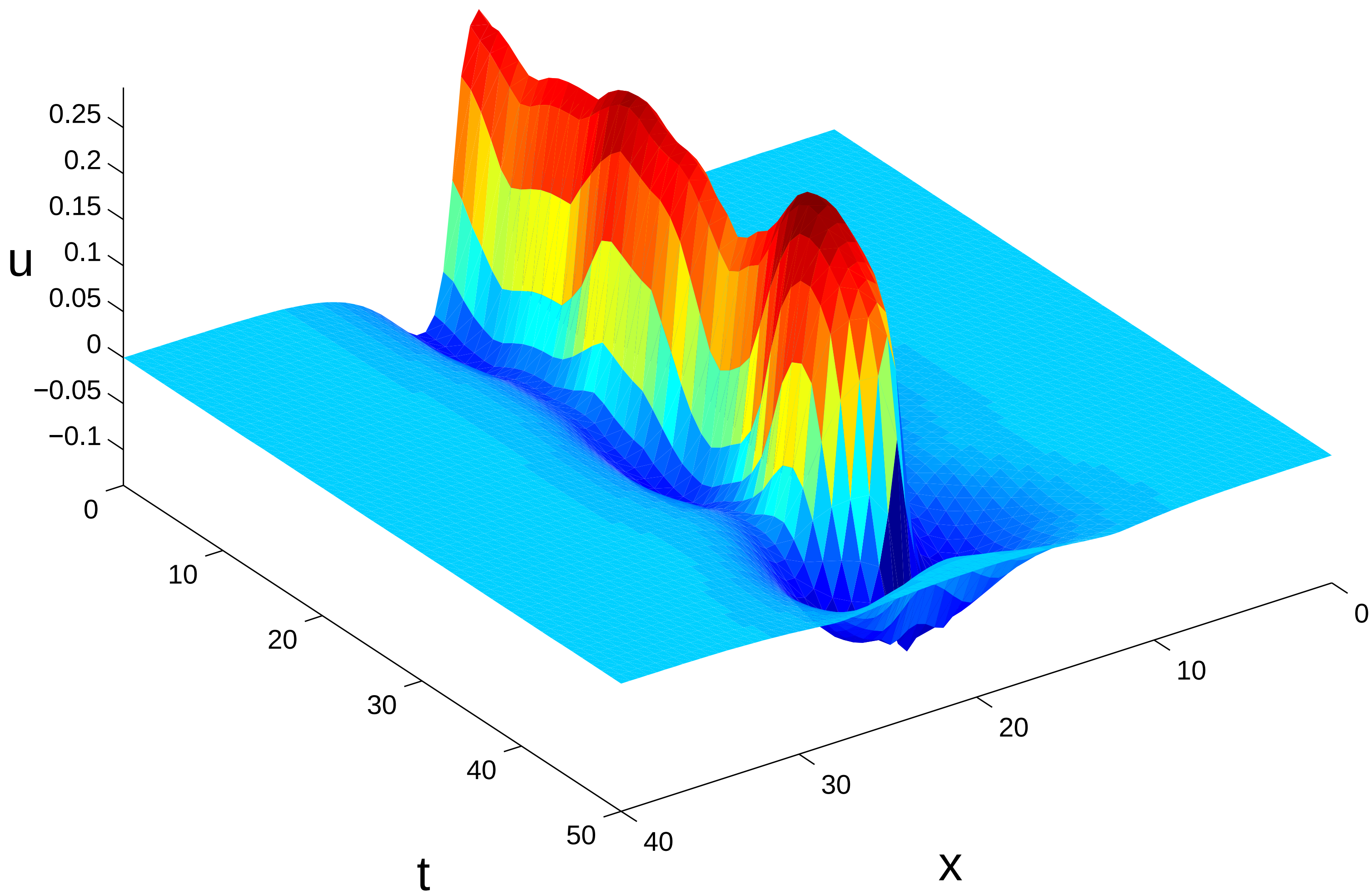
$$\eta(t) = \alpha e^{-\alpha t},\ \eta_a(t) = e^{-t},\ w(x) = e^{-|x|}/2,\ w_a(x) = e^{-|x|/\sigma_a}/(2\sigma_a)$$



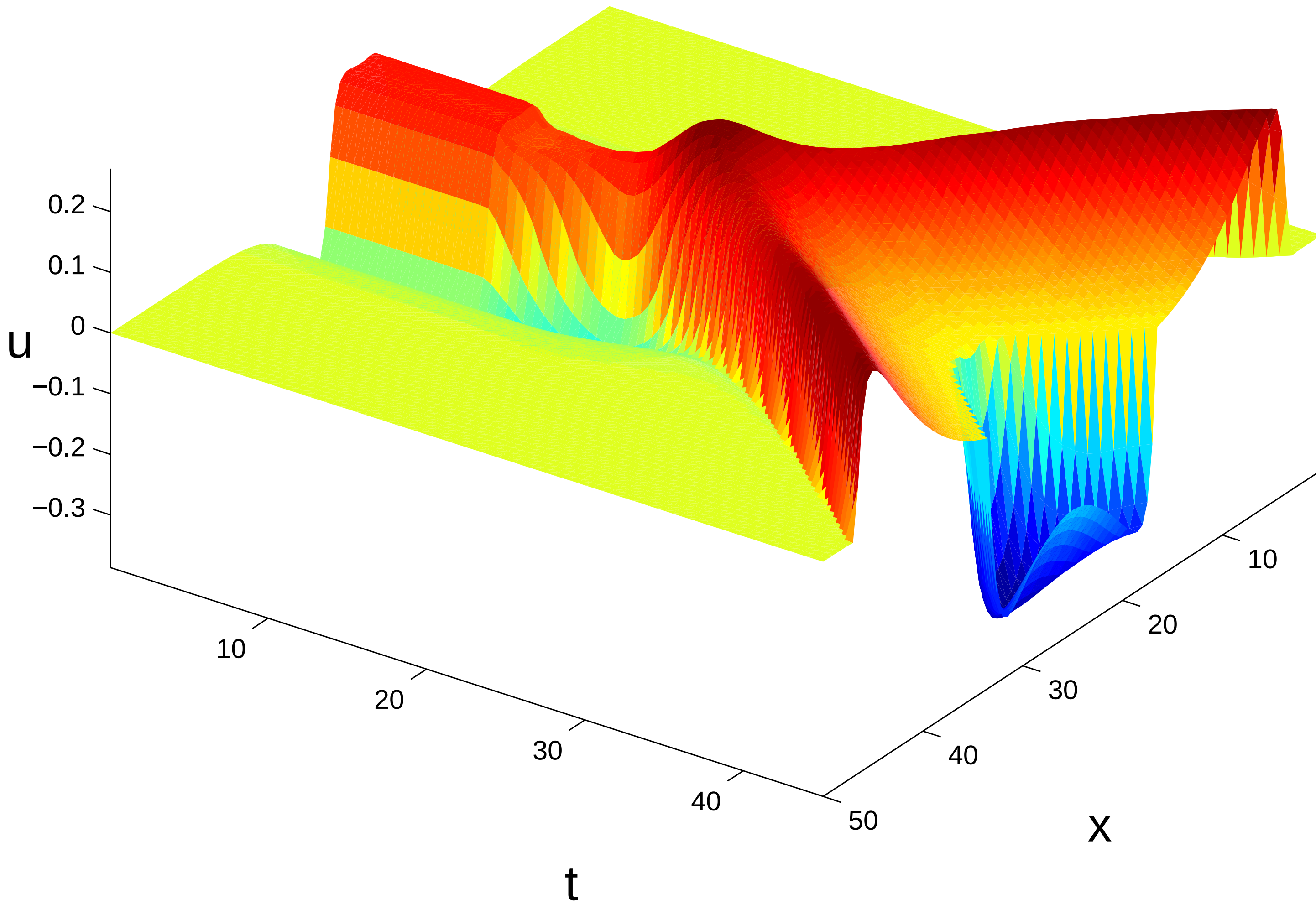

- Let $\lambda = \nu + i\omega$ and plot Re $\mathcal{E}(\lambda) = 0 =$ Im $\mathcal{E}(\lambda)$




$v \to \infty$ to recover work of Pinto and Ermentrout [SIAM J Appl Math, II]



u
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0
10
20
30
40
50
t
0
10
20
30
40
x


u
0.2
0.1
0
−0.1
−0.2
−0.3
10
20
30
40
t
0
10
20
30
40
50
x

# Summary

- Evans function for integral neural field equations with

  Arbitrary synaptic footprints
  Arbitrary synaptic response
  Space dependent delays

  [For a Heaviside]

# Summary

- Evans function for integral neural field equations with

  Arbitrary synaptic footprints
  Arbitrary synaptic response
  Space dependent delays

  [For a Heaviside]

- Usual properties for $\mathcal{E}(\lambda)$:

  $\mathcal{E}(\lambda) = 0$ iff $\lambda$ is an eigenvalue of $\mathcal{L}$
  Order of the roots = multiplicity of eigenvalues
  $\mathcal{E}(\lambda)$ is analytic

# Summary

- Evans function for integral neural field equations with

  Arbitrary synaptic footprints
  Arbitrary synaptic response
  Space dependent delays

  [For a Heaviside]

- Usual properties for $\mathcal{E}(\lambda)$:

  $\mathcal{E}(\lambda) = 0$ iff $\lambda$ is an eigenvalue of $\mathcal{L}$
  Order of the roots = multiplicity of eigenvalues
  $\mathcal{E}(\lambda)$ is analytic

- Front, pulse and bump stability for a number of examples

# Further work

- Anisotropy and inhomogeneity [Bressloff *et al.*]

# Further work

- Anisotropy and inhomogeneity [Bressloff *et al.*]
- Multi-layered (2D) networks and thalamo-cortical interactions.

$$u(x,t) = \int_{-\infty}^{\infty} w(x-y) \int_{-\infty}^{t} \eta(t-s) f(v(y,s)) \mathrm{d}s \mathrm{d}y$$

$$\boxed{v(h,u) = \frac{v_s u + ghs}{\tau^{-1} + u}}$$

$s = 1$ if $v(h,u)$ crosses $v_h$ from below and $s = 0$ if $v$ crosses $v_h$ from above.

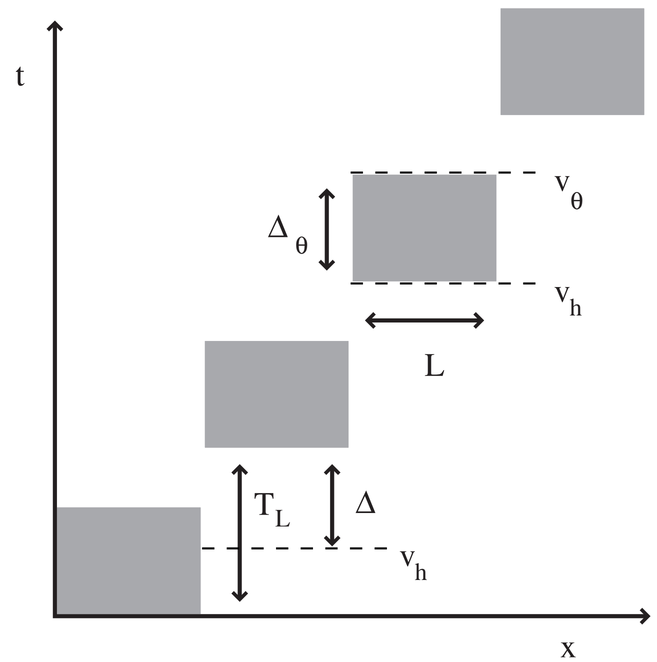
$$\dot{h} = \begin{cases} -h/\tau_h^- & v \geq v_h \\ (1-h)/\tau_h^+ & v < v_h \end{cases}$$

i.e. a model of a network with a slow T-type calcium current, with an exact solution for a Heaviside.

S Coombes PRE 67 (2003) 041910

# An Inhibitory TC network

- $w(x) = \Theta(\sigma - |x|)/2\sigma$

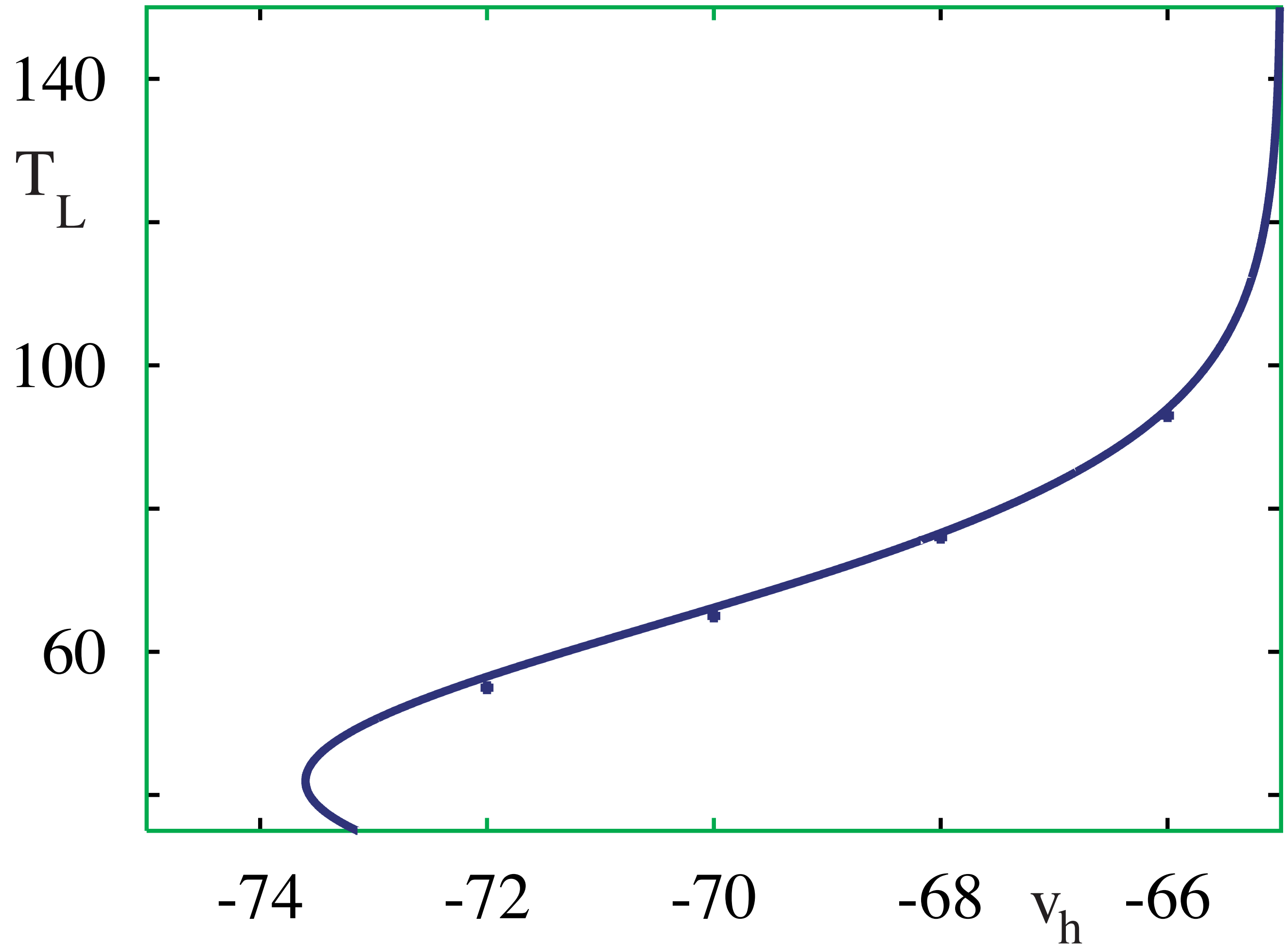



- Solitary lurcher

$$u(x,t) = Q(t, \mathrm{min}(t, \Delta_\theta))W(x)$$

$$Q(t,a) = \int_0^a \eta(t-s)\mathrm{d}s, \qquad W(x) = \int_x^{x+L} w(y)\mathrm{d}y$$

- Closed form solution ...




Slow wave is stable