Coupled Cell Systems

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Ian Stewart *Warwick*

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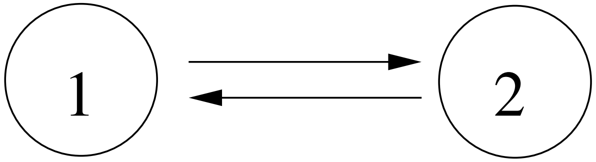
Marcus Pivato *Trent*

Andrew Török *Houston*

Yunjiao Wang *Houston*

Two Identical Cells



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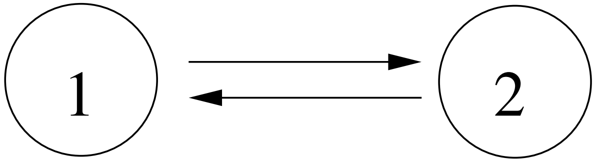

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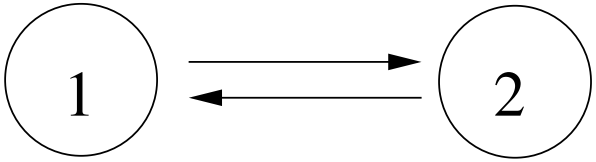

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- Fix$(\sigma) = \{x_1 = x_2\}$ is flow invariant
  Synchrony is a robust phenomenon
- Time-periodic solutions can exist where two cells oscillate a half-period out of phase

$$x_2(t) = x_1(t + \frac{1}{2})$$

Symmetry Overview

- Basic questions for symmetric differential equations

  (a) What is meant by symmetry for a DiffEq $\dot{x} = f(x)$?

  (b) What kinds of symmetry can solutions have?

  (c) How does sol'n symmetry change with parameters?

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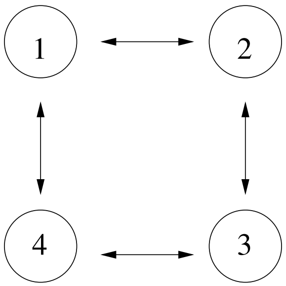
- **Related**: Network architecture is modeling assumption

Fixed-Point Subspaces

- $\Sigma \subset \Gamma$ is a subgroup
- Fixed-point subspace: $\mathrm{Fix}(\Sigma) = \{x : \sigma x = x \quad \forall \sigma \in \Sigma\}$
- $\mathrm{Fix}(\Sigma)$ is flow-invariant: $\sigma f(x) = f(\sigma x) = f(x)$

Symmetry and Synchrony

- Coupled cell systems described by graph



$$\dot{x}_i = g(x_i, x_{i-1}, x_{i+1})$$

$$g(x, y, z) = g(x, z, y)$$

D_4 symmetry

Output from different cells can be compared

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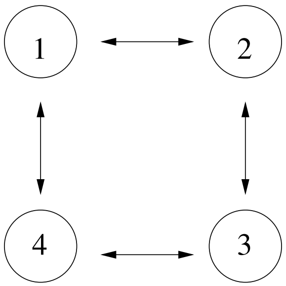

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Example: $\sigma = (2\ 4)$ $\quad x_2(0) = x_4(0) \Longrightarrow x_2(t) = x_4(t)$

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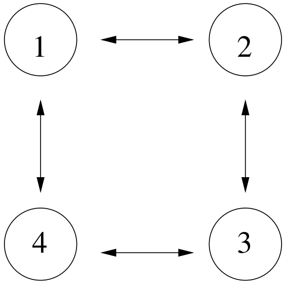

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- Fixed-point subspaces are synchrony subspaces

Example: $\sigma = (2\ 4)$ $\quad x_2(0) = x_4(0) \Longrightarrow x_2(t) = x_4(t)$

- Question: Are all synchrony spaces fixed-point spaces?

Answer: No

Spatio-Temporal Symmetries

- **Question**: Assume Γ is finite

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- Let $x(t)$ be a time-periodic solution
  - $K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\}$ **space symmetries**
  - $H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symm's**

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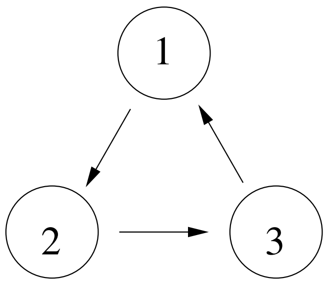
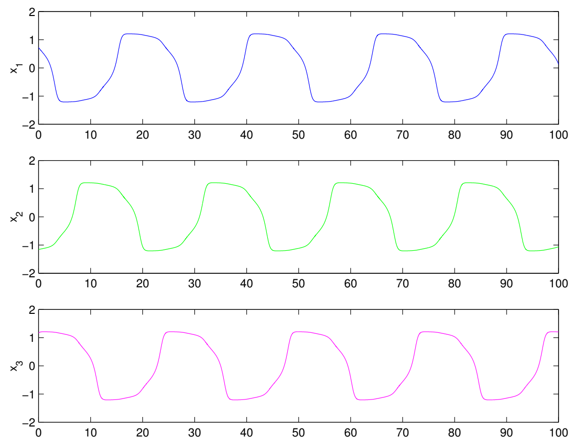
- **Facts**:
  - $h \in H \implies \theta \in \mathbf{S}^1$ such that $hx(t) = x(t+\theta)$
  - H/K is cyclic

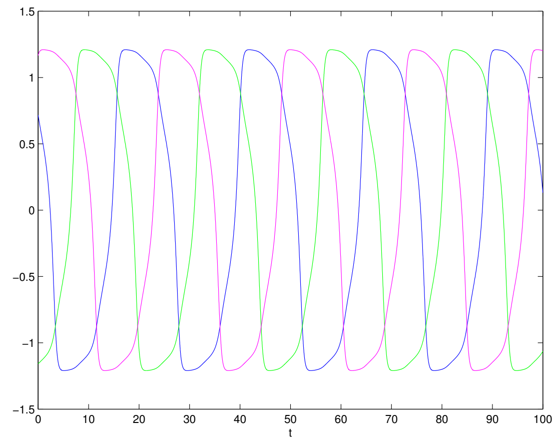
3-Cell Directed Ring: Rotating Wave

How do spatio-temporal symmetries manifest themselves in coupled cell systems? Answer: phase synchrony

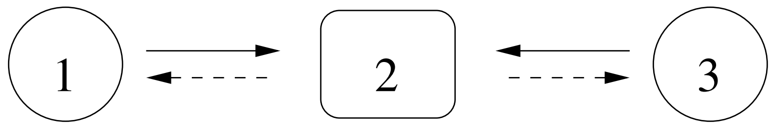


- One-dimensional internal dynamics. Phase space is $\mathbf{R}^3$
- $K = \mathbf{1}$; $H = \mathbf{Z}_3$





Another Three-Cell System



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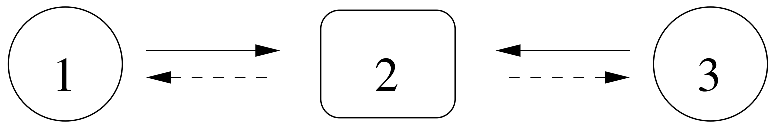

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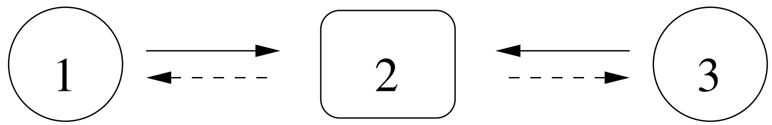

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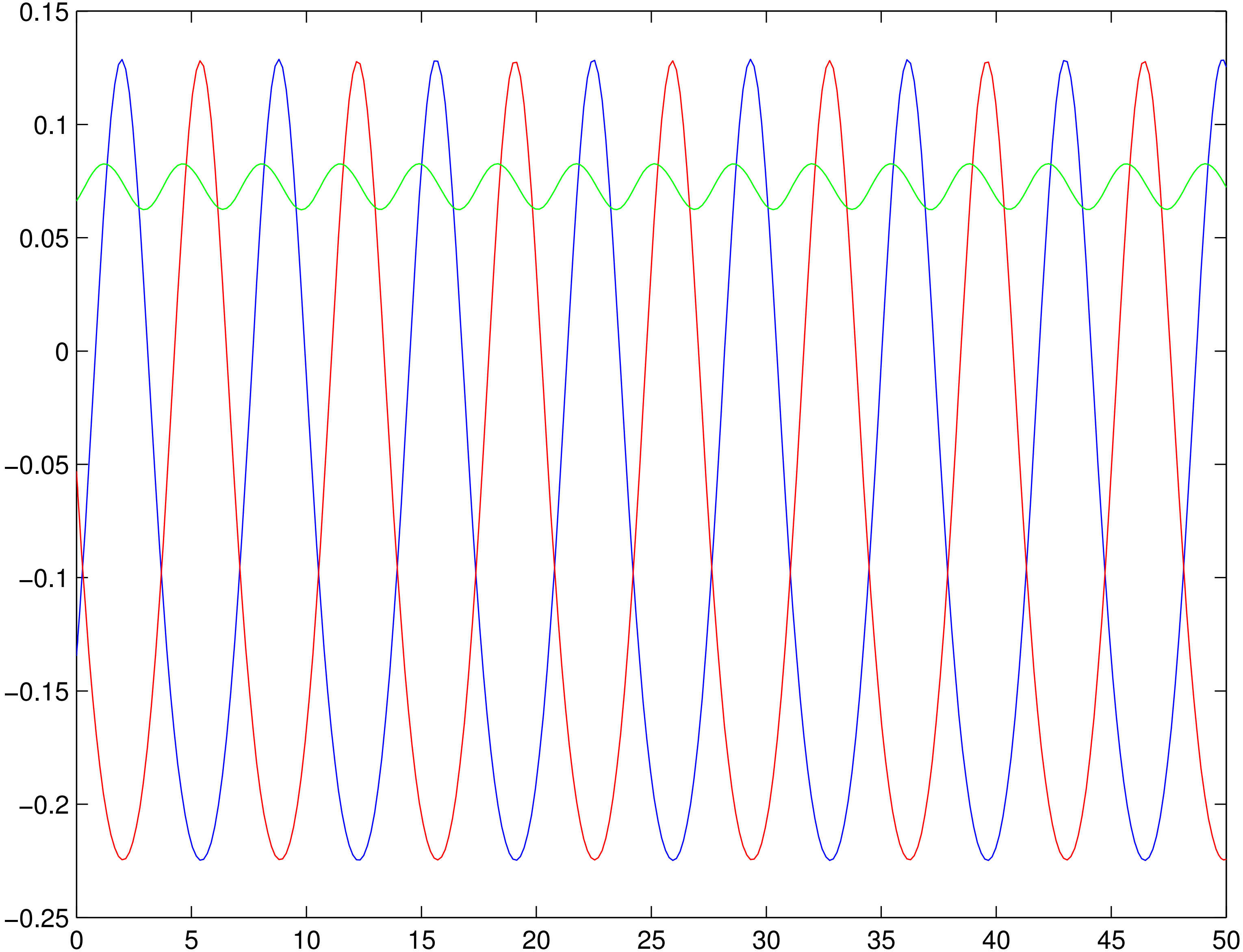
- Out-of-phase periodic solutions ($H = \mathbf{Z}_2(\sigma)$, $K = \mathbf{1}$):

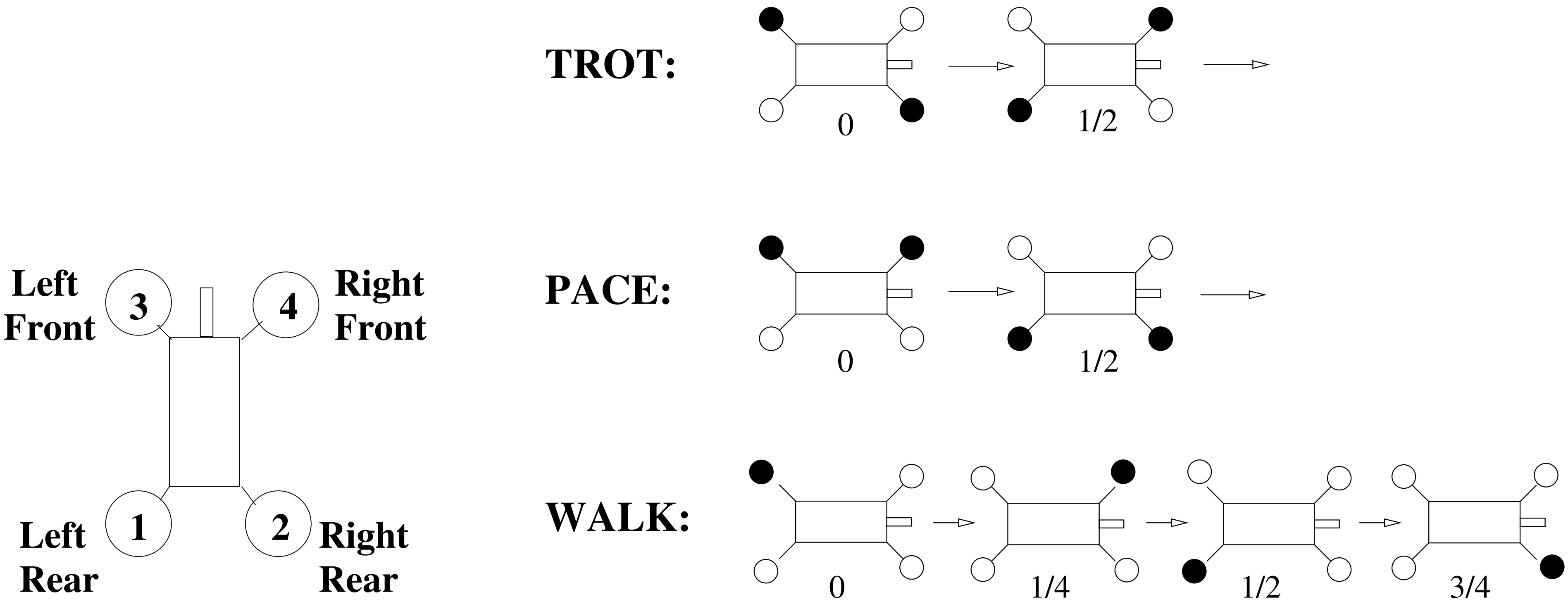
$$\sigma X(t) = X\left(t + \tfrac{1}{2}\right)$$

$$x_3(t) = x_1\left(t + \tfrac{1}{2}\right) \quad \text{and} \quad x_2(t) = x_2\left(t + \tfrac{1}{2}\right)$$

Another Three-Cell System (2)



Quadrupedal Gaits

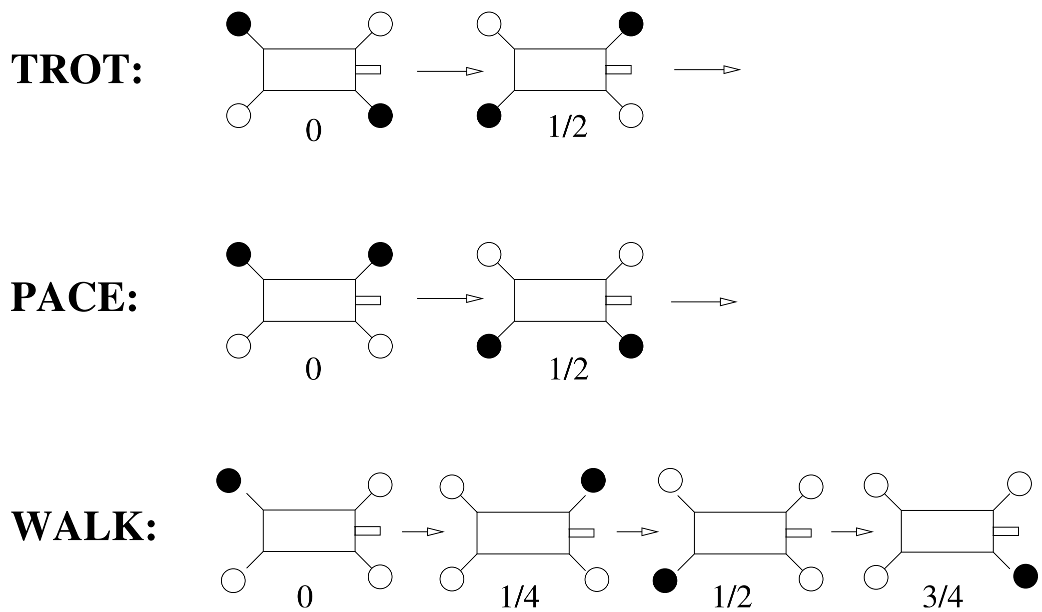


- Black disk indicates time when foot hits ground
- Trot   Thanks to: Sue Morris at http://www.classicaldressage.co.uk

G., Stewart, Buono, and Collins (1999, 2000)

Gait Symmetries

Gait	Spatio-temporal symmetries
Trot	(Left/Right, $\frac{1}{2}$) and (Front/Back, $\frac{1}{2}$)
Pace	(Left/Right, $\frac{1}{2}$) and (Front/Back, 0)
Walk	(Figure Eight, $\frac{1}{4}$)



Collins and Stewart (1993)

Central Pattern Generators (CPG)

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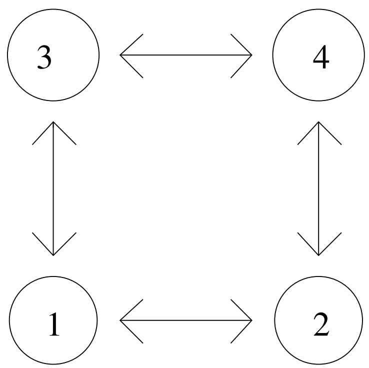
- Simplest network

  One cell 'signals' each leg

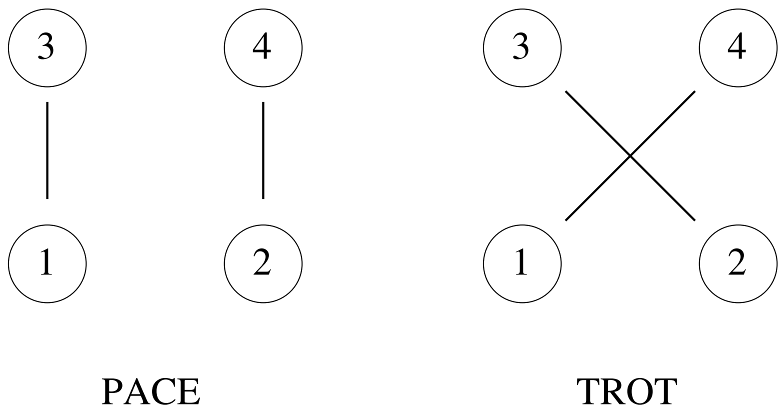


Four Cells Do Not Suffice

- $\Gamma =$ symmetry group of network
- Network produces walk. There is a four-cycle

$$(1\ 3\ 2\ 4) \in \Gamma$$

- Four-cycle permutes pace to trot



- CPG cannot be modeled by four-cell network where each cell gives rhythmic pulsing to one leg

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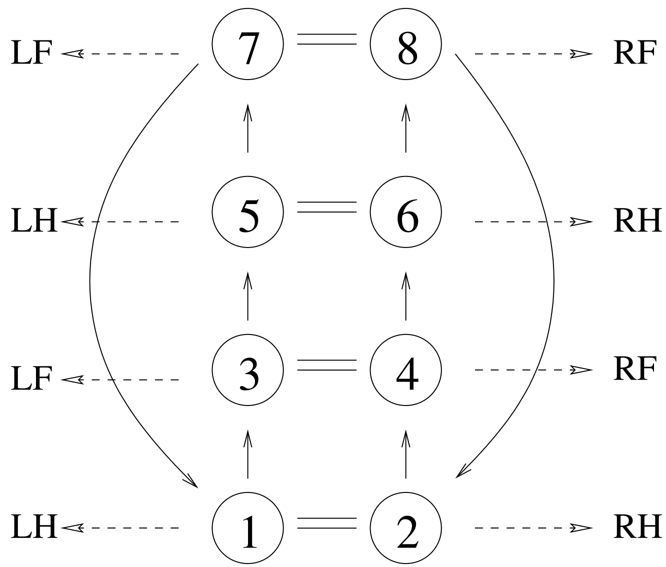

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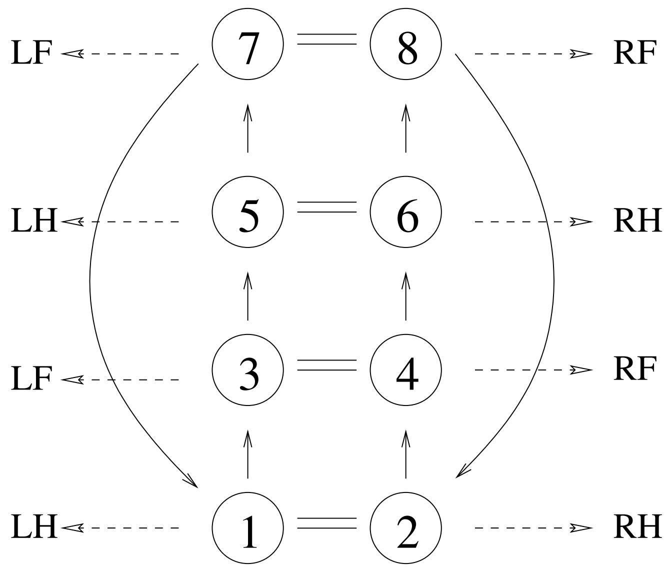


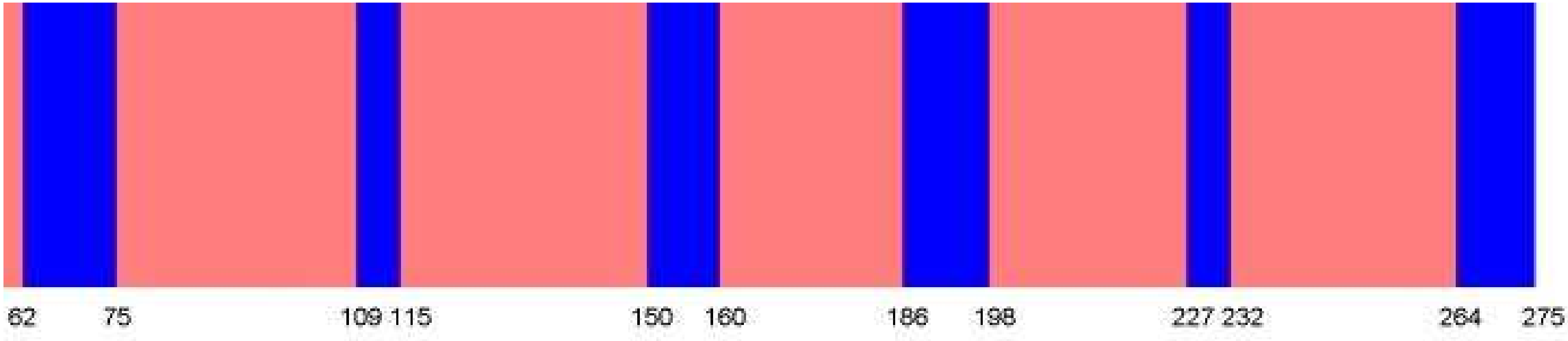
- $\Gamma = \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ is **abelian**

Primary Gaits: $H = \Gamma = \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$

K	Γ/K	Phase Diagram	Gait
Γ	$\mathbf{1}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	pronk
$< \omega >$	$\mathbf{Z}_2$	$\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$	pace
$< \kappa\omega >$	$\mathbf{Z}_2$	$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	trot
$< \kappa, \omega^2 >$	$\mathbf{Z}_2$	$\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	bound
$< \kappa\omega^2 >$	$\mathbf{Z}_4$	$\begin{pmatrix} \pm\frac{1}{4} & \pm\frac{3}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$	walk$^{\pm}$
$< \kappa >$	$\mathbf{Z}_4$	$\begin{pmatrix} 0 & 0 \\ \pm\frac{1}{4} & \pm\frac{1}{4} \end{pmatrix}$	jump$^{\pm}$

- Primary gaits occur by Hopf bifurcation from stand

The Jump



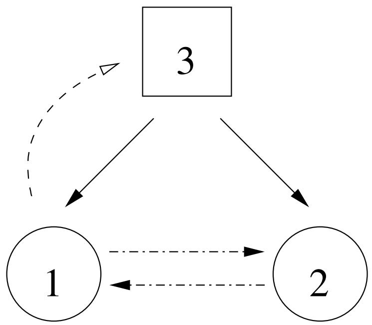
- Average Right Rear to Right Front = 31.2 frames
- Average Right Front to Right Rear = 11.4 frames
- $\frac{31.2}{11.4} = 2.74$

Coupled Cell Theory

- input sets and input isomorphisms
- network architecture $=$ symmetry groupoids
- balanced colorings and synchrony subspaces
- quotient networks

Stewart, G., and Pivato (2003); G., Stewart, and Török (2004)

Asymmetric Three-Cell Network



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- $Y = \{x : x_1 = x_2\}$ is flow-invariant

  Synchrony spaces exist in networks without symmetry

  Restrict equations $\dot{x}_1, \dot{x}_2$ to Y

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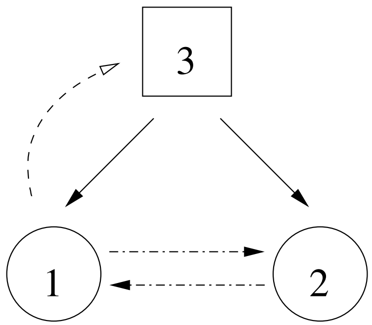

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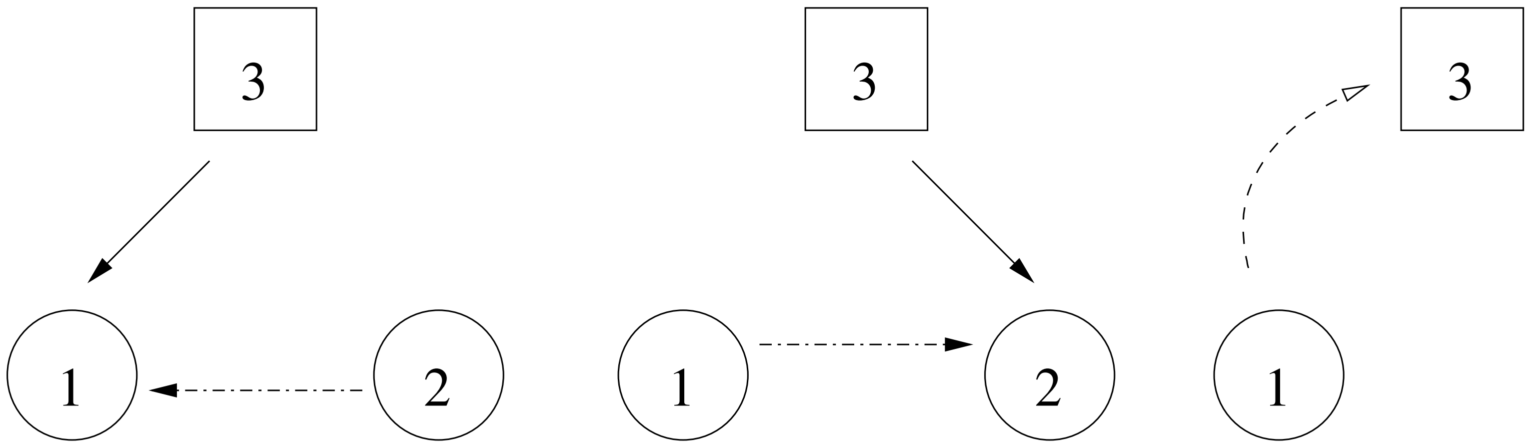
- Cells 1 and 2 are **identical within the network**

Input Sets

- Input set of cell j: Cell j & cells i that connect to j



- Key idea: cells 1, 2 have isomorphic input sets

Coupled Cell Network Definition

- A set of *cells* $\mathcal{C} = \{1, \ldots, N\}$
- An equivalence relation on cells
- Each cell c has *input terminal* $I(c)$ with incoming *arrows*
- An equivalence relation on arrows
- Equivalent arrows have equivalent tail and head cells

A coupled cell network is represented by a graph

- For each class of cells choose node symbol $\bigcirc, \Box, \triangle$
- For each class of arrows choose arrow symbol $\rightarrow, \Rightarrow, \rightsquigarrow$

Symmetry Groupoid

- An **input isomorphism** is a bijection $\beta : I(c) \to I(d)$ that preserves arrow types
- $\mathcal{B}_G =$ set of all input isomorphisms; $\mathcal{B}_G$ is a groupoid
- Groupoid is like group; but product not always defined
- Coupled cell systems: ODEs that commute with $\mathcal{B}_G$

Synchrony Subspaces

- Color cells in $\mathcal{C}$ (red, blue, maroon, etc)

$$\Delta = \{x : x_c = x_d \quad \text{whenever} \quad c \text{ and } d \quad \text{have same color}\}$$

Stewart, G., and Pivato (2003)

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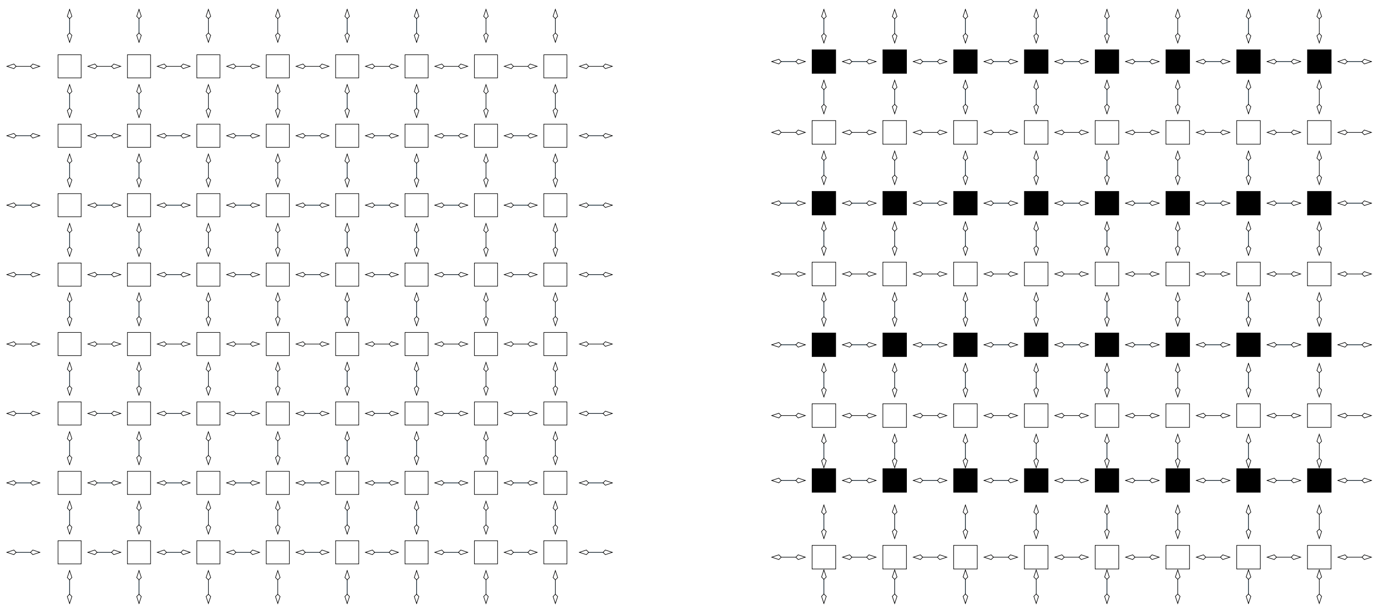
- **Theorem**: **synchrony subspace** $\Longleftrightarrow$ **balanced**

Stewart, G., and Pivato (2003)

Example: Lattice Dynamical Systems

- Consider square lattice with nearest neighbor coupling
- Form a two-color balanced relation

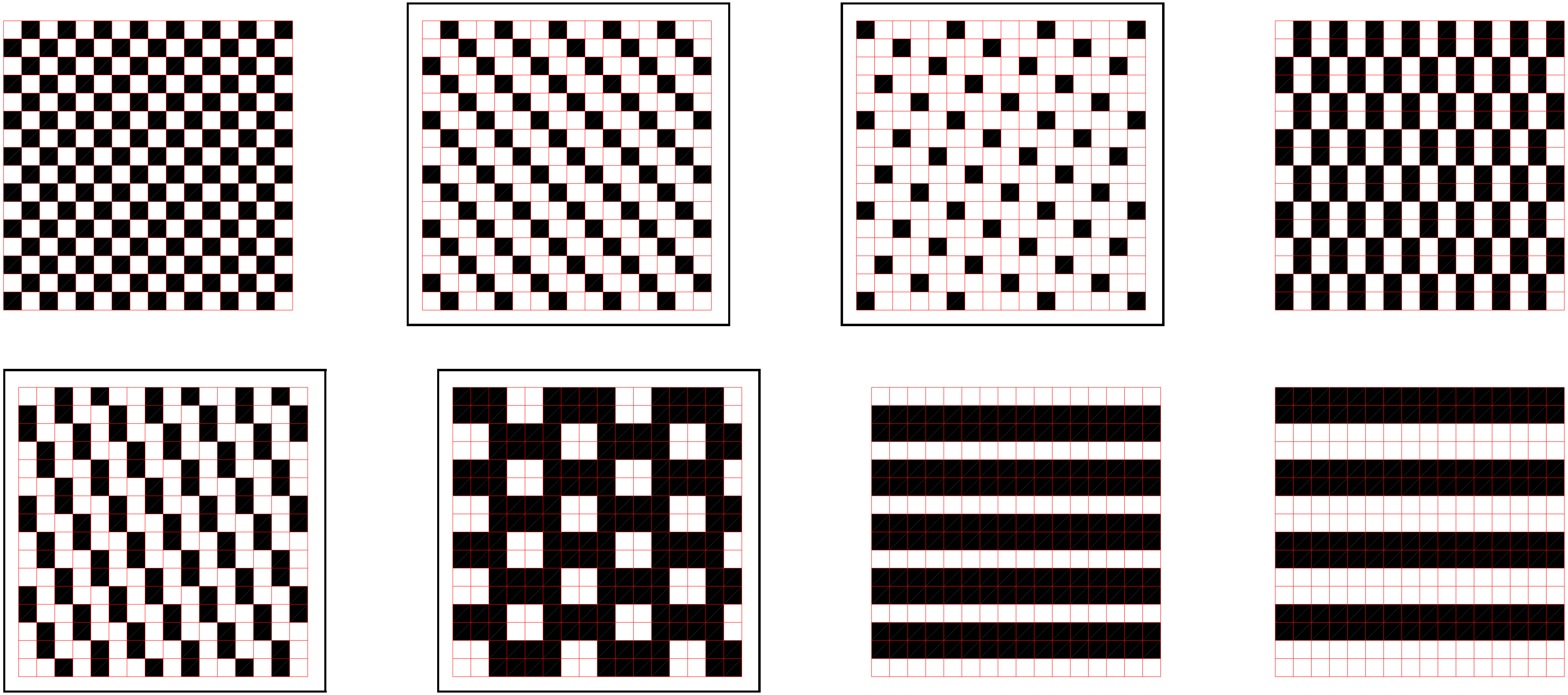


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

Stewart, G. and Nicol (2003)

Lattice Dynamical Systems (2)

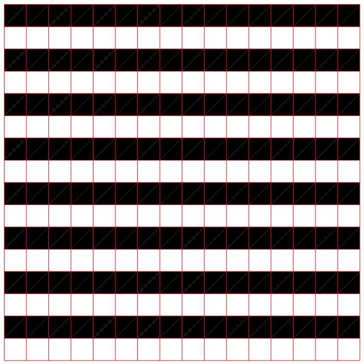
There are eight isolated balanced two-colorings on square lattice with nearest neighbor coupling

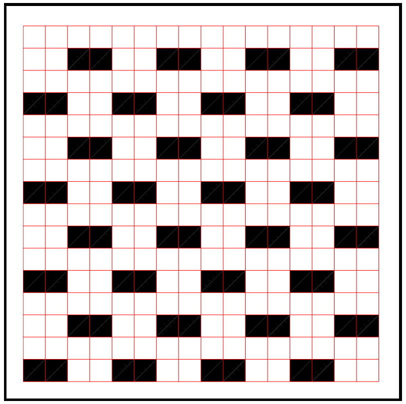


Wang and G. (2004) ☐ indicates nonsymmetric solution

Lattice Dynamical Systems (3)

- There are two infinite families of balanced two-colorings





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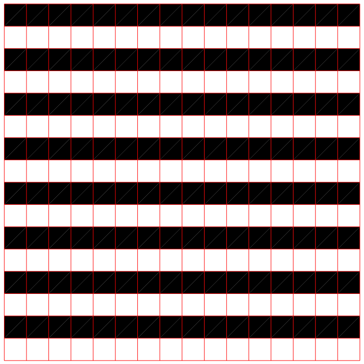
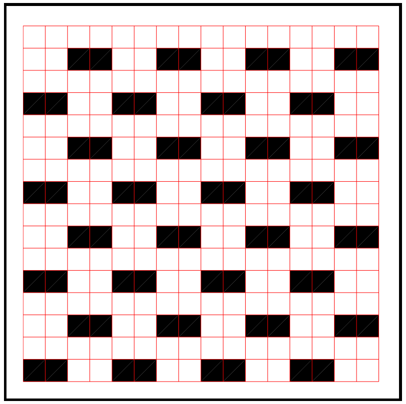
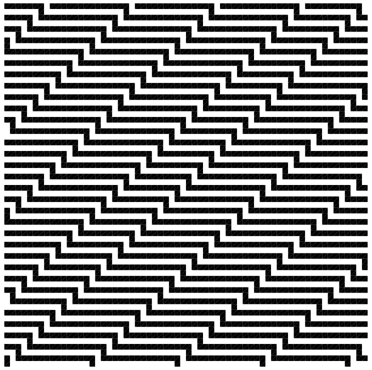
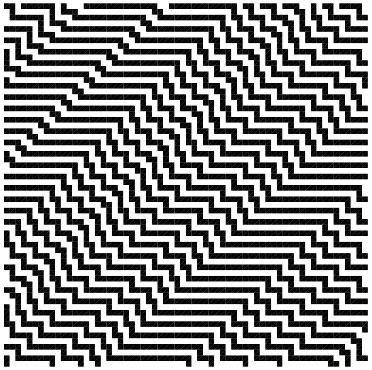




- A continuum of different synchrony subspaces exist





Lattice Dynamical Systems (4)

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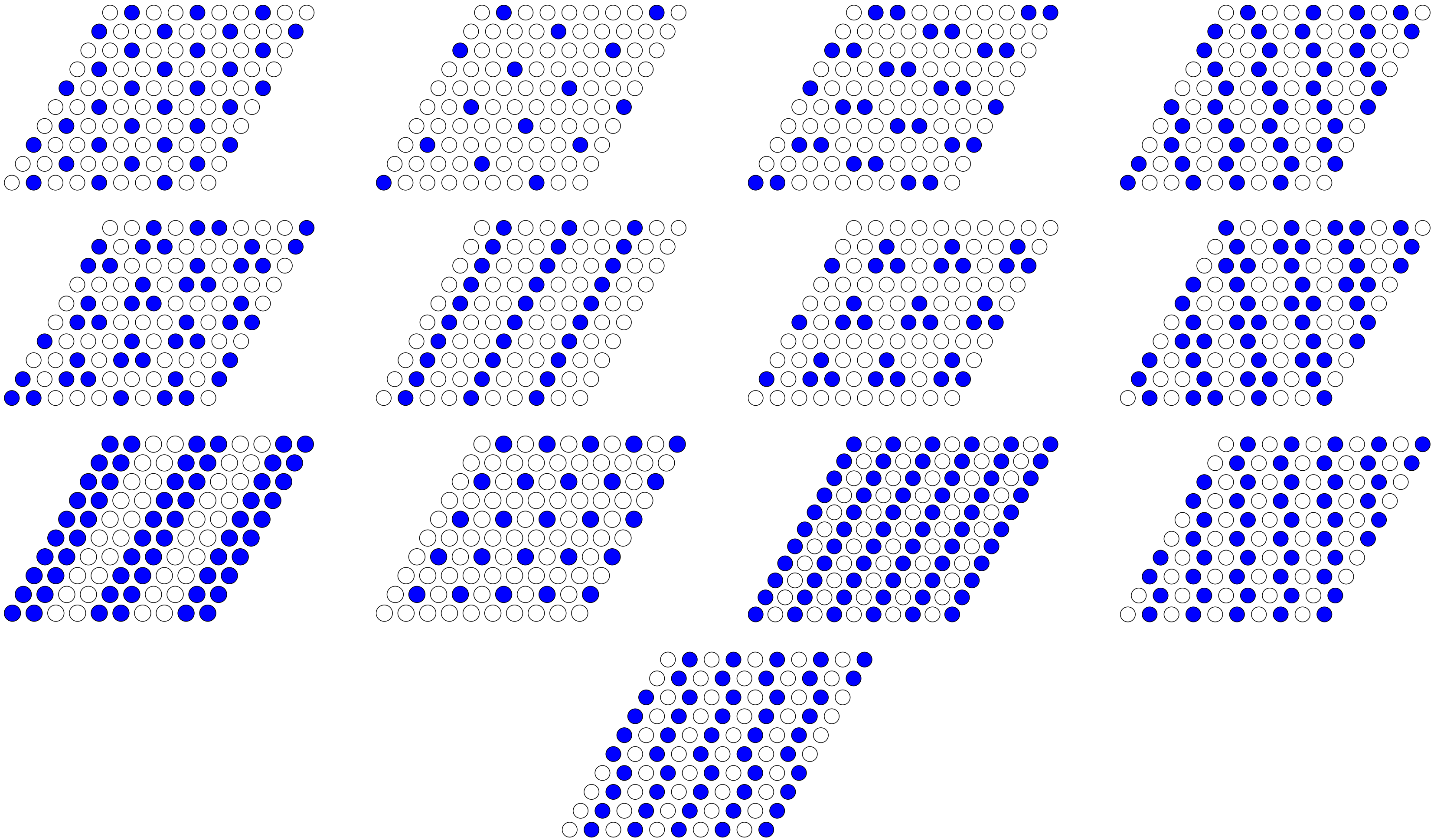
- **Lemma**: Each balanced two coloring leads to equilibria in one parameter bifurcations
- Architecture is important

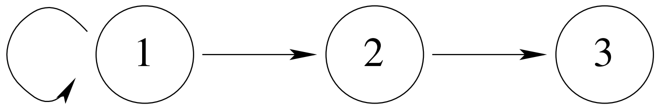
  No infinite families with next nearest neighbor coupling

Hexagonal Lattice: NNN Coupling

There are 13 two-color patterns of synchrony in hex lattice with nearest and next nearest neighbor coupling



Three-Cell Feed-Forward Network



$$\begin{aligned} \dot{x}_1 &= g(x_1, x_1) \\ \dot{x}_2 &= g(x_2, x_1) \\ \dot{x}_3 &= g(x_3, x_2) \end{aligned}$$

$$J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

$\alpha =$ linearized internal $\qquad \beta =$ linearized coupling

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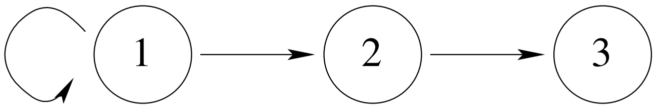
- 

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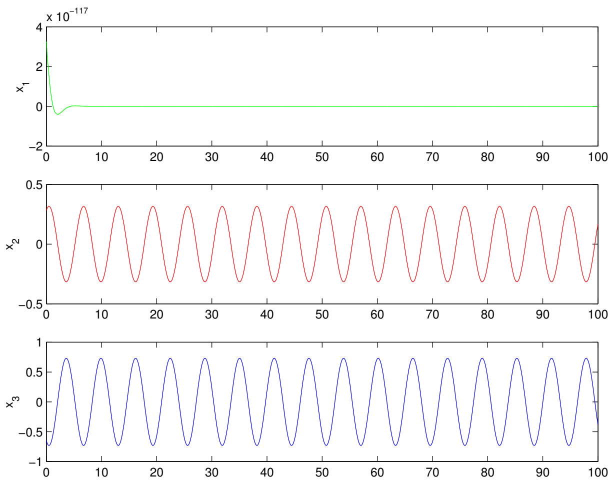
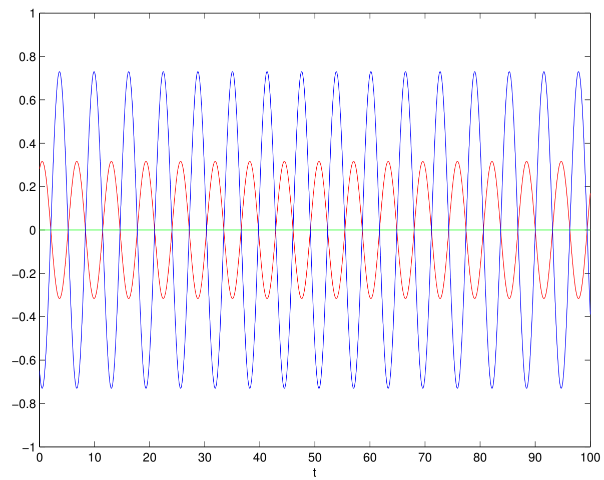
- Network supports solution by Hopf bifurcation where $x_1(t)$ equilibrium $\qquad x_2(t), x_3(t)$ time periodic



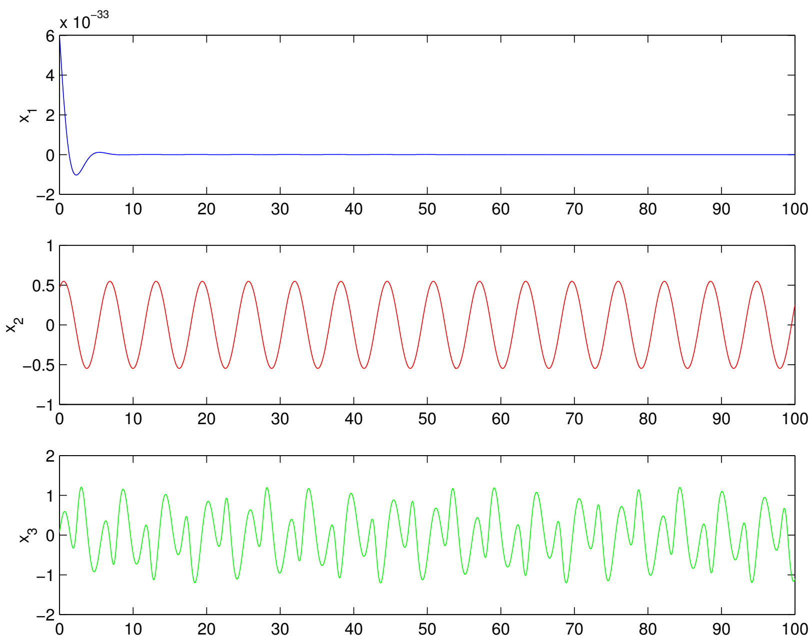


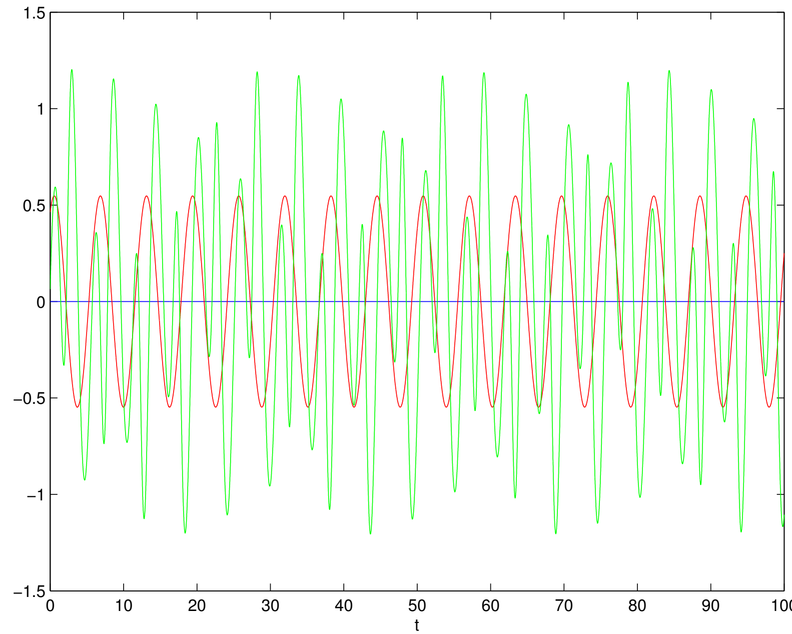
$x_2(t) \approx \lambda^{1/2} \qquad x_3(t) \approx \lambda^{1/6}$

Three-Cell Feed-Forward Network (2)

- Network supports solution where

  $x_1(t)$ equilibrium, $x_2(t)$ time periodic, $x_3(t)$ quasiperiodic





Patterns in Hyperbolic Equilibria

- Let $x_0 = (x_1^0, \ldots, x_N^0)$ be a hyperbolic equilibrium

  Color cells c, d same color iff $x_c^0 = x_d^0$

  $\Delta = \{x : x_c = x_d$ if c and d have same color$\}$

G., Stewart, and Török (2003)

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- Theorem: Coloring is rigid iff balanced

- Conjecture: Hyperbolic periodic solutions can have rigid phase shift synchrony only when there is a symmetric quotient network

G., Stewart, and Török (2003)

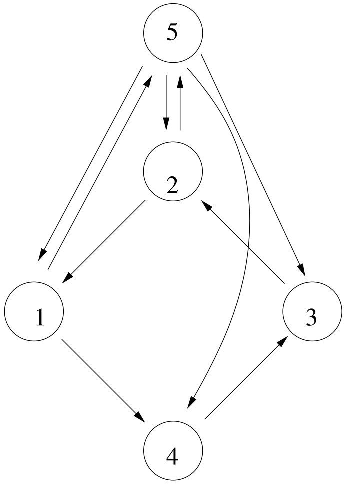
Quotient Cell Systems

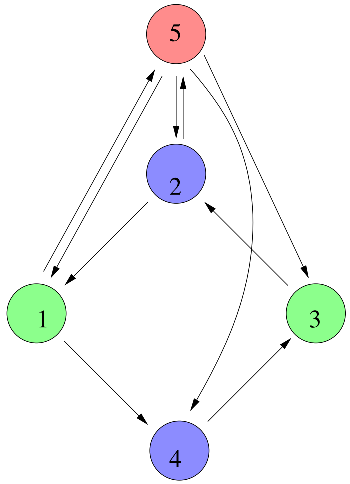
- Given cell network $\mathcal{C}$ and balanced coloring $\bowtie$
- Define *quotient network* $\mathcal{C}_{\bowtie}$ by
  - $\mathcal{C}_{\bowtie} = \{\overline{c} : c \in \mathcal{C}\} = \mathcal{C}/\bowtie$
  - Quotient cells equivalent if $\mathcal{C}$ cells equivalent
  - Quotient arrows are projections of $\mathcal{C}$ arrows
  - Quotient arrows equivalent if $\mathcal{C}$ arrows equivalent
- Thm: $\mathcal{C}$-admissible DE restricted to $\Delta_{\bowtie}$ is $\mathcal{C}_{\bowtie}$-admissible

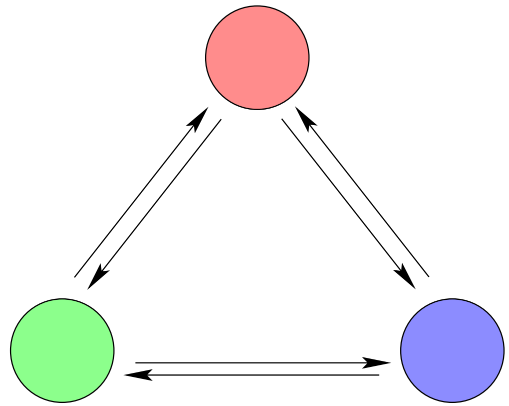
  Every $\mathcal{C}_{\bowtie}$-admissible DE on $\Delta_{\bowtie}$ lifts to $\mathcal{C}$-admissible DE

G., Stewart, and Török (2003)

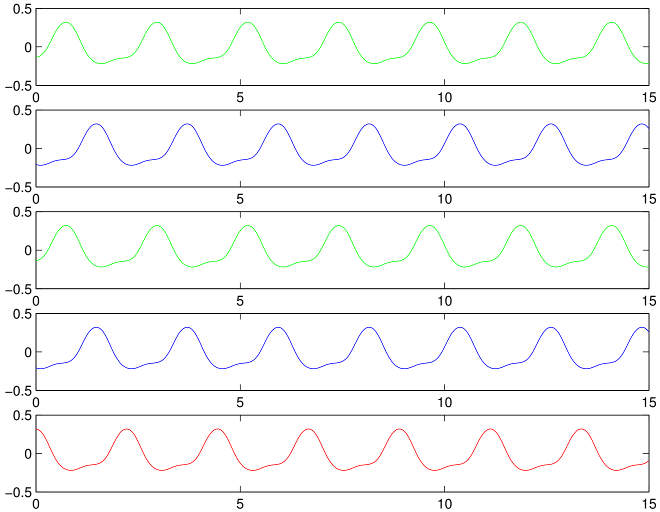
Asymmetric Five-Cell Network

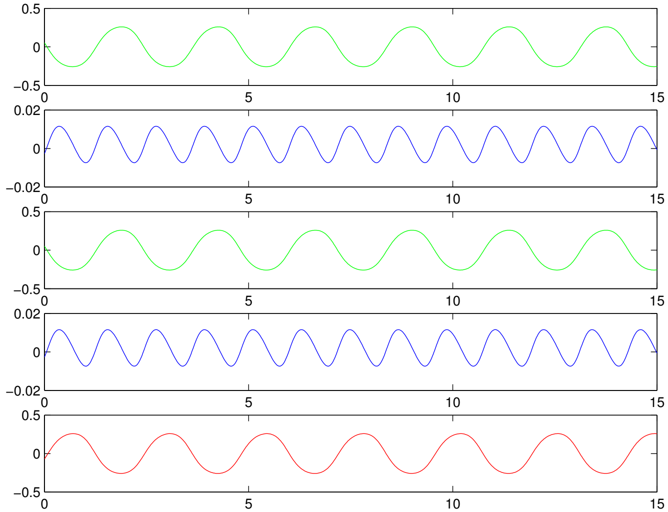






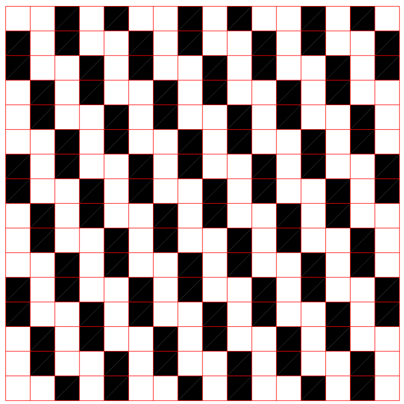
- Quotient is bidirectional 3-cell ring with $\mathbf{D}_3$ symmetry
- One-parameter synchrony-breaking Hopf yields

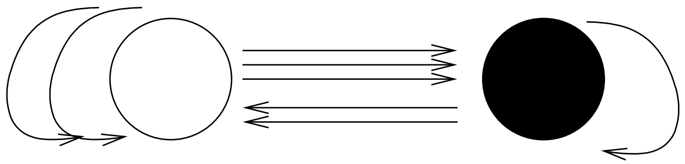




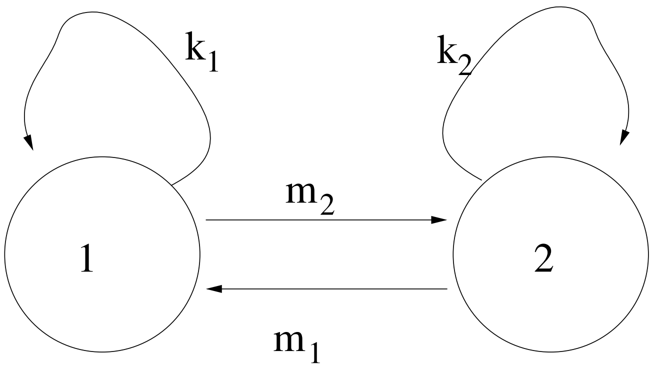
Two Color Quotient Networks

- Every balanced two coloring has two-cell quotient





Two-Color Branching Lemma



$$\ell = k_1 + m_1 = k_2 + m_2$$

$$\begin{aligned}\dot{x}_1 &= g(x_1, \overline{\underbrace{x_1, \ldots, x_1}_{k_1}, \underbrace{x_2, \ldots, x_2}_{m_1}}) \\ \dot{x}_2 &= g(x_2, \overline{\underbrace{x_2, \ldots, x_2}_{k_2}, \underbrace{x_1, \ldots, x_1}_{m_2}})\end{aligned}$$

$x_1 = x_2$ is flow-invariant

- Let $\alpha =$ linearized internal and $\beta =$ linearized coupling

$$\text{Jacobian} = \begin{bmatrix} \alpha + k_1\beta & m_1\beta \\ m_2\beta & \alpha + k_2\beta \end{bmatrix}$$

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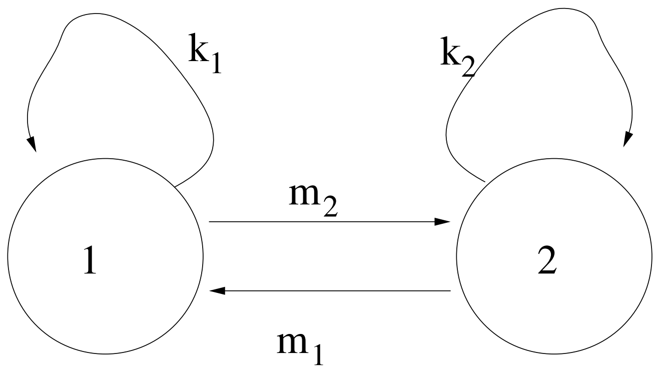

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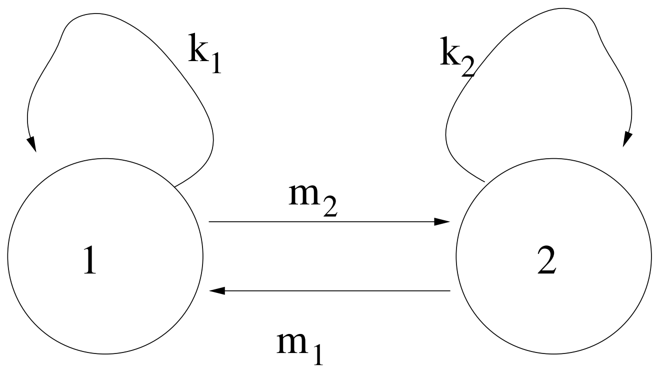
- Eigenvalues are $\alpha + \ell\beta$ $((1,1))$ and $\alpha + (k_1 + k_2 - \ell)\beta$

Two-Color Branching Lemma



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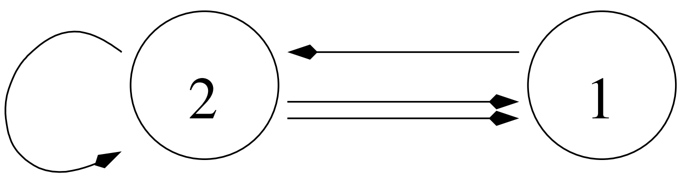
- Eigenvalues are $\alpha + \ell\beta$ $((1,1))$ and $\alpha + (k_1 + k_2 - \ell)\beta$
- Vary α — get synchrony-breaking bifurcation

Two-Color Synchrony-Breaking Hopf

- Unique synchrony-breaking Hopf bifurcation
- Periodic sol'ns are synchronous on cells of same color
- Near bifurcation — to first order
  - Opposite color cells $\approx$ one-half period out of phase
  - Ratio of amplitudes of opposite color cells $\approx m_1/m_2$



$m_1 = 2$ $\quad m_2 = 1$