FINITE TIME EXTINCTION FOR THE RICCI FLOW ON MANIFOLDS WITHOUT ASPHERICAL SUMMANDS

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0. Introduction

In this lecture we will explain why the Ricci flow becomes extinct in finite time on 3–manifolds without aspherical summands. This was shown by Perelman in [Pe3] and by Colding–Minicozzi in [CM2]. Our treatment here follows [CM2]; see also [Pe3] for applications to the elliptic part of geometrization.

On a homotopy 3–sphere there is a natural way of constructing minimal surfaces and that comes from the min–max argument where the minimal of all maximal slices of sweep–outs is a minimal surface; cf. [CD]. In [CM2] we looked at how the area of this min–max surface changes under the flow. Geometrically the area measures a kind of width of the 3–manifold (see the figure below) and as we will see for 3–manifolds without aspherical summands (like a homotopy 3–sphere) the area becomes zero in finite time corresponding to that the solution becomes extinct in finite time[1].

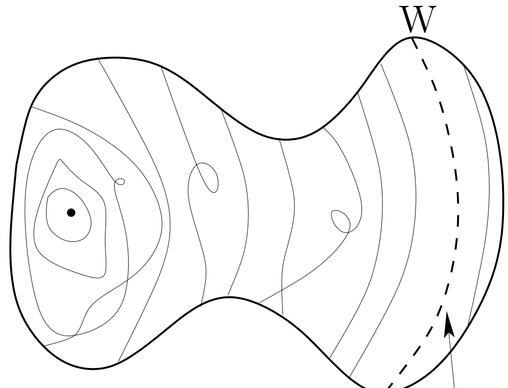

Figure 1. The sweep–out, the min–max surface, and the width W.

Let M^3 be a smooth closed orientable 3–manifold and let $g(t)$ be a one–parameter family of metrics on M evolving by Hamilton's Ricci flow (see [Ha]), so

$$\partial_t g = -2\,\mathrm{Ric}_{M_t}\,. \tag{0.1}$$

Unless otherwise stated we will assume throughout that M is prime and non–aspherical (so $\pi_k(M) \neq \{0\}$ for some $k > 1$). If M is prime but not irreducible, then $M = \mathbf{S}^2 \times \mathbf{S}^1$ (proposition 1.4 in [Hr]) so

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[1] It may be of interest to compare our notion of width, and the use of it, to a well–known approach to the Poincaré conjecture. This approach asks to show that for any metric on a homotopy 3–sphere a min–max type argument produces an <u>embedded</u> minimal 2–sphere. Note that in the definition of the width it play no role whether the minimal 2–sphere is embedded or just immersed, and thus, the analysis involved in this was settled a long time ago. This well–known approach has been considered by many people, including Freedman, Meeks, Pitts, Rubinstein, Schoen, Simon, Smith, and Yau; see [CD].

$\pi_3(M) = \mathbf{Z}$. Otherwise, if M is irreducible, then the sphere theorem implies that $\pi_2(M) = 0$ (corollary 3.9 in [Hr]). In the second case, the Hurewicz isomorphism theorem then implies that $\pi_3(M) \neq \{0\}$ (since M is non–aspherical). Therefore, in either case, by suspension, the space of maps from $\mathbf{S}^2$ to M is not simply connected.

Fix a continuous map $\beta : [0,1] \to C^0 \cap L_1^2(\mathbf{S}^2, M)$ where $\beta(0)$ and $\beta(1)$ are constant maps so that β is in the nontrivial homotopy class $[\beta]$. We define the width $W = W(g, [\beta])$ by

$$W(g) = \min_{\gamma \in [\beta]} \max_{s \in [0,1]} \mathrm{E}(\gamma(s)) \, . \tag{0.2}$$

One could equivalently define the width using the area rather than the energy, but the energy is somewhat easier to work with. As for the Plateau problem, this equivalence follows using the uniformization theorem and the inequality $\mathrm{Area}(u) \leq \mathrm{E}(u)$ (with equality when u is a branched conformal map); cf. lemma 4.12 in [CM1].

The next theorem gives an upper bound for the derivative of the width $W(g(t))$ under the Ricci flow which forces the solution $g(t)$ to become extinct in finite time (see paragraph 4.4 of [Pe2] for the precise definition of extinction time when surgery occurs).

Theorem 0.3. ([CM2]). Let M^3 be a closed orientable prime non–aspherical 3–manifold equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow, the width $W(g(t))$ satisfies

$$\frac{d}{dt} W(g(t)) \leq -4\pi + \frac{3}{4(t+C)} W(g(t)) \, , \tag{0.4}$$

in the sense of the limsup of forward difference quotients. Hence, $g(t)$ must become extinct in finite time.

The 4π in (0.4) comes from the Gauss–Bonnet theorem and the $3/4$ comes from the bound on the minimum of the scalar curvature that the evolution equation implies. Both of these constants matters whereas the constant C depend on the initial metric and the actual value is not important.

To see that (0.4) implies finite extinction time rewrite (0.4) as

$$\frac{d}{dt} \left(W(g(t)) \, (t+C)^{-3/4} \right) \leq -4\pi \, (t+C)^{-3/4} \tag{0.5}$$

and integrate to get

$$(T+C)^{-3/4} \, W(g(T)) \leq C^{-3/4} \, W(g(0)) - 16 \, \pi \, \left[(T+C)^{1/4} - C^{1/4} \right] . \tag{0.6}$$

Since $W \geq 0$ by definition and the right hand side of (0.6) would become negative for T sufficiently large we get the claim.

Arguing as in 1.5 of [Pe3] (or alternatively using Remark 2.14), we get as a corollary finite extinction time for the Ricci flow on all 3–manifolds without aspherical summands.

Corollary 0.7. ([Pe3] and [CM2]). Let M^3 be a closed orientable 3–manifold whose prime decomposition has only non–aspherical factors and is equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow with surgery, $g(t)$ must become extinct in finite time.

1. Upper bound for the rate of change of area of minimal 2–spheres

Suppose that $\Sigma \subset M$ is a closed immersed surface (not necessarily minimal), then using (0.1) an easy calculation gives (cf. page 38–41 of [Ha])

$$\frac{d}{dt}_{t=0} \mathrm{Area}_{g(t)}(\Sigma) = - \int_{\Sigma} [R - \mathrm{Ric}_M(\mathbf{n}, \mathbf{n})] \, . \tag{1.1}$$

If Σ is also minimal, then

$$\frac{d}{dt}_{t=0} \text{Area}_{g(t)}(\Sigma) = -2 \int_\Sigma \mathrm{K}_\Sigma - \int_\Sigma [|A|^2 + \text{Ric}_M(\mathbf{n}, \mathbf{n})] = - \int_\Sigma K_\Sigma - \frac{1}{2} \int_\Sigma [|A|^2 + R] \, . \tag{1.2}$$

Here K_Σ is the (intrinsic) curvature of Σ, $\mathbf{n}$ is a unit normal for Σ (our Σ's below will be $\mathbf{S}^2$'s and hence have a well–defined unit normal), A is the second fundamental form of Σ so that $|A|^2$ is the sum of the squares of the principal curvatures, Ric_M is the Ricci curvature of M, and R is the scalar curvature of M. (The curvature is normalized so that on the unit $\mathbf{S}^3$ the Ricci curvature is 2 and the scalar curvature is 6.) To get (1.2), we used that by the Gauss equations and minimality of Σ

$$\mathrm{K}_\Sigma = \mathrm{K}_M - \frac{1}{2} |A|^2 \, , \tag{1.3}$$

where K_M is the sectional curvature of M on the two–plane tangent to Σ.

The next lemma gives an upper bound for the rate of change of area of minimal 2–spheres.

Lemma 1.4. If $\Sigma \subset M^3$ is a branched minimal immersion of the 2–sphere, then

$$\frac{d}{dt}_{t=0} \text{Area}_{g(t)}(\Sigma) \leq -4\pi - \frac{\text{Area}_{g(0)}(\Sigma)}{2} \min_M R(0) \, . \tag{1.5}$$

Proof. Let $\{p_i\}$ be the set of branch points of Σ and $b_i > 0$ the order of branching at p_i. By (1.2)

$$\frac{d}{dt}_{t=0} \text{Area}_{g(t)}(\Sigma) \leq - \int_\Sigma K_\Sigma - \frac{1}{2} \int_\Sigma R = -4\pi - 2\pi \sum b_i - \frac{1}{2} \int_\Sigma R \, , \tag{1.6}$$

where the equality used the Gauss–Bonnet theorem with branch points. □

2. Extinction in finite time

We begin by recalling a result on harmonic maps which gives the existence of minimal spheres realizing the width $W(g)$. The results of Sacks and Uhlenbeck give the harmonic maps but potentially allow some loss of energy. This energy loss was ruled out by Siu and Yau (using also arguments of Meeks and Yau), see Chapter VIII in [ScYa]. For our purposes, the most convenient statement of this is given in theorem 4.2.1 of [Jo].

Proposition 2.1. Given a metric g on M and a nontrivial $[\beta] \in \pi_1(C^0 \cap L_1^2(\mathbf{S}^2, M))$, there exists a sequence of sweep–outs $\gamma^j : [0,1] \to C^0 \cap L_1^2(\mathbf{S}^2, M)$ with $\gamma^j \in [\beta]$ so that

$$W(g) = \lim_{j \to \infty} \max_{s \in [0,1]} \mathrm{E}(\gamma_s^j) \, . \tag{2.2}$$

Furthermore, there exist $s_j \in [0,1]$ and branched conformal minimal immersions $u_0, \dots, u_m : \mathbf{S}^2 \to M$ so that, as $j \to \infty$, the maps $\gamma_{s_j}^j$ converge to u_0 weakly in L_1^2 and uniformly on compact subsets of $\mathbf{S}^2 \setminus \{x_1, \dots, x_k\}$, and

$$W(g) = \sum_{i=0}^m \mathrm{E}(u_i) = \lim_{j \to \infty} \mathrm{E}(\gamma_{s_j}^j) \, . \tag{2.3}$$

Finally, for each $i > 0$, there exists a point x_{k_i} and a sequence of conformal dilations $D_{i,j} : \mathbf{S}^2 \to \mathbf{S}^2$ about x_{k_i} so that the maps $\gamma_{s_j}^j \circ D_{i,j}$ converge to u_i.

Remark 2.4. It is implicit in Proposition 2.1 that $W(g) > 0$. This can, for instance, be seen directly using [Jo]. Namely, page 125 in [Jo] shows that if $\max_s \mathrm{E}(\gamma_s^j)$ is sufficiently small (depending on g), then γ^j is homotopically trivial.

We will also need a standard additional property for the min–max sequence of sweep–outs γ^j of Proposition 2.1 which can be achieved by modifying the sequence as in section 4 of [CD] (cf. proposition 4.1 on page 85 in [CD]). Loosely speaking this is the property that any subsequence $\gamma^k_{s_k}$ with energy converging to $W(g)$ converges (after possibly going to a further subsequence) to the union of branched immersed minimal 2–spheres. Precisely this is that we can choose γ^j so that: Given $\epsilon > 0$, there exist J and $\delta > 0$ (both depending on g and γ^j) so that if $j > J$ and

$$\mathrm{E}(\gamma^j_s) > W(g) - \delta \,, \tag{2.5}$$

then there is a collection of branched minimal 2–spheres $\{\Sigma_i\}$ with

$$\mathrm{dist}\,(\gamma^j_s, \cup_i \Sigma_i) < \epsilon \,. \tag{2.6}$$

Here, the distance means varifold distance (see, for instance, section 4 of [CD]). Below we will use that, as an immediate consequence of (2.6), if F is a quadratic form on M and Γ denotes γ^j_s, then

$$\left| \int_\Gamma [\mathrm{Tr}(F) - F(\mathbf{n}_\Gamma, \mathbf{n}_\Gamma)] - \sum_i \int_{\Sigma_i} [\mathrm{Tr}(F) - F(\mathbf{n}_{\Sigma_i}, \mathbf{n}_{\Sigma_i})] \right| < C\, \epsilon\, \|F\|_{C^1}\, \mathrm{Area}(\Gamma) \,. \tag{2.7}$$

In the proof of the result about finite extinction time we will also need that the evolution equation for $R = R(t)$, i.e. (see, for instance, page 16 of [Ha]),

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{3}\, R^2 \,, \tag{2.8}$$

implies by a straightforward maximum principle argument that at time $t > 0$

$$R(t) \geq \frac{1}{1/[\min R(0)] - 2t/3} = -\frac{3}{2(t+C)} \,. \tag{2.9}$$

In the derivation of (2.9) we implicitly assumed that $\min R(0) < 0$. If this was not the case, then (2.9) trivially holds with $C = 0$, since, by (2.8), $\min R(t)$ is always non–decreasing. This last remark is also used when surgery occurs. This is because by construction any surgery region has large (positive) scalar curvature.

Proof. (of Theorem 0.3) By the remark following the statement of the theorem it is enough to show (0.4). Fix a time τ. Below $\tilde{C}$ denotes a constant depending only on τ but will be allowed to change from line to line. Let $\gamma^j(\tau)$ be the sequence of sweep–outs for the metric $g(\tau)$ given by Proposition 2.1. We will use the sweep–out at time τ as a comparison to get an upper bound for the width at times $t > \tau$. The key for this is the following claim (the first inequality in (2.10) below): Given $\epsilon > 0$, there exist J and $\bar{h} > 0$ so that if $j > J$ and $0 < h < \bar{h}$, then

$$\begin{aligned} &\mathrm{Area}_{g(\tau+h)}(\gamma^j_s(\tau)) - \max_s \mathrm{E}_{g(\tau)}(\gamma^j_s(\tau)) \\ &\qquad \leq [-4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau+C)}\, \max_s \mathrm{E}_{g(\tau)}(\gamma^j_s(\tau))]\, h + \tilde{C}\, h^2 \,. \end{aligned} \tag{2.10}$$

To see why (2.10) implies (0.4), we use the definition of the width to get

$$W(g(\tau+h)) \leq \max_{s\in[0,1]} \mathrm{Area}_{g(\tau+h)}(\gamma^j_s(\tau)) \,, \tag{2.11}$$

and then take the limit as $j \to \infty$ (so that $\max_s \mathrm{E}_{g(\tau)}(\gamma^j_s(\tau)) \to W(g(\tau))$) in (2.10) to get

$$\frac{W(g(\tau+h)) - W(g(\tau))}{h} \leq -4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau+C)}\, W(g(\tau)) + \tilde{C}\, h \,. \tag{2.12}$$

Taking $\epsilon \to 0$ in (2.12) gives (0.4). It remains to prove (2.10). First, let $\delta > 0$ and J, depending on ϵ (and on τ), be given by (2.5)–(2.7). If $j > J$ and $\mathrm{E}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta$, then let $\cup_i \Sigma_{s,i}^j(\tau)$ be the collection of minimal spheres in (2.7). Combining (1.1), (2.7) with $F = \mathrm{Ric}_M$, and Lemma 1.4 gives

$$\begin{aligned}\frac{d}{dt}_{t=\tau} \mathrm{Area}_{g(t)}(\gamma_s^j(\tau)) &\leq \frac{d}{dt}_{t=\tau} \mathrm{Area}_{g(t)}(\cup_i \Sigma_{s,i}^j(\tau)) + \tilde{C}\,\epsilon\, \|\mathrm{Ric}_M\|_{C^1}\, \mathrm{Area}_{g(t)}(\gamma_s^j(\tau)) \\ &\leq -4\pi - \frac{\mathrm{E}_{g(\tau)}(\gamma_s^j(\tau))}{2} \min_M R(\tau) + \tilde{C}\,\epsilon \\ &\leq -4\pi + \frac{3}{4(\tau + C)} \max_s \mathrm{E}_{g(\tau)}(\gamma_s^j(\tau)) + \tilde{C}\,\epsilon\,,\end{aligned} \tag{2.13}$$

where the last inequality used the lower bound (2.9) for $R(\tau)$. Since the metrics $g(t)$ vary smoothly and every sweep–out γ^j has uniformly bounded energy, it is easy to see that $\mathrm{E}_{g(\tau+h)}(\gamma_s^j(\tau))$ is a smooth function of h with a uniform C^2 bound independent of both j and s near $h = 0$ (cf. (1.1)). In particular, (2.13) and Taylor expansion gives $\bar{h} > 0$ (independent of j) so that (2.10) holds for s with $\mathrm{E}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta$. In the remaining case, we have $\mathrm{E}(\gamma_s^j(\tau)) \leq W(g) - \delta$ so the continuity of $g(t)$ implies that (2.10) automatically holds after possibly shrinking $\bar{h} > 0$. □

Remark 2.14. When M is reducible, then the factors in the prime decomposition must split off in a uniformly bounded time. This follows from a (easy) modification of the proof of Theorem 0.3. Namely, each (non–trivial) factor in the prime decomposition gives rise to a 2–sphere which does not bound a 3–ball in M and, hence, to a stable minimal 2–sphere in this isotopy class by [MeSiYa]. Applying the argument of the proof of Theorem 0.3 to these minimal 2–spheres, we see that the minimal area in this isotopy class must go to zero in finite time as claimed.

3. Perelman's proof of the finite time extinction

Suppose for simplicity that M^3 is a homotopy 3–sphere. In [Pe3] Perelman looks at the space $\Lambda(M)$ of closed curves in M, that is maps from $\mathbf{S}^1$ to M, and for each such curve c he looks at the infimum of areas of all disks that span c and denote it by $A(c)$. He then considers a non–trivial homotopy class in $\pi_*(\Lambda(M), M)$. For a given representative for such a homotopy he looks at the supremum of $A(c)$ where c lies in the representative. He then takes the infimum over all representatives of the given homotopy class. This is the quantity that he considers in place of our width and he shows that it becomes zero in finite time corresponding to that the solution become extinct in finite time.

In his case there are several complications that arise that do not occur when one looks at the width. One of these is some regularity issues, another is that he is forced to, at the same time as he lets the metric flow by the Ricci flow, let the curves evolve by curve shortening and it requires work to make sense of this.

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