Cluster Expansions, Caustics and Counting Graphs

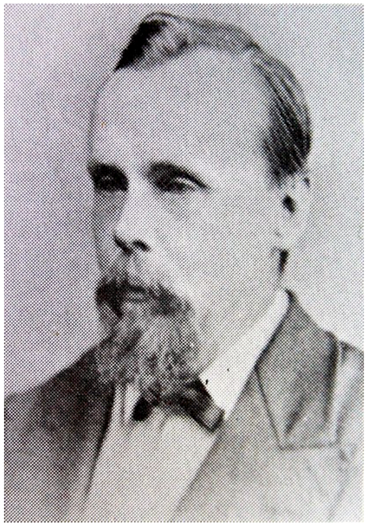
MSRI Workshop on

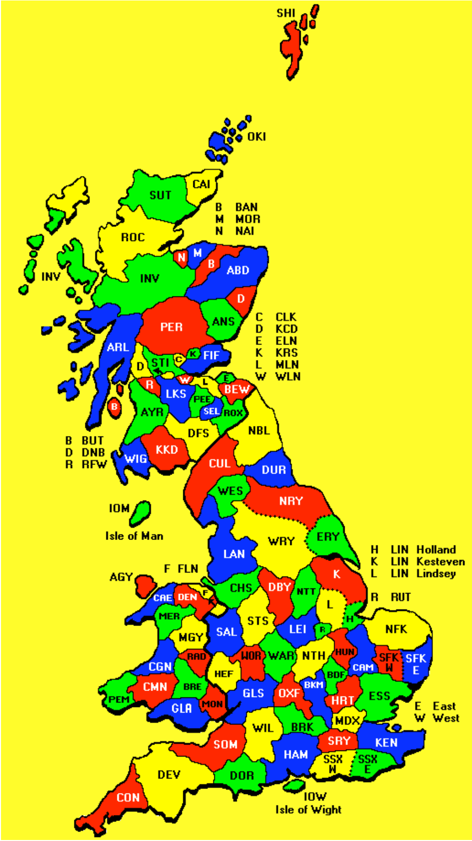
RMT & its Applications

September 14, 2010

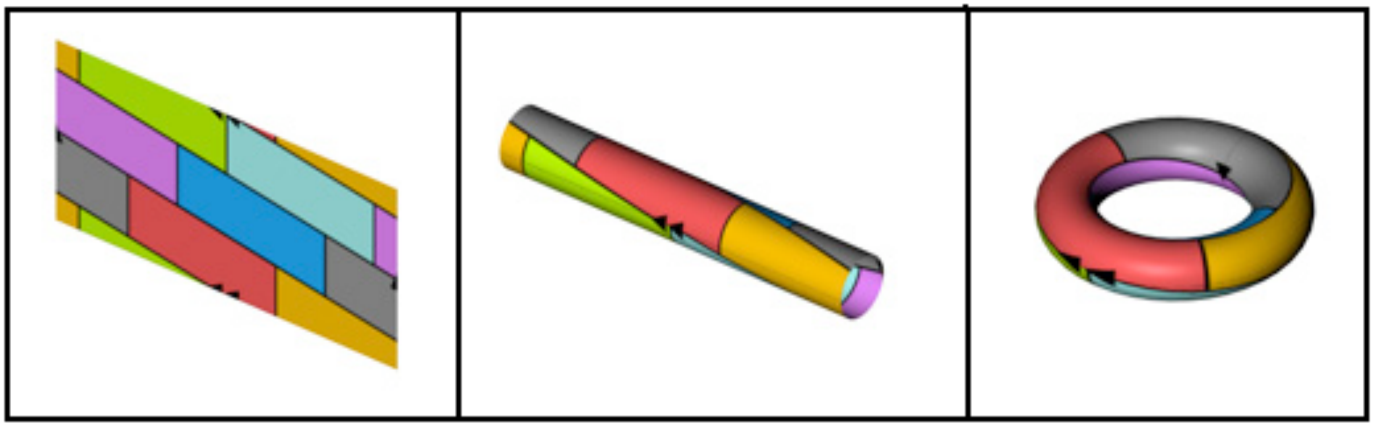
Four Color Theorem

- Francis Guthrie (1852) South African botanist, student at University College London
- Augustus de Morgan
- Arthur Cayley (1878)
- Computer-aided proof by Kenneth Appel & Wolfgang Haken (1976)





Generalizations



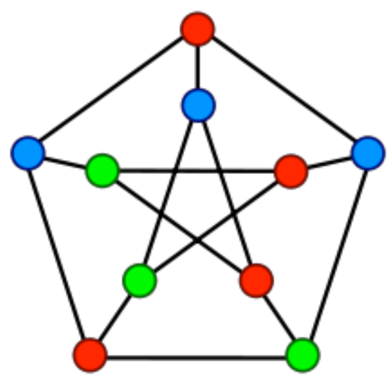
- Heawood's Conjecture (1890) The *chromatic number*, p, of an orientable Riemann surface of *genus* g is

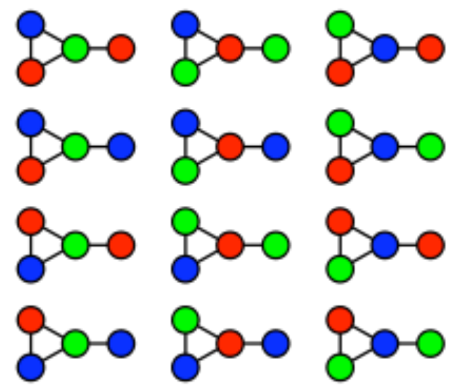
$$p = \{7 + (1+48g)^{1/2}\}/2$$

- Proven, for $g \geq 1$ by Ringel & Youngs (1969)

Generalizations

- Graph Coloring (dual problem): Replace each region ("country") by a vertex (its "capital") and connect the capitals of contiguous countries by an edge. The four color theorem is equivalent to saying that
- The vertices of every planar graph can be colored with just four colors so that no edge has vertices of the same color; i.e.,
- Every planar graph is 4-partite.
- Non-planar:





Combinatorics of Maps

- This subject goes back at least to the work of Tutte in the ‘60s and was motivated by the goal of classifying and algorithmically constructing graphs with specified properties.

William Thomas Tutte (1917 –2002)
British, later Canadian, mathematician and codebreaker.

A census of planar maps (1963)

g - Maps

A *map* D on a compact, oriented and connected surface X is a pair $D = (K(D), [\imath])$ where

- $K(D)$ is a connected 1-complex;
- $[\imath]$ is an isotopical class of inclusions $\imath : K(D) \to X$;
- the complement of $K(D)$ in X is a disjoint union of open cells (faces);
- the complement of the vertices in $K(D)$ is a disjoint union of open segments (edges).

A *g-map* is a map in which the surface X is the closed, oriented Riemann surface of genus g and which in addition carries a labeling (ordering) of the vertices.

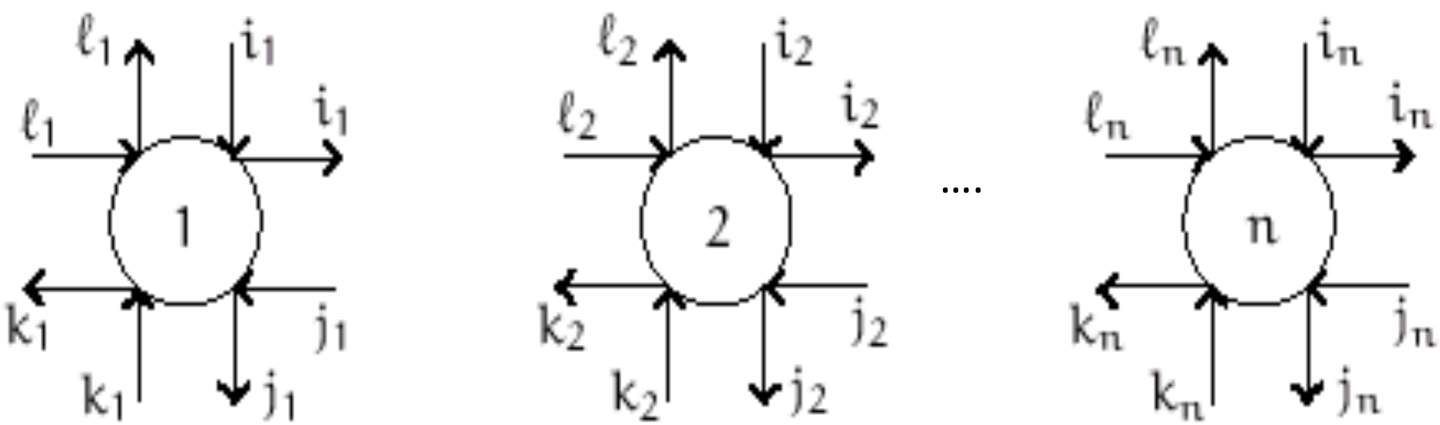
Constellations: the Permutation Model

ν = 2 case

A *4-valent **diagram*** consists of

- n (4-valent) vertices;
- a labeling of the vertices by the numbers 1,2,…,n;
- a labeling of the edges incident to the vertex s

  (for s = 1 , …, n) by letters i_s , j_s , k_s and l_s where this alphabetic order corresponds to the cyclic order of the edges around the vertex).



Random Matrix Partition Functions

- Unitary Ensembles
- Partition Function & Free Energy
- Equilibrium Measures
- Rigorous Asymptotics
- The Genus Expansion

Random Matrix Measures

- M ε $\mathcal{H}_n$, n x n Hermitian matrices
- Family of measures on $\mathcal{H}_n$ (Unitary Ensembles)

$$d\mu_{t_{2\nu}} = \frac{1}{\widetilde{Z}_N^{(n)}(t_{2\nu})} \qquad \exp\left\{-N \ \mathrm{Tr}[V_\nu(M, t_{2\nu})]\right\} dM$$

$$\text{Weight (potential)} \quad : \quad V_\nu(\lambda;\ t_{2\nu}) = \frac{1}{2}\lambda^2 + t_{2\nu}\lambda^{2\nu}$$

$$\text{normalizable for} \quad : \quad \Re t_{2\nu} > 0$$

- t = 0: *Gaussian Unitary Ensemble* (GUE)

The Partition Function

- Descent from Matrices to Eigenvalues
- A tau function mediates this transition
- *Fine Scaling:* A = N½ M

$$\widetilde{Z}_N^{(n)}(t_{2\nu}) = \int \exp\{-N \,\mathrm{Tr}[V_\nu(M, t_{2\nu})]\}\, dM$$

$$Z_N^{(n)}(t_{2\nu}) = \int \cdots \int \exp\left\{-N^2\left[\frac{1}{N}\sum_{j=1}^{n} V_\nu(\lambda_j;\ t_{2\nu}) - \frac{1}{N^2}\sum_{j\neq\ell}\log|\lambda_j - \lambda_\ell|\right]\right\} d^n\lambda.$$

$$\widetilde{Z}_N^{(n)}(t_{2\nu}) = c(n, N) Z_N^{(n)}(t_{2\nu})$$

$$\tau_{n,N}^2(t) \doteq \frac{Z_N^{(n)}(t)}{Z_N^{(n)}(0)} = \frac{\widetilde{Z}_N^{(n)}(t)}{\widetilde{Z}_N^{(n)}(0)} = \mathbb{E}_{\mu_0}\left(\exp\left\{-\frac{t}{N^{\nu-1}}\mathrm{Tr}[A^{2\nu}]\right\}\right)$$

The Free Energy for Large N

- One wants to study the asymptotic behavior of $1/N^2 \log \tau^2_{n,N}$ (t) as n,N → ∞ with x = n/N fixed near 1.
- Leading order:

$$Z_N^{(n)}(t_{2\nu}) = \int \cdots \int \exp\left\{-N^2\left[\frac{1}{N}\sum_{j=1}^{n} V_\nu(\lambda_j;\ t_{2\nu}) - \frac{1}{N^2}\sum_{j\neq\ell}\log|\lambda_j-\lambda_\ell|\right]\right\} d^n\lambda.$$

$$\sup_{\mu\in\mathbf{A}} \left\{-\int V(\lambda)d\mu(\lambda) + \int\int \log|\lambda-\eta| d\mu(\lambda)d\mu(\eta)\right\}$$

$$\mathbf{A} = \text{Borel probability measures on } \mathbb{R}.$$

The Free Energy for Large N

Leading order:

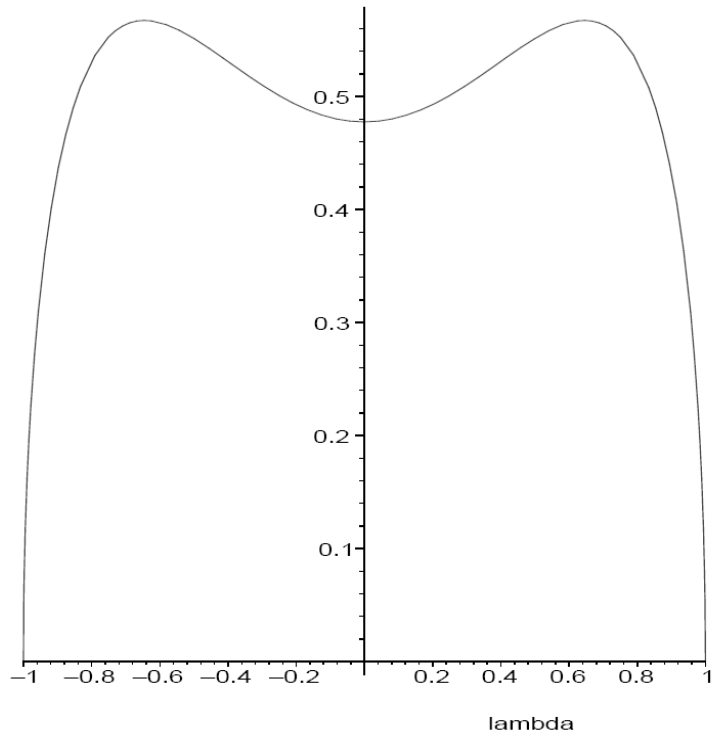
$$\sup_{\mu \in \mathbf{A}} \left\{ - \int V(\lambda) d\mu(\lambda) + \int \int \log|\lambda - \eta| d\mu(\lambda) d\mu(\eta) \right\}$$

$$\mathbf{A} \quad = \quad \text{Borel probability measures on } \mathbb{R}.$$

Maximizer : *equilibrium measure* μ_{eq}

t = 0 ($V = \frac{1}{2} \lambda^2$) → Wigner semicircle law

Introduction of z_0(t)



Equilibrium measure for $V = \frac{1}{2}\lambda^2 + \lambda^4$

- In general, set $z_0(t) = \beta^2/4$ where $[-\beta(t),\beta(t)] = \mathrm{supp}(\mu_{eq})$

Rigorous Asymptotics [EM '03]

$$N^{-2} \log \tau_{n,N}^2(t) \;=\; e_0(x,t) + \frac{1}{N^2} e_1(x,t) + \frac{1}{N^4} e_2(x,t) + \cdots + \frac{1}{N^{2g}} e_g(x,t) + \cdots$$

- uniformly valid as $N \to \infty$ for $x \approx 1$, Re $t > 0$, $|t| < T$.
- $e_g(x,t)$ locally analytic in x, t near $t=0$, $x \approx 1$.
- Coefficients only depend on the endpoints of the support of the equilibrium measure.
- The asymptotic expansion of t-derivatives may be calculated through term-by-term differentiation.
  The large *N* asymptotics of all matrix correlations are calculable from derivatives of the free energy.

The Genus Expansion

$$N^{-2}\log\tau_{n,N}^{2}(t) \quad = \quad e_0(x,t) + \frac{1}{N^2}e_1(x,t) + \frac{1}{N^4}e_2(x,t) + \cdots + \frac{1}{N^{2g}}e_g(x,t) + \cdots$$

$$e_g(t) = \sum_{j\geq 1}\frac{1}{j!}(-t)^j\kappa_g(j) \qquad (x = 1)$$

in which each of the coefficients $\kappa_g(j)$ is the number of g-maps with j 2ν-valent vertices.

- Information about generating functions for graphical enumeration is encoded in asymptotic correlation functions for the spectra of random matrices and vice-versa. Bessis, Itzykson, Zuber, *Adv Appld Math* (1980)
- Riemann-Hilbert Analysis Fokas, Its, Kitaev, *CMP* (1991)
- *Nonlinear Steepest Descent* Deift, Kreicherbauer, McLaughlin, *J Approx Th*, (1998), Deift, Kreicherbauer, McLaughlin, Venakides, Zhou, *CPAM* (1999)

The Genus Expansion

$$N^{-2}\log\tau_{n,N}^{2}(t) \quad = \quad e_0(x,t)+\frac{1}{N^2}e_1(x,t)+\frac{1}{N^4}e_2(x,t)+\cdots+\frac{1}{N^{2g}}e_g(x,t)+\cdots$$

$$e_g(t)=\sum_{j\geq 1}\frac{1}{j!}(-t)^j\kappa_g(j)$$

in which each of the coefficients $\kappa_g(j)$ is the number of g-maps with j 2ν-valent vertices.

<u>Goals</u>

- To place the derivation of the genus expansion on a firm analytical footing. (E-McLaughlin, 2003)
- To understand the structure of the $e_g(t)$ as global analytic functions of t, in particular to exhibit their sole dependence on the endpoints of support of $\mu_{eq}(t)$. (E-McLaghlin-Pierce, 2008)
- To explicitly write down closed form expressions for the $e_g(t)$ *and related generating functions* suitable for the extraction of explicit or asymptotic combinatorial information.

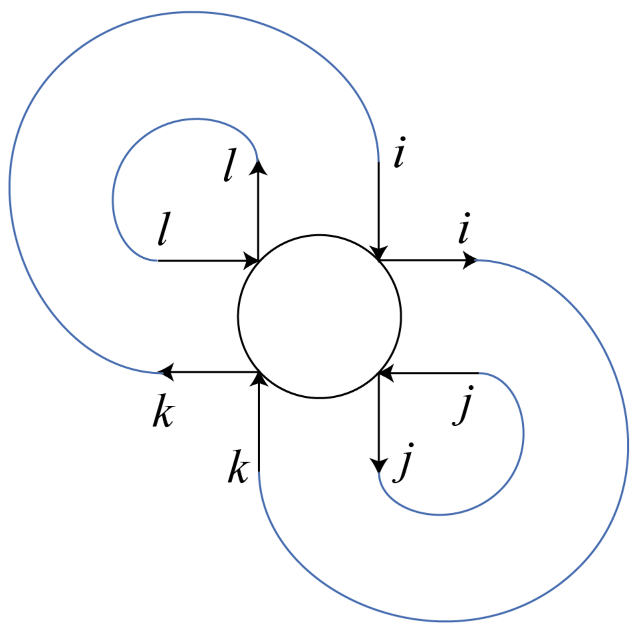
Four-Valent Planar Maps (BIPZ '78)

For potentials V of the form $V = \frac{1}{2}\lambda^2 + t_4\lambda^4$,

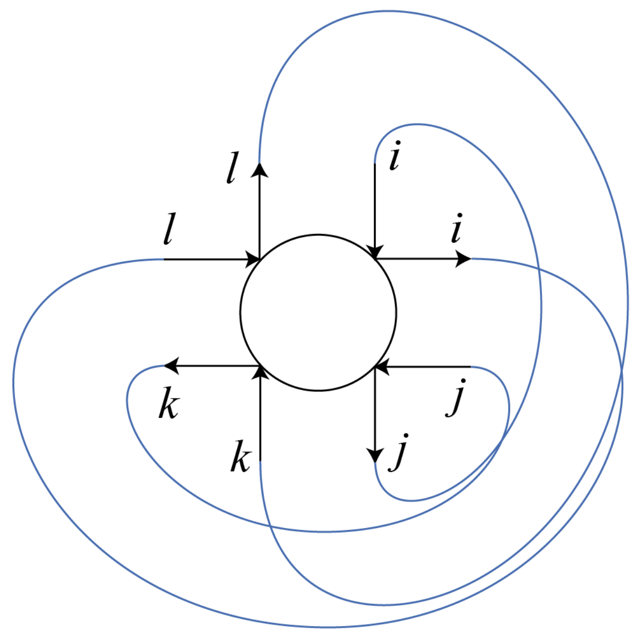
$$e_0 = \frac{1}{24}(z_0 - 1)(z_0 - 9) + \tfrac{1}{2}\log(z_0)$$

$$e_0(t_4) = \sum_{j=1}^{\infty} \kappa_4(n) \frac{(-t_4)^n}{n!} \qquad \kappa_4(n) = (12)^n \frac{(2n-1)\,!}{(n+2)\,!}$$

One Vertex Maps



g=0



g=1

Other Approaches

- Loop Equations Chekov, Eynard, Orantin, Prats-Ferrer, … Iterative equations for the recursive derivation of the e_g as functions of multiple times.

- For a derivation of the loop equations based on Riemann-Hilbert analysis (E-McLaughlin MSRI vol 55)

- Intersection Numbers on Decorated Moduli Spaces Mumford, Witten, Kontsevich

- Hurwitz Numbers, Hodge Integrals & GW Theory Okounkov, Pandharipande, Ekedahl, Lando, Shapiro, Vainshtein

Structure of e_0 [EMP '08]

For potentials V of the form $V = \frac{1}{2}\lambda^2 + t_{2\nu}\lambda^{2\nu}$,

$$e_0 = \eta\left(z_0 - 1\right)\left(z_0 - r\right) + \tfrac{1}{2}\log(z_0)$$

$$\eta = \frac{\left(\nu - 1\right)^2}{4\nu\left(\nu + 1\right)}, \quad r = \frac{3\left(\nu + 1\right)}{\nu - 1}$$

$$e_0\left(t_{2\nu}\right) = \sum_{j=1}^{\infty} \kappa_{2\nu}\left(n\right) \frac{\left(-t_{2\nu}\right)^n}{n!} \qquad \kappa_{2\nu}(n) = (c_\nu)^n \frac{\left(\nu n - 1\right)!}{\left(\left(\nu - 1\right)n + 2\right)!}$$

$$c_\nu = 2\nu \binom{2\nu - 1}{\nu - 1}$$

Low Genus [EMP '08]

$$e_0 = \tfrac{1}{2}\log(z_0) + \eta(z_0 - 1)(z_0 - r)$$

$$e_1 = -\tfrac{1}{12}\log\left(\nu - (\nu - 1)z_0\right)$$

$$e_2 = \frac{(z_0 - 1)Q_4(z_0)}{\left(\nu - (\nu - 1)z_0\right)^5}$$

$$= \frac{1}{240} - \frac{8\nu^3 + 71\nu^2 + 80\nu + 12}{2880\left(\nu - (\nu - 1)z_0\right)^2} + \frac{\nu\left(31\nu^2 + 98\nu + 40\right)}{1440\left(\nu - (\nu - 1)z_0\right)^3}$$

$$- \frac{\nu^2(22\nu + 25)}{576\left(\nu - (\nu - 1)z_0\right)^4} + \frac{7\nu^3}{360\left(\nu - (\nu - 1)z_0\right)^5}$$

$$(\nu = 2) := \frac{1}{720}\frac{(z_0 - 1)^3\left(3z_0^2 - 21z_0 - 82\right)}{(2 - z_0)^5}$$

Rationality of Higher e_g

$$e_0 = \tfrac{1}{2}\log(z_0) + \eta(z_0 - 1)(z_0 - r)$$

$$e_1 = -\tfrac{1}{12}\log\left(\nu - (\nu - 1)z_0\right)$$

$$e_2 = \frac{(z_0 - 1)Q_4(z_0)}{\left(\nu - (\nu - 1)z_0\right)^5}$$

$$\vdots$$

$$e_g = \frac{(z_0 - 1)^r Q_{5g-5-r}(z_0)}{\left(\nu - (\nu - 1)z_0\right)^{5g-5}} \quad \left(\text{where } r = \max\left\{1, \left\lfloor \frac{2g-1}{\nu - 1} \right\rfloor\right\}\right)$$

$$= C^{(g)} + \frac{c_0^{(g)}(\nu)}{\left(\nu - (\nu - 1)z_0\right)^{2g-2}} + \ldots + \frac{c_{3g-3}^{(g)}(\nu)}{\left(\nu - (\nu - 1)z_0\right)^{5g-5}}$$

BIZ Conjecture ('80)

8. CONCLUSION

It would of course be very interesting to obtain $e_H(g)$ in closed form for any value of H. The method of this paper enabled us to do so up to $H = 2$, but works in the general case, although it requires an increasing amount of work. We conjecture a general expression of the form

$$e_H = \frac{(1-a^2)^{2H-1}}{(2-a^2)^{5(H-1)}} P_H(a^2), \qquad H \geq 2, \tag{8.1}$$

with P_H a polynomial in a^2, the degree of which could be obtained by a careful analysis of the above procedure. From (7.35) its value for $a^2 = 1$

$$e_g = \frac{(z_0 - 1)^r Q_{5g-5-r}(z_0)}{\left(\nu - (\nu - 1) z_0\right)^{5g-5}} \quad \left(\text{where } r = \max\left\{1, \left\lfloor \frac{2g-1}{\nu - 1} \right\rfloor\right\}\right)$$

$$= C^{(g)} + \frac{c_0^{(g)}(\nu)}{\left(\nu - (\nu-1)z_0\right)^{2g-2}} + \ldots + \frac{c_{3g-3}^{(g)}(\nu)}{\left(\nu - (\nu-1)z_0\right)^{5g-5}}$$

What is z_0 (t) ?

1. Eigenvalue density edges:
   $z_0(t) = \beta^2/4$ where $[-\beta(t), \beta(t)] = \mathrm{supp}(\mu_{eq})$

2. Generating Function for Generalized Catalan Numbers
   EMP '08

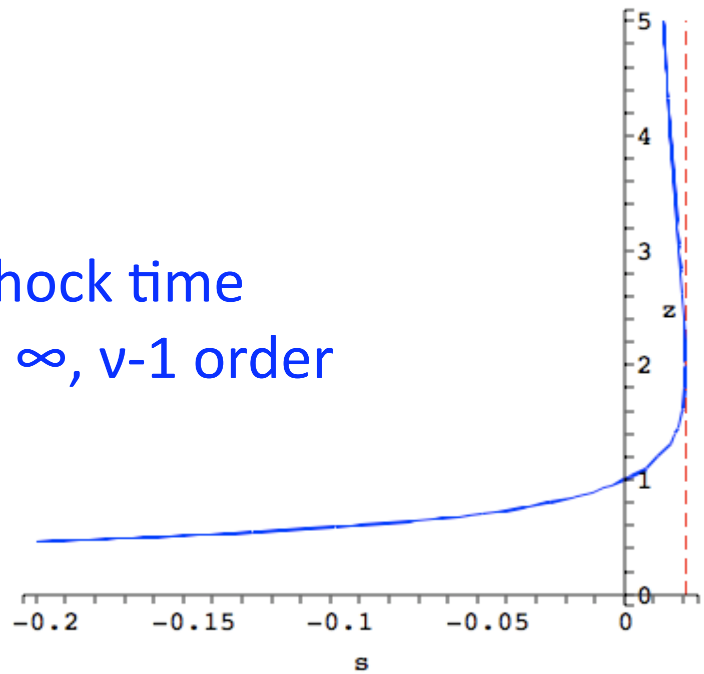
$$
\begin{aligned}
z_0(t) &= \sum_{j \geq 0} c_\nu^j \zeta_j (-t)^j \quad \text{where} \\
c_\nu &= 2\nu \begin{pmatrix} 2\nu - 1 \\ \nu - 1 \end{pmatrix} = (\nu + 1) \begin{pmatrix} 2\nu \\ \nu + 1 \end{pmatrix} \quad \text{and} \\
\zeta_j &= \frac{1}{j} \begin{pmatrix} \nu j \\ j - 1 \end{pmatrix} = \frac{1}{(\nu - 1) j + 1} \begin{pmatrix} \nu j \\ j \end{pmatrix}.
\end{aligned}
$$

What is z_0 (t) ?

3. A self-similar soln. of the inviscid Burgers eqn
   $f_s = c_\nu f^\nu f_w$, $f(s,w) = w z_0(w^{\nu-1} s)$, $f(0,w) = w$

$$
\begin{aligned}
z_0'(s) &= c_\nu z_0(s)^\nu \left(z_0(s) + (\nu - 1) s z_0'(s)\right) \\
1 &= z_0(s) - c_\nu s z_0(s)^\nu \\
z_0' &= \frac{c_\nu z_0^{\nu+1}}{\nu - (\nu - 1) z_0}.
\end{aligned}
$$

Finite shock occurs at $z_0 = \nu/(\nu-1)$ with shock time $s_c = (\nu-1)^{\nu-1}/c_\nu \nu^\nu$; all other shocks are at ∞, ν-1 order ramification point.



What is z_0 (t) ?

4. Leading order of recursion coeffs. for OPRL of weight exp(-NV)

$$
\begin{aligned}
\pi_{n+1,N}(\lambda) &= \lambda \pi_{n,N}(\lambda) - b^2_{n,N}(t)\pi_{n-1,N}(\lambda) \\
b^2_{n,N}(t) &= x(z_0(s) + \frac{1}{n^2} z_1(s) + \frac{1}{n^4} z_2(s) + \cdots) \\
s &= -x^{\nu-1}t \\
b^2_{n,N}(0) &= n/N = x \\
z_g^{(j)}(0) &= \frac{d^j z_g}{ds^j}|_{s=0} \\
&= {}^{\#}\{\text{two-legged } g\text{-maps with } j \ 2\nu\text{-valent vertices }\}
\end{aligned}
$$

Solves continuum limit of dPI hierarchy (Cresswell & Joshi (1999))

dPI (ν = 2) is Freud's equation:

$$4tb^2_{n,N}(t)\left(b^2_{n-1,N}(t) + b^2_{n,N}(t) + b^2_{n+1,N}(t)\right) + b^2_{n,N}(t) = \frac{n}{N}.$$

The "string equation" of 2DQG: Douglas-Shenker, Brezin-Kazakov, Gross-Migdal '80's

Calculations: g=1,2 EMP '08

$$z_1(s) = \frac{(\nu-1)\nu(z_0-1)z_0\left(-\nu^2+(\nu-1)(\nu+2)z_0\right)}{12\left(\nu-(\nu-1)z_0\right)^4}$$

$$\begin{aligned} z_2(t) = & \frac{1}{1440}(\nu-1)\nu(z_0-1)z_0\,[(2\nu^6-14\nu^7+24\nu^8) \\ & +(-12\nu^3+148\nu^4-546\nu^5+758\nu^6-252\nu^7-96\nu^8)z_0 \\ & +(264\nu^2-1510\nu^3+25551\nu^4-500\nu^5-1789\nu^6+840\nu^7+144\nu^8)z_0^2 \\ & +(-536\nu+1396\nu^2+912\nu^3-4596\nu^4+2492\nu^5+1296\nu^6-868\nu^7-96\nu^8)z_0^3 \\ & +(168+234\nu-1467\nu^2+558\nu^3+1902\nu^4-1446\nu^5-267\nu^6+294\nu^7+24\nu^8)z_0^4] \\ & \cdot(\nu-(\nu-1)z_0)^{-9}. \end{aligned}$$

Calculations g=3 EMP '08

$$
\begin{aligned}
z_3(t) = {} & \frac{\nu(\nu-1)}{362880}\frac{z_0(z_0-1)}{(\nu-(\nu-1)z_0)^{14}} \\
& \big[(\nu-2)(\nu-3)(\nu-4)(\nu-5)(\nu-6)(124-147\nu+35\nu^2) \\
& \quad + (\nu-3)(\nu-2)(\nu-1)(104160+47584\nu-332550\nu^2+270697\nu^3-83226\nu^4+8923\nu^5)(z_0-1) \\
& \quad + 3(\nu-2)(\nu-1)^2(312480+744980\nu-1245750\nu^2+373091\nu^3+1085920\nu^4-485414\nu^5 \\
& \qquad\qquad + 67225\nu^6)(z_0-1)^2 \\
& \quad + (\nu-2)(\nu-1)^3(-1562400-7251840\nu+290690\nu^2+11468057\nu^3-2824078\nu^4-3154302\nu^5 \\
& \qquad\qquad + 1078663\nu^6)(z_0-1)^3 \\
& \quad + (\nu-2)(\nu-1)^4(1562400+10781280\nu+12588010\nu^2-10677353\nu^3-11255921\nu^4+3006363\nu^5 \\
& \qquad\qquad + 1779986\nu^6)(z_0-1)^4 \\
& \quad + 3(\nu-1)^5(624960+5411808\nu+10100796\nu^2-1315908\nu^3-9371695\nu^4-973573\nu^5+1835799\nu^6 \\
& \qquad\qquad + 308858\nu^7)(z_0-1)^5 \\
& \quad + (\nu-1)^6(-624960-6823584\nu-20098900\nu^2-16851720\nu^3+3867117\nu^4+8356442\nu^5 \\
& \qquad\qquad + 2223760\nu^6+119824\nu^7)(z_0-1)^6 \\
& \quad + 5(\nu-1)^7(17856+235296\nu+939236\nu^2+1505064\nu^3+1032603\nu^4+285860\nu^5+24472\nu^6 \\
& \qquad\qquad + 64\nu^7)(z_0-1)^7\big] .
\end{aligned}
$$

Low Genus

$$z_1 = \frac{z_0(z_0-1)P_1(z_0)}{\left(\nu-(\nu-1)z_0\right)^4}$$

$$z_2 = \frac{z_0(z_0-1)P_4(z_0)}{\left(\nu-(\nu-1)z_0\right)^9}$$

$$z_3 = \frac{z_0(z_0-1)P_7(z_0)}{\left(\nu-(\nu-1)z_0\right)^{14}}$$

Rationality of z_g

$$z_1 = \frac{z_0(z_0-1)P_1(z_0)}{\left(\nu-(\nu-1)z_0\right)^4}$$

$$z_2 = \frac{z_0(z_0-1)P_4(z_0)}{\left(\nu-(\nu-1)z_0\right)^9}$$

$$z_3 = \frac{z_0(z_0-1)P_7(z_0)}{\left(\nu-(\nu-1)z_0\right)^{14}}$$

$$\vdots$$

$$z_g = \frac{z_0(z_0-1)P_{3g-2}(z_0)}{\left(\nu-(\nu-1)z_0\right)^{5g-1}} = z_0\left\{\frac{a_0^{(g)}(\nu)}{\left(\nu-(\nu-1)z_0\right)^{2g}} + \ldots + \frac{a_{3g-1}^{(g)}(\nu)}{\left(\nu-(\nu-1)z_0\right)^{5g-1}}\right\}$$

Philosophy

- *Rational generating functions* are generating functions, $A(s) = a_1 s + a_2 s^2 + \ldots$, for sequences that satisfy a *linear* recursion relation:
- $a_{n+k} = c_1\, a_{n+k-1} + c_2\, a_{n+k-2} + \ldots + c_k\, a_n \quad (a_1, \ldots a_{k-1}$ given)
- $(c_1 s + c_2 s^2 + \ldots + c_k s^k)\, A(s) = P(s) + A(s) \quad (\deg P = k-1)$
- Conversely, if $A(s) = N(s)/D(s)$ with $D(s)=d_1 s+\ldots +d_m s^m$; then $[s^n]\, A(s)\bullet D(s)=d_0 a_n+\ldots+ d_j a_{n-j} +\ldots+d_m a_{n-m}$ for all $n >$ deg N.
- *But* $e_g, z_g \in \mathbb{Q}(v)\,(z_0(s))$, *are* *not rational* ; Divide and Conquer:
- z_0 drives asymptotics of z_g and e_g (universality of alg-log type)

Recursion Formulae & Finite Determinacy

- Derived Generating Functions

$$z_g^{(j)}(s) \quad = \quad c_\nu^j z_0^{j\nu+1} \left(\sum_{\ell=0}^{3g-1+j} \frac{a_\ell^{(g,j)}(\nu)}{(\nu - (\nu-1)z_0)^{2g+j+\ell}} \right).$$

- Coefficient Extraction

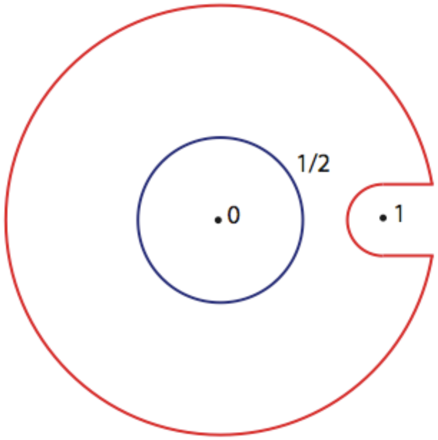
$$\begin{aligned}
z_g^{(j)}(0) &= \#\{\text{two-legged } g\text{-maps with } j \ 2\nu\text{-valent vertices }\} \\
&= \frac{1}{2\pi i} \oint_{s\sim 0} \frac{z_g^{(j)}(s)}{s} ds = \frac{1}{2\pi i} \oint_{z\sim 1} \frac{(\nu - (\nu-1)z) z_g^{(j)}(z)}{z(z-1)} dz \\
&= c_\nu^j \sum_{\ell=0}^{3g-1+j} a_\ell^{(g,j)}(\nu) \ \text{ and} \\
a_\ell^{(g,j)}(\nu) &= [(j-1)\nu - (2g+\ell+(j-2))] a_\ell^{(g,j-1)}(\nu) + \nu[2g+\ell+(j-2)] a_{\ell-1}^{(g,j-1)}(\nu) \ \text{ with} \\
a_0^{(0,0)}(\nu) &= 1 \qquad a_\ell^{(0,0)}(\nu) = 0 \ \text{ for } \ \ell > 0 \qquad a_\ell^{(g,j)}(\nu) = 0 \ \text{ for } \ \ell < 0
\end{aligned}$$

Enumerative Asymptotics

- Shock at $z_0 = \nu/(\nu-1)$ at time $s_c = (\nu-1)^{\nu-1}/(c_\nu \nu^\nu)$
- Rescale: $s = s_c \hat{s}$ → $[\nu/(\nu-1) - z_0] = [2\nu(1-\hat{s})]^{1/2} [1 + O(1-\hat{s})]$

$$z_g = \left(\frac{\nu}{\nu-1} - \sqrt{2\nu(1-\hat{s})}\right)\left\{\frac{1}{\left(\sqrt{2\nu}(\nu-1)\right)^{2g}}\frac{a_0^{(g)}(\nu)}{(1-\hat{s})^{\frac{2g}{2}}} + \frac{1}{\left(\sqrt{2\nu}(\nu-1)\right)^{2g+1}}\frac{a_1^{(g)}(\nu)}{(1-\hat{s})^{\frac{2g+1}{2}}} + \cdots + \frac{1}{\left(\sqrt{2\nu}(\nu-1)\right)^{5g-1}}\frac{a_{3g-1}^{(g)}(\nu)}{(1-\hat{s})^{\frac{5g-1}{2}}}\right\}\{1 + \mathcal{O}(1-\hat{s})\}$$

- Asymptotic growth of coefficients as n → ∞



Hankel Contour

$$[s^n]z_g = \frac{1}{2\pi i}\oint_{s\sim 0}\frac{z_g(s)}{s^{n+1}}ds = s_c^{-n}[\hat{s}^n]z_g$$

$$[\hat{s}^n]\, z_g \sim \frac{\nu}{\nu-1}\frac{a_{3g-1}^{(g)}(\nu)}{\left(\sqrt{2\nu}(\nu-1)\right)^{5g-1}}\frac{n^{\frac{5g-3}{2}}}{\Gamma\left(\frac{5g-1}{2}\right)} + \mathcal{O}(n^{\frac{5g-5}{2}})$$

Relations to Other Enumeration Problems

- *Rooted map* (Tutte) distinguish, vertex, adjacent edge and side of edge
- $M_{n,g}$ = the number of rooted maps on a genus *g* Riemann surface with exactly n edges

$$M_{n,g} \sim t_g n^{5(g-1)/2} 12^n \text{ as } n \longrightarrow \infty$$

Bender, Gao, Richmond (2008)

$$t_0 = \frac{2}{\sqrt{\pi}} \quad t_1 = \frac{1}{24} \quad t_2 = \frac{7}{\sqrt{4320\pi}}$$

$$t_g = -\frac{1}{2^{g-2}6^{\frac{g}{2}}} \cdot \frac{1}{\Gamma(\frac{5g-1}{2})}\alpha_g$$

where α_g is the g^{th} coefficient in the asymptotic expansion at $-\infty$ of the PI transcendent

Universal Asymptotics ? Gao (1993)

[9] for a survey). But it seems very difficult to obtain any nice explicit formula for the exact number of non-planar maps (cf. [1, 4, 11, 12, 13, 16, 17, 22, 23]). Instead, people started working on asymptotic formulas for such numbers. In 1986–1988, E. A. Bender, E. R. Canfield, and N. C. Wormald [3, 10] studied various classes of rooted maps on general surfaces and they obtained asymptotic formulas for the number of rooted maps, rooted smooth maps and rooted 2-c maps (Throughout this paper, k-connected is abbreviated to k-c.). Letting $T_g(n)$ ($P_g(n)$) be the number of orientable (non-orientable) rooted maps (in a certain class) with n edges of type g, they observed that these asymptotic formulas fit the following pattern:

$$T_g(n) \sim t_g(\beta n)^{5(g-1)/2}\,\gamma^n,$$
$$P_g(n) \sim p_g(\beta n)^{5(g-1)/2}\,\gamma^n,$$

where t_g and p_g are positive constants defined in [3], β and γ are independent of n and g (but they depend on the class of maps), and

$$a_n \sim b_n \quad \text{means} \quad \frac{a_n}{b_n} \to 1 \quad \text{as} \quad n \to \infty.$$

But this pattern is not satisfied by the triangular maps and a large class of degree restricted maps (cf. [13–15]). Instead, they satisfy the following modified pattern:

$$T_g(n) \sim \alpha t_g(\beta n)^{5(g-1)/2}\,\gamma^n, \tag{1}$$
$$P_g(n) \sim \alpha p_g(\beta n)^{5(g-1)/2}\,\gamma^n, \tag{2}$$

where α is the gcd of the face valencies of the class of maps.

We shall show that pattern (1) and (2) are also satisfied by the following classes of maps:

1. loopless maps;
2. simple maps, i.e., maps without loops or multiple edges;
3. 3-c triangular maps.

Asymptotic formulas like (1) and (2) played an important role in proving some asymptotic properties of maps such as 0–1 laws for submaps of maps and chromatic properties of maps (cf. [5–7]). It is believed that there should be some purely combinatorial explanation for pattern (1) and (2), but no such one has been found yet. (1) and (2) are derived through some delicate asymptotic analysis about "typical recursions" which are satisfied by many classes of maps and were first described in [3]. We

$$[s^k]e_g \quad \sim \quad \frac{c^{(g)}_{3g-1}(\nu)}{\left(\sqrt{2\nu}(\nu-1)\right)^{5g-1}}\,\frac{k^{\frac{5g-7}{2}}}{\Gamma\left(\frac{5g-5}{2}\right)}\,s_c^{-k}$$

Some General Classes

A Table for Parameters α, β, and γ w.r.t. Edges

Classes of maps	α	β	γ	References
All maps	1	1	12	[3]
2-c maps	1	3	27/4	[10]
3-c maps	1	9	4	[8]
Smooth maps	1	$(3/2)^{1/2}$	$5+2\sqrt{6}$	[3]
Loopless maps	1	3/2	256/27	
Simple maps	1	$(3/2)^2$	8	
Triangular maps	3	$(1/3)6^{1/5}$	$2^{2/3}\times 3^{1/2}$	[13]
2-c triangular maps	3	$(2/3)6^{1/5}$	$3\times 2^{-1/3}$	[14]
3-c triangular maps	3	$(4/3)6^{1/5}$	$(8/3)2^{-1/3}$	
2d-regular maps	2d	$(8d)^{1/5}(d-1)/d$	$\frac{d}{d-1}\left(\frac{d-1}{2}\binom{2d}{d}\right)^{1/d}$	[15]
Loopless 2d-regular maps	2d	$(8d)^{1/5}(d-1)/d$	$\frac{d}{d-1}\left(\frac{d-1}{2}\binom{2d}{d}\right)^{1/d}$	

$$[s^k]e_g \sim \frac{c_{3g-1}^{(g)}(\nu)}{\left(\sqrt{2\nu}(\nu-1)\right)^{5g-1}} \frac{k^{\frac{5g-7}{2}}}{\Gamma\left(\frac{5g-5}{2}\right)} s_c^{-k}$$

Comparison between $M_{n,g}$ and $[s^k]\, e_g$

$$
\begin{aligned}
M_{n,g}^{(2\nu)} \quad & \sim \quad 2\nu\ t_g \left((8\nu)^{1/5}\, \frac{\nu-1}{\nu} n \right)^{\frac{5(g-1)}{2}} \left(\frac{\nu}{\nu-1} \left(\frac{\nu-1}{2} \binom{2\nu}{\nu} \right)^{1/\nu} \right)^{n} \\
n = \nu k \implies \quad & = \quad 2\nu\ t_g\, (8\nu)^{\frac{g-1}{2}} \left((\nu-1)k \right)^{\frac{5(g-1)}{2}} \left(\frac{\nu^{\nu}}{(\nu-1)^{\nu}} \left(\frac{\nu-1}{2} \binom{2\nu}{\nu} \right) \right)^{k} \\
& = \quad 2^{g}\nu\ t_g\, (2\nu)^{\frac{g-1}{2}} \left((\nu-1)k \right)^{\frac{5(g-1)}{2}} (2\nu s_c)^{-k} \\
\text{un-rooting} \quad & \implies \quad \sim 2^{g-2}\ t_g\, (2\nu)^{\frac{g-1}{2}} (\nu-1)^{\frac{5(g-1)}{2}}\, k^{\frac{5g-7}{2}}\, (s_c)^{-k}
\end{aligned}
$$

$$
[s^k]e_g \quad \sim \quad \frac{c_{3g-1}^{(g)}(\nu)}{\left(\sqrt{2\nu}(\nu-1)\right)^{5g-1}} \frac{k^{\frac{5g-7}{2}}}{\Gamma\left(\frac{5g-5}{2}\right)} s_c^{-k}
$$

Relations to Other Enumeration Problems

- Bender, Gao & Richmond (2008)
- Following Goulden & Jackson (2008)
- Garoufalides, Marino (2008)

Painleve I

$$
\begin{aligned}
y'' &= 6y^2 + \xi, \\
y(\xi) &\sim \sqrt{\frac{-\xi}{6}} \left(1 + \sum_{g=1}^{\infty} \alpha_g (-\xi)^{-5g/2} \right) \quad \text{as} \;\; \xi \to -\infty \\
\alpha_{g+1} &= \frac{25g^2 - 1}{8\sqrt{6}} \alpha_g - \frac{1}{2} \sum_{m=1}^{g} \alpha_m \alpha_{g+1-m} \\
\alpha_0 &= 1.
\end{aligned}
$$

Double-Scaling Limits

$$
\begin{aligned}
b^2_{N,N} &- \frac{\nu}{\nu-1} \\
&= \left(z_0 - \frac{\nu}{\nu-1}\right) + \sum_{g=1}^{\infty} z_g N^{-2g} \\
&= \left(z_0 - \frac{\nu}{\nu-1}\right) \\
&+ \quad z_0 \sum_{g=1}^{\infty} \left\{ \frac{a_0^{(g)}(\nu)}{(\nu-(\nu-1)z_0)^{2g}} + \frac{a_1^{(g)}(\nu)}{(\nu-(\nu-1)z_0)^{2g+1}} + \cdots + \frac{a_{3g-1}^{(g)}(\nu)}{(\nu-(\nu-1)z_0)^{5g-1}} \right\} N^{-2g}. \\
&= -\left(\frac{-2c_\nu \gamma_1 \nu^{\nu+1}}{(\nu-1)^{\nu+2}}\xi\right)^{1/2} \left\{ (\nu-1) - \frac{\nu}{\nu-1} \sum_{g=1}^{\infty} a_{3g-1}^{(g)}(\nu) \left(\frac{-2c_\nu \gamma_1 \nu^{\nu+1}}{(\nu-1)^{\nu+2}}\xi\right)^{-5g/2} \right\} N^{-2/5} \\
&+ \quad \mathcal{O}(N^{-4/5})
\end{aligned}
$$

$(\nu - (\nu-1)\, z_0) \sim N^{\delta}$ such that highest order terms have a common factor in N that is independent of *g* :

$\delta = -2/5$ ➔ $N^{4/5}\,(s - s_c) = \gamma^{(\nu)}\, \xi$ where $s_c = (\nu-1)^{\nu-1}/(c_\nu\, \nu^{\nu})$

New Recursion Relations

$$
\begin{aligned}
a^{g+1}_{3(g+1)-1}(\nu) &= \frac{\nu^3\left(25g^2-1\right)}{6} a^{(g)}_{3g-1}(\nu) + \frac{\nu}{2}\sum_{m=1}^{g} a^{(m)}_{3m-1}(\nu) a^{(g-m+1)}_{3(g-m+1)-1}(\nu) \\
a^{(1)}_{2}(\nu) &= \frac{\nu^2}{6} \\
a^{(g)}_{3g-1} &= -2^{5g-1}\,(2/3)^{g/2}\,\alpha_g \quad \text{for} \quad g \geq 1
\end{aligned}
$$

- Coincides with with the recursion for PI in the case ν = 2.

$$
\begin{aligned}
y'' &= 6y^2 + \xi, \\
y(\xi) &\sim \sqrt{\frac{-\xi}{6}} \left(1 + \sum_{g=1}^{\infty} \alpha_g (-\xi)^{-5g/2}\right) \quad \text{as} \quad \xi \to -\infty \\
\alpha_{g+1} &= \frac{25g^2-1}{8\sqrt{6}}\alpha_g - \frac{1}{2}\sum_{m=1}^{g} \alpha_m \alpha_{g+1-m} \\
\alpha_0 &= 1.
\end{aligned}
$$

Connecting RHPs & Non-Hermetian OPs

- Fokas, Its, Kitaev 1992
- Kamvissis, Rachmanov 2005
- Duits, Kuijlaars 2006
- Bertola 2006

Elements of Proof

I. Extended Riemann-Hilbert Analysis
II. Deconstruct the extended continuum limit in terms weighted Dyck bridges
III. A long recursive calculation (generalized binomial inversion ?)

Element I: Uniformizing the Equlibrium Measure

- For $\lambda = 2\, z_0^{1/2}\, \eta$

$$d\mu_{V_t}(\eta) = \frac{2}{\pi}\chi_{(-1,1)}(\eta)\left\{ z_0 + (1-z_0)\left[\frac{(2\eta)^{2\nu-2}}{\binom{2\nu-1}{\nu-1}} + \sum_{j=1}^{\nu-1} 2\frac{\binom{2j-1}{j-1}}{\binom{2\nu-1}{\nu-1}}(2\eta)^{2\nu-2-2j}\right]\right\}\sqrt{(\eta+1)(1-\eta)}\; d\eta.$$

$$= z_0\, \hat{u}_{Gauss}(\eta) + (1-z_0)\, \hat{u}_{mon(\nu)}(\eta)$$

- Each measure continues to the complex η as a differential whose *square* is a *holomorphic quadratic differential*.

Element I: Free Energy for "Large Time"

Theorem There is a constant $\Delta > 0$ such that for (complex) t with $\Re(t) \geq 0$, $|\Im(t)| < \Delta$ one has a uniformly valid asymptotic expansion

$$\log \tau^2_{N,N}(t) = N^2 e_0(t) + e_1(t) + \frac{1}{N^2} e_2(t) + \cdots \tag{1}$$

as $N \to \infty$. Also, the recurrence coefficients for the monic orthogonal polynomials with weight $\exp(-NV(\lambda))$ have a full asymptotic expansion, uniformly valid for (complex) t with $\Re(t) \geq 0$, $|\Im(t)| < \Delta$, of the form

$$b^2_{N,N}(t) \quad = \quad z_0(-t) + \frac{1}{N^2} z_1(-t) + \frac{1}{N^4} z_2(-t) + \cdots \tag{2}$$

as $N \to \infty$. The meaning of these expansions is: if you keep terms up to order N^{-2h}, the error term is bounded by CN^{-2h-2}, where the constant C is independent of t in the domain $\{(\Re t \geq 0; -\Delta < \Im t < \Delta\}$. Moreover, in this domain, for each ℓ, the functions $e_\ell(t)$ and $z_\ell(-t)$ are analytic functions of t and the asymptotic expansion of derivatives of $\log(Z_N(t))$ and $b^2_{N,N}(t)$ may be calculated via term-by-term differentiation of the series.

See also Bleher-Its "Asymptotics of the partition of a random matrix model" Ann. Inst. Fourier **55** (2005)

Element II: Toda-Hirota Representation

$$\begin{aligned}
\pi_{n+1,N}(\lambda) &= \lambda\pi_{n,N}(\lambda) - b_{n,N}^2(t)\pi_{n-1,N}(\lambda). \\
b_n^2(\theta) &= \frac{1}{2}\frac{d^2}{d\theta_1^2}\log\left[\tau_{n,1}^2(\theta_1,\theta)\right]_{\theta_1=0} \quad \text{where} \\
\tau_{n,1}^2(\theta_1,\theta) &= Z_1^{(n)}(\theta_1,\theta)/Z_1^{(n)}(0,0) \text{ and} \\
Z_N^{(n)}(t_1,t) &= \int\cdots\int \exp\left\{-N\sum_{j=1}^{n}\left(\frac{1}{2}\lambda_j^2 + t\lambda_j^{2\nu} + t_1\lambda_j\right)\right\}\mathcal{V}(\lambda)d^n\lambda \\
\frac{1}{2}\frac{db_n^2}{d\theta} &= \sum_{\{\text{Dyck bridges}\}}\left[\prod_{m=1}^{\nu+1} b^2_{n+\ell_m(w)+1} - \prod_{m=1}^{\nu+1} b^2_{n+\ell_m(w)}\right] \\
b_{n,N}^2 &= n\left(z_0(s) + \frac{1}{n^2}z_1(s) + \frac{1}{n^4}z_2(s) + \cdots\right) \\
b_{n+\ell,N}^2 &= n\left(wz_0(sw^{\nu-1}) + \cdots + \frac{1}{n^{2g}}w^{1-2g}z_g(sw^{\nu-1}) + \cdots\right) \quad \text{where} \\
w &\sim \left(1 + \frac{\ell}{n}\right)
\end{aligned}$$

Element II: Cluster Expansion

$$\frac{d}{ds}f(s,w)\Big|_{w=1} = \sum_{\{\text{Db's}\}} n f^{\nu+1}\left\{\prod_{m=1}^{\nu+1}\left[1+\frac{f_w}{f}\left(\frac{\ell_m+1}{n}\right)+\frac{f_{w^{(2)}}}{2f}\left(\frac{\ell_m+1}{n}\right)^2+\cdots+\frac{f_{w^{(h)}}}{h!f}\left(\frac{\ell_m+1}{n}\right)^h+\cdot\cdot\right.\right.$$

$$\left.\left. - \prod_{m=1}^{\nu+1}\left[1+\frac{f_w}{f}\frac{\ell_m}{n}+\frac{f_{w^{(2)}}}{2f}\left(\frac{\ell_m}{n}\right)^2+\cdots+\frac{f_{w^{(h)}}}{h!f}\left(\frac{\ell_m}{n}\right)^h+\cdots\right]\right\}\Big|_{w=1}$$

Element II: Continuum Limit (+)

$$f_s = F^{(\nu)}\left(n^{-1}; f, f_w, \dots, f_{w^m}, \dots\right) \doteq$$

$$c_\nu f^\nu f_w + \frac{1}{n^2} F_1^{(\nu)}(f, f_w, f_{ww}, f_{www}) + \cdots$$

$$+ \frac{1}{n^{2g}} F_g^{(\nu)}(f, f_w, f_{w^{(2)}}, \cdots, f_{w^{(2g+1)}}) + \cdots$$

for (s, w) near $(0, 1)$ and initial data given by $f(0, w) = w$.

$$F_g^{(\nu)} = \sum_{\lambda:|\lambda|=2g+1 \ni \ \ell(\lambda)\leq \nu+1} \frac{d_\lambda^{(\nu,g)}}{\prod_j r_j(\lambda)!} f^{\nu-\ell(\lambda)+1} \prod_j \left(\frac{f_{w^{(j)}}}{j!}\right)^{r_j(\lambda)} \quad \text{where}$$

$$|\lambda| = \sum_i \lambda_i$$

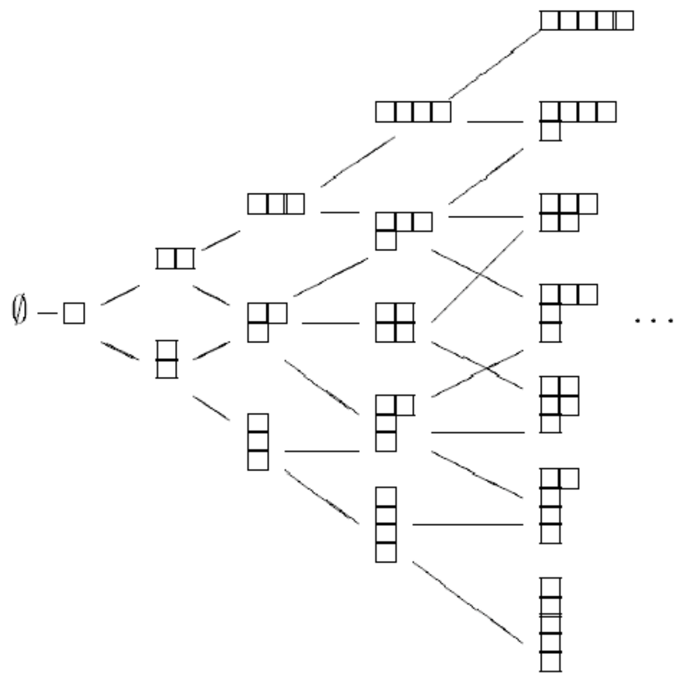
$$r_j(\lambda) = \#\{\lambda_i | \lambda_i = j\}$$

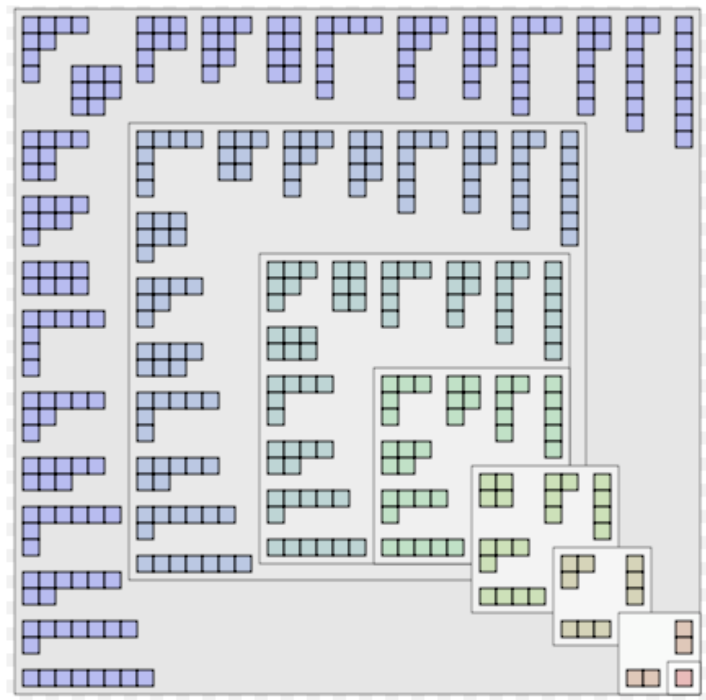
$$\ell(\lambda) = \sum_j r_j(\lambda)$$

$F_g^{(\nu)}$ resembles a *Faa di Bruno* formula :

$$\frac{d^n}{dw^n} g(f(w)) = \sum_{\ell=0}^{n} g^{(\ell)}(f(w))\, B_{n,\ell}\left(f'(w), f''(w), \dots, f^{(n-\ell+1)}(w)\right)$$

Element II: The Young Graph





$$d_\lambda^{(\nu,g)} = \sum_{\substack{(\nu+1,\nu,\dots,2,1)\subseteq\mu\subseteq(2\nu,2\nu-1,\dots,\nu)\\ \mu\in\mathcal{R}}} 2\, m_\lambda\left(\mu_1-\eta_1,\dots,\mu_{\nu+1}-\eta_{\nu+1}\right)$$

$(\eta_1,\dots,\eta_{\nu+1}) = (2\nu, 2\nu-2,\dots,2,0)$ where

$$m_\lambda(x_1,x_2,\dots) = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n} + \dots.$$

Element III:
Weakly Nonlinear Higher Order Asymptotics

$$\begin{aligned}
f(s,w) &= f_0(s,w) + \frac{1}{n^2} f_1(s,w) + \cdots + \frac{1}{n^{2g}} f_g(s,w) + \cdots . \\
\frac{df_0}{ds} &= c_\nu (f_0)^\nu (f_0)_w \\
f_0(0,w) &= w, \\
\frac{df_g}{ds} &= c_\nu \left((f_0)^\nu (f_g)_w + \nu (f_0)^{\nu-1} (f_0)_w f_g \right) + \text{Forcing}_g, \ \text{ for } g > 0. \\
\text{Forcing}_g &= \left(\frac{c_\nu}{\nu+1} \frac{\partial}{\partial w} \sum_{\substack{0 \le k_j < g \\ k_1 + \cdots + k_{\nu+1} = g}} f_{k_1} \cdots f_{k_{\nu+1}} \right) \\
&+ F_1^{(\nu)}[2g-2] + F_2^{(\nu)}[2g-4] + \cdots + F_g^{(\nu)}[0] \ \text{ where} \\
F_\ell^{(\nu)}[2r] &= \text{the coefficient of } n^{-2r} \text{ in } F_\ell^{(\nu)}
\end{aligned}$$

Element III: Propagator Representation

$$
\begin{aligned}
f_g(s,w) &= w^{1-2g} z_g(w^{\nu-1}s) \\
f(s,1) &= z_0(s) + \frac{1}{n^2} z_1(s) + \frac{1}{n^4} z_2(s) + \cdots \\
z_g(s) & \quad \text{is an abelian function of} \\
z_0 & \quad \text{with singlularities only possible at } z_0 = 0, \nu/(\nu-1). \\
z_g(s) &= z_g(z_0(s)) \\
&= \frac{z_0(s)^{2(1-g)}}{\nu-(\nu-1)z_0(s)} \int_1^{z_0(s)} \frac{(\nu-(\nu-1)z)}{c_\nu z^{\nu+3-2g}} \text{Forcing}_g(z)dz
\end{aligned}
$$

Element III:

Proposition

(i) z_g is regular at $z_0 = 0$ and in fact vanishes at least simply there.

(ii) Forcing_g may be written as a sum of terms each of which has the form

$$f_0^{\nu-m+1}(f_{k_1})_{w^{(j_1)}}(f_{k_2})_{w^{(j_2)}}\cdots(f_{k_m})_{w^{(j_m)}}$$

where $0 < m \leq \nu + 1$ and $j_1 + \cdots + j_m = 2(g - k_1 - k_2 - \cdots - k_m) + 1$, with $k_i < g$.

(iii) For $(k, n) \neq (0, 0)$,

$$(f_k)_{w^{(n)}}(s, 1) \quad = \quad z_0 \left\{ \frac{b_0^{(k,n)}}{(\nu - (\nu - 1)z_0)^{2k+n}} + \cdots + \frac{b_{3k+n-1}^{(k,n)}}{(\nu - (\nu - 1)z_0)^{5k+2n-1}} \right\}.$$

Element III: Upshot

- For z_g the solvability conditions (meromorphic w/o residues) are always satisfied.
- For e_g the solvability conditions are satisfied for all $g > 1$
- At $g=0$ and $g=1$, for e_g, these conditions require choosing a branch of the logarithm.
- For $g > 0$, z_g is rational in z_0 ;
- For $g > 1$, e_g is rational in z_0 .

- z_g (resp. e_g) all have the same envelope of holomorphy.

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