Universality Limits for Random Matrices via Classical Complex Analysis.

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Abstract Let μ be a finite positive measure on the real line, with orthonormal polynomials $\{p_n\}$, and reproducing kernel

$$K_n(x,y)=\sum_{k=0}^{n-1}p_k(x)\,p_k(y)\,.$$

We also need the normalized cousin

$$\tilde{K}_n(x,y)=\mu'(x)^{1/2}\,\mu'(y)^{1/2}\,K_n(x,y)\,.$$

For Hermitian matrices, (the so-called unitary case), the universality limit in the bulk can be reduced to

$$\lim_{n\to\infty}\frac{K_n\left(\xi+\frac{u}{\tilde{K}_n(\xi,\xi)},\xi+\frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}=\frac{\sin\pi(u-v)}{\pi(u-v)}.$$

We discuss proofs of this that use classical methods of complex analysis and functional analysis. The essential feature, is that the functions of u,v in the left-hand side of the limit are uniformly bounded for u,v in compact subsets of the plane, and hence form a normal family. To show that the limit is the sinc kernel, one shows that subsequential limits have a reproducing kernel type property for Paley-Wiener space, that holds only for the sinc kernel.

This method can be applied to fixed measures and varying measures, and adapted to the edge of the spectrum. It can be used to show that universality holds in linear Lebesgue measure, *meas*, without any local or global conditions, in the set $\{\mu'>0\}=\{\xi:\mu'(\xi)>0\}$:

Theorem *Let μ be a measure with compact support and with infinitely many points in the support. Let $\varepsilon>0$ and $r>0$. Then as $n\to\infty$,*

$$meas\left\{\xi\in\{\mu'>0\}:\sup_{|u|,|v|\le r}\left|\frac{K_n\left(\xi+\frac{u}{\tilde{K}_n(\xi,\xi)},\xi+\frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}-\frac{\sin\pi(u-v)}{\pi(u-v)}\right|\ge\varepsilon\right\}$$
$$\to\quad 0.$$

When there is some local regularity of μ, one can establish universality pointwise:

Theorem *Let μ be a measure with compact support Let J be an open interval in which μ is absolutely continuous, and μ' is positive and continuous.*

Then

$$\lim_{n\to\infty, n\notin\mathcal{E}} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi,\xi\right)} = \frac{\sin\pi\left(u-v\right)}{\pi\left(u-v\right)}$$

uniformly for u, v *in compact subsets of* $\mathbb{C}$*, where* $\mathcal{E}$ *is a set of density 0.*

We also discuss related results of Findley, Simon, Totik, on fixed measures, and results on varying measures established by Eli Levin and the speaker.