A boundary value transformation for an inverse problem arising in magnetometry

Dmitry Glotov

The following problem arises in magnetometry: given the measurements of the magnetic field on the boundary of some domain, determine the location, orientation, and magnitudes of the sources inside the domain. The devices that measure the magnetic field may be classified as *vector* and *scalar* magnetometers. The vector magnetometers measure the full magnetic field while the scalar magnetometers are only capable of determining the field's magnitude. The range of applications of vector magnetometers is limited as some of them require frequent calibration and high maintenance and others produce measurements only for small fields. The scalar magnetometers are robust, accurate, and affordable devices but the measurements that they produce are not complete as they cannot determine the field's direction. The results presented in this talk are motivated by the need to bridge the gap between these two types of devices. In particular, we introduce a method that allows to convert the scalar boundary data into the more meaningful vector measurements.

In the absence of currents, the magnetic field $\mathbf{B}$ is modeled as the gradient of a magnetic *scalar* potential u, that is, $\mathbf{B} = \nabla u$ for some u satisfying

$$\Delta u = f. \tag{1}$$

The source term f is considered in the form

$$f = \textstyle\sum_{j=1}^{M} a_j \delta_{x^j} + \sum_{j=1}^{N} b^j \cdot D\delta_{y^j}. \tag{2}$$

Such representation of f corresponds to M monopoles located at x^j and N dipoles located at y^j. There is vast literature for the *inverse source problem for the Poisson equation* that may be formulated as follows: given the values of potential u satisfying (1) with the source in the form (2) and the normal derivative $\partial u/\partial \nu$ on the boundary of some domain Ω, find M, N, x_j, a^j, $j = 1, \ldots, M$, y^j, b^j, $j = 1, \ldots, N$. This form of data corresponds to the measurements produced by the vector magnetometers. Our focus is on the boundary data in the form $p = |\nabla u|^2$ that corresponds to the measurements produced by the scalar magnetometers. We pose the following question:

Question. *Given the values of p and ∇p on the boundary of a domain Ω, determine whether there exists a unique harmonic function u defined in a neighborhood of $\partial\Omega$ such that*

$$\left\{ \begin{array}{rcl} |\nabla u|^2 & = & p, \\ \nabla^2 u(\nabla u) & = & \frac{1}{2}\nabla p, \end{array} \right. \qquad \textit{on } \partial\Omega. \tag{3}$$

Find u and its gradient on $\partial\Omega$, if such u exists.

We address these issues in the case of dimension two. We prove that, under some additional constraints, the solution of the above problem is unique and provide necessary conditions for the existence of solutions. We introduce an algorithm for solving this problem numerically. Finally, we discuss the possibility of extending this method to dimension three.