A boundary value transformation for an inverse problem arising in magnetometry

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Direct problem for Poisson equation

Let Ω be a bounded domain in $\mathbb{R}^n$. Consider Poisson equation:

$$\Delta u = f, \quad \text{in } \Omega, \tag{1}$$

with Dirichlet boundary data

$$u = g, \quad \text{on } \partial\Omega, \tag{2}$$

or Neumann boundary data

$$\frac{\partial u}{\partial \nu} = h, \quad \text{on } \partial\Omega, \tag{3}$$

Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ or $h \in H^{-1/2}(\partial\Omega)$, the *direct problem* specified by (1), (2) or (1), (3) has unique solution $u \in H^1(\Omega)$.

Green's representation

Let Ω be a domain for which the divergence theorem holds and $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then, for $y \in \Omega$,

$$u(y) = \int_{\partial\Omega} u \, \frac{\partial G_1}{\partial \nu} \, ds + \int_{\Omega} G_1 \Delta u \, dx$$

or

$$u(y) = -\int_{\partial\Omega} G_2 \, \frac{\partial u}{\partial \nu} \, ds + \int_{\Omega} G_2 \Delta u \, dx,$$

for $G_1 = \Gamma + h_1$ or $G_2 = \Gamma + h_2$, where

$$\Gamma(x-y) = \begin{cases} \dfrac{1}{n(2-n)\omega_n} |x-y|^{2-n}, & n > 2 \\ \dfrac{1}{2\pi} \log|x-y|, & n = 2; \end{cases}$$

$h_1, h_2 \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ are harmonic and chosen so that $G_1|_{\partial\Omega} = 0$ or $\left.\dfrac{\partial G_2}{\partial \nu}\right|_{\partial\Omega} = 0$.

Inverse source problem

Suppose the source term f has the form

$$f = \sum_{j=1}^{M} a_j \delta_{x^j} + \sum_{j=1}^{N} b^j \cdot D\delta_{y^j}, \tag{4}$$

for some M, $N \in \mathbb{N}$; $a_j \in \mathbb{R}$, $x^j \in \Omega$, for $j = 1, \dots, N$; and $b^j \in \mathbb{R}^n$, $y^j \in \Omega$, for $j = 1, \dots, M$, where δ_x is the Dirac delta function at x.

The *inverse source problem*: given the values of potential u and its normal derivative $\partial u/\partial \nu$ on the boundary of the domain, find the number, location, and magnitudes of the point sources, that is, find N, M, a_j, x^j, b^j, and y^j.

Ohe-Ohnaka (1994, 1995), Yamatani-Ohnaka (1997), el Badia-Ha Duong (2000), Inui-Yamatani-Ohnaka (2003), Naro-Ando (2003), el Badia (2005), Ling-Han-Yamamoto (2005), Ikehata (2007), Yamatani-Ohe-Ohnaka (2007)

Boundary data

We focus our attention on the boundary data involving $|Du|$, the absolute value of the gradient of the solution.

Example

Let $\Omega = B_1(0)$ and $f = \chi_{B_R(0)\setminus B_r(0)}$, $r < R < 1$, i.e. f is a volume density. Then $p = |Du|^2$ is constant.

Require additional data in the form of $Dp = D|Du|^2 = 2\langle D^2u, Du\rangle$.

Question

Given the values of p and Dp on $\partial\Omega$, determine whether there exists a unique harmonic function u defined in a neighborhood of $\partial\Omega$ such that

$$\begin{cases} |Du|^2 = p, \\ \langle D^2u, Du\rangle = \frac{1}{2}Dp, \end{cases} \qquad \text{on } \partial\Omega. \tag{5}$$

Find u and its gradient on $\partial\Omega$, if such u exists.

Applications

Magnetometry: geomagnetic survey; identification of pollution sources in the environment; inverse electroencephalography/magnetoencephalography (EEG/MEG); exploration of space; detection of archeological sites; marine magnetic anomaly detection.

Vector magnetometers measure the magnetic field (expensive, high maintenance). Examples: fluxgate magnetometers, Superconducting Quantum Interface Devices, Spin-Exchange-Relaxation-Free atomic magnetometers.

Scalar magnetometers measure the magnitude of the magnetic field only (highly accurate, robust, reliable). Examples: proton precession magnetometers, cesium vapor magnetometers.

Inverse *gravity* problem: determine the density given the measurement of the force on the given surface.

Linear system for $n = 2$

In 2d, let $x = (x_1, x_2)$ and $p = u_1^2 + u_2^2$. Here, subscripts of u and p denote partial derivatives. Then

$$p_1 = 2(u_1 u_{11} + u_2 u_{12}),$$
$$p_2 = 2(u_1 u_{12} + u_2 u_{22}).$$

Using the fact that u is harmonic, we get

$$\begin{aligned} u_1 p_1 - u_2 p_2 &= 2(u_1^2 u_{11} - u_2^2 u_{22}) &= 2pu_{11}, \\ u_2 p_1 + u_1 p_2 &= 2(u_2^2 u_{12} + u_1^2 u_{21}) &= 2pu_{12}. \end{aligned} \tag{6}$$

These are linear equations for u_1, u_2, u_{11}, u_{12} in terms of p, p_1, p_2.

Equivalent system of ODEs

At each $x \in \partial\Omega$, τ is tangent to $\partial\Omega$ at x and ν is the outward pointing unit normal to $\partial\Omega$ at x.
Parametrize the boundary $\gamma : [0, T] \to \partial\Omega$ and introduce a time variable t.
Setting $y(t) = (y_1(t), y_2(t)) = (u_\tau(\gamma(t)), u_\nu(\gamma(t)))$, under the additional assumption that $p > 0$, system (6) becomes

$$\dot{y} = A(t)\, y, \tag{7}$$

with

$$A(t) = \begin{pmatrix} a_1(t) & -a_2(t) \\ a_2(t) & a_1(t) \end{pmatrix} \tag{8}$$

where

$$a_1(t) = \frac{p_\tau(t)}{2p(t)} \quad \text{and} \quad a_2(t) = \frac{p_\nu(t)}{2p(t)}. \tag{9}$$

Well-posedness of the ODE

Theorem

The solution of the initial value problem for (7) *exists for all* $t \in \mathbb{R}$. *The space of solutions has dimenension two. More precisely, if* (y_1, y_2) *is a solution of* (7)–(8), *then so is* $(-y_2, y_1)$. *Furthermore, the solutions are* T*-periodic if and only if* $\int_0^T a_1(t)\,dt = 0$ *and* $\int_0^T a_2(t)\,dt = 2n\pi$, $n \in \mathbb{N}$.

Lemma

Suppose u *is a solution of* (1),(4) *in* Ω, *that is,*

$$u(x) = u_0(x) + \sum_{j=1}^{M} a_j \Gamma(x - x^j) + \sum_{j=1}^{N} b^j \cdot D\Gamma(x - y^j), \qquad (10)$$

Let $p = |Du|^2$ *and assume* $p > 0$ *on* $\partial\Omega$. *Then*

$$-\frac{1}{2\pi} \int_{\partial\Omega} \frac{1}{2p} \frac{\partial p}{\partial \nu}\, d\tau = M + 2N. \qquad (11)$$

Non-uniqueness

If (y_1, y_2) is any solution of the ODEs system (7)–(8), then $(u_\tau, u_\nu) = (y_1, y_2)$ satisfies (5).

Example

Fix $r > 0$, $b \in \mathbb{R}^2$ with $b \neq 0$, suppose $\Omega = B_r(0)$, and let $u = b \cdot D\Phi$ where $\Phi(x) = -\frac{1}{2\pi} \log |x|$, i.e.,

$$u(x) = -\frac{1}{2\pi} \frac{b \cdot x}{|x|^2}.$$

Then $p(x) = |Du(x)|^2 = \dfrac{1}{4\pi^2} \dfrac{|b|^2}{|x|^2}$ and, for $x \in \partial B_r(0)$,

$$p(x) = \frac{|b|^2}{4\pi^2 r^4}, \quad \frac{\partial}{\partial \tau} p(x) = 0, \quad \text{and} \quad \frac{\partial}{\partial \nu} p(x) = \frac{\partial p}{\partial r} = -\frac{|b|^2}{\pi^2 r^5},$$

all constant on the set $\partial B_r(0)$.

Additional constraint

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with smooth boundary. Suppose u is harmonic in a neighborhood containing $\partial\Omega$ and is such that $p = |Du|^2 > 0$ on $\partial\Omega$ and $\int_{\partial\Omega} u_\nu \, d\tau \neq 0$. If $\tilde{u}$ is another harmonic function satisfying $|D\tilde{u}|^2 = p$ and $2\langle D^2\tilde{u}, D\tilde{u}\rangle = Dp$ on $\partial\Omega$, then $\tilde{u} = \pm u + C$ for some constant C.

When u is a solution of (1), (4), the following relation holds

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\tau = \sum_{j=1}^{N} a_j .$$

No physical interpretation when u is the magnetic potential. Possible when u is either the gravitational or electic potential.

Numerical example

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_1^2 + x_2^2 \leq 1\}$, N be the number of mesh points, and let $\theta = 2\pi/N$ be the dicretization angle. The points $(x_j, y_j) = (\cos j\theta, \sin j\theta)$, for $j = 1, \ldots, N$, form a uniform mesh on the boundary $\partial\Omega$.

Unknowns: $u_\tau^j = u_\tau(x_j, y_j)$, $u_\nu^j = u_\nu(x_j, y_j)$, $u_{\tau\tau}^j = u_{\tau\tau}(x_j, y_j)$, $u_{\tau\nu}^j = u_{\tau\nu}(x_j, y_j)$.

Center finite-difference approximation for the derivatives of u of second order:

$$\begin{aligned} u_{\tau\tau}^j &= \tfrac{1}{2}(-\cot\theta\, u_\tau^{j-1} + u_\nu^{j-1} + \cot\theta\, u_\tau^{j+1} + u_\nu^{j+1}), \\ u_{\tau\nu}^j &= \tfrac{1}{2}(-u_\tau^{j-1} - \cot\theta\, u_\nu^{j-1} - u_\tau^{j+1} + \cot\theta\, u_\nu^{j+1}). \end{aligned} \tag{12}$$

Periodic boundary conditions: $u_\tau^0 = u_\tau^N$, $u_\nu^0 = u_\nu^N$, $u_\tau^{N+1} = u_\tau^1$, $u_\nu^{N+1} = u_\nu^1$.

Matrix form: $AX = 0$

The unknown vector is $X = (u_\tau^1, u_\nu^1, \ldots, u_\tau^N, u_\nu^N)^t$. The matrix A is in the block diagonal form in which each block corresponds to the variables (u_τ^j, u_ν^j) at each $j = 1, \ldots, N$:

$$A = \begin{pmatrix} B^1 & U^1 & & & L^1 \\ L^2 & B^2 & U^2 & & \\ & \ddots & \ddots & \ddots & \\ & & L^{N-1} & B^{N-1} & U^{N-1} \\ U^N & & & L^N & B^N \end{pmatrix}, \text{ where } B^j = \begin{pmatrix} p_\tau^j & -p_\nu^j \\ p_\nu^j & p_\tau^j \end{pmatrix}, \text{ and}$$

$$U^j = \begin{pmatrix} p^j \cot\theta & -p^j \\ p_j & -p^j \cot\theta \end{pmatrix}, L^j = \begin{pmatrix} p^j \cot\theta & -p^j \\ p^j & p^j \cot\theta \end{pmatrix}.$$

Equation $\sum_{j=1}^N u_\tau^j = 0$ is the discrete version of the additional constraint $\int_{\partial\Omega} u_\tau \, d\tau = 0$. Introducing this constraint amounts to appending the row $(1, 0, 1, 0, \ldots, 1, 0)$ to matrix A.

Convergence

Potential with two monopoles and one dipole that corresponds to setting $u_0 \equiv 0$, $N = 2$, $a_1 = 0.2$, $x^1 = (-0.2, 0.1)$, $a_2 = 0.1$, $x^2 = (0.1, -0.3)$, $M = 1$, $b^1 = (.25, .15)$, and $y^1 = (0.5, 0.4)$ in (10)

Observed rate of convergence:

$$\|\tilde{u} - u\| + \|(\tilde{u}_\tau - u_\tau, \tilde{u}_\nu - u_\nu)\| \leq C\theta^2 \|(u_\tau, u_\nu)\|,$$

where $\theta = T/N$ is the discretization angle, and $\|\cdot\|$ is the norm in either L^2 or L^∞, u is the exact solution, $\tilde{u}$ is the approximation.

Convergence

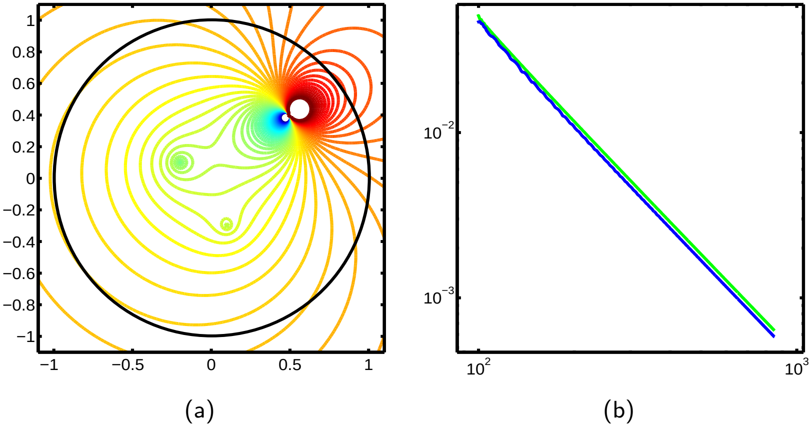

Figure: (a) Contour plot of the solution; (b) loglog plot of the error in L^2 and sup norms.