Inverse Scattering From Cusp

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Motivating example

- Riemann surface with cusp
- This is a classical topic. However, let us recall the basic facts.
- Any hyperbolic manifolds are constructed by the **action of discrete groups** on the unit disc or the upper-half plane.

The upper-half space model

- Consider the upper-half space model of the 2-dim. hyperbolic space, i.e.

$$\mathbf{H}^2 = \{z = x + iy\,;\, y > 0\}$$

equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

Action of $SL(2, \mathbf{R})$

$$SL(2, \mathbf{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$g \cdot z = \frac{az + b}{cz + d}, \quad z \in \mathbf{H}^2$$

The quotient space

- Take a discrete subgroup $\Gamma \subset SL(2, \mathbf{R})$, (called **Fuchsian group**) and consider the quotient space

$$\mathcal{M}_\Gamma = \Gamma \backslash \mathbf{H}^2$$

by the action

$$\Gamma \times \mathbf{H}^2 \ni (\gamma, z) \to \gamma \cdot z$$

Geometric finiteness

Γ (or $\mathcal{M}_\Gamma$) is geometrically finite

$\iff \mathcal{M}_\Gamma$ is a finite sided convex polygon

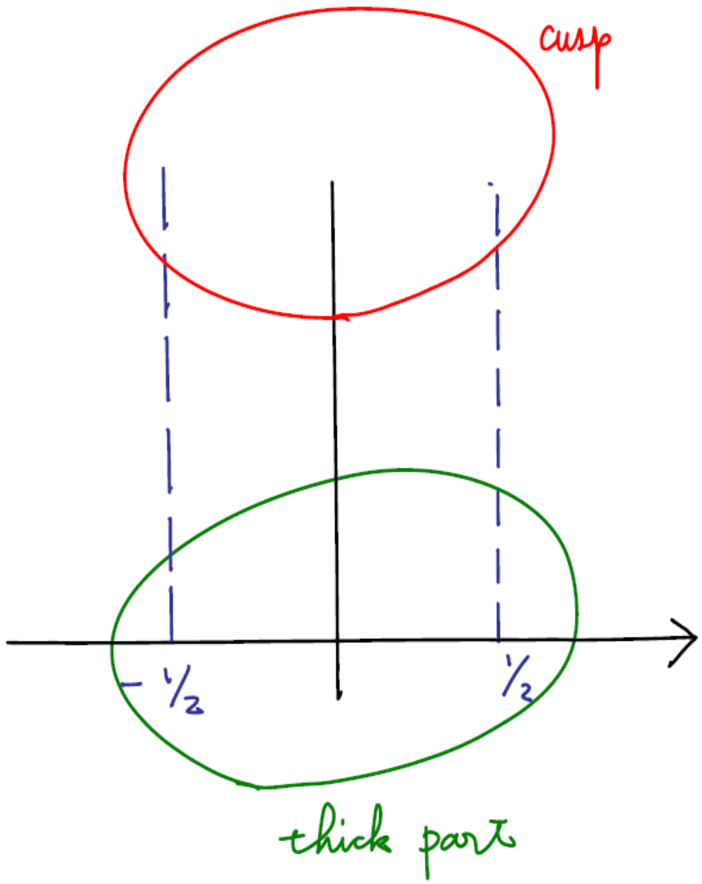
$\iff \Gamma$ is finitely generated

Example 1. Translation

$$\Gamma : z \longrightarrow z + 1$$

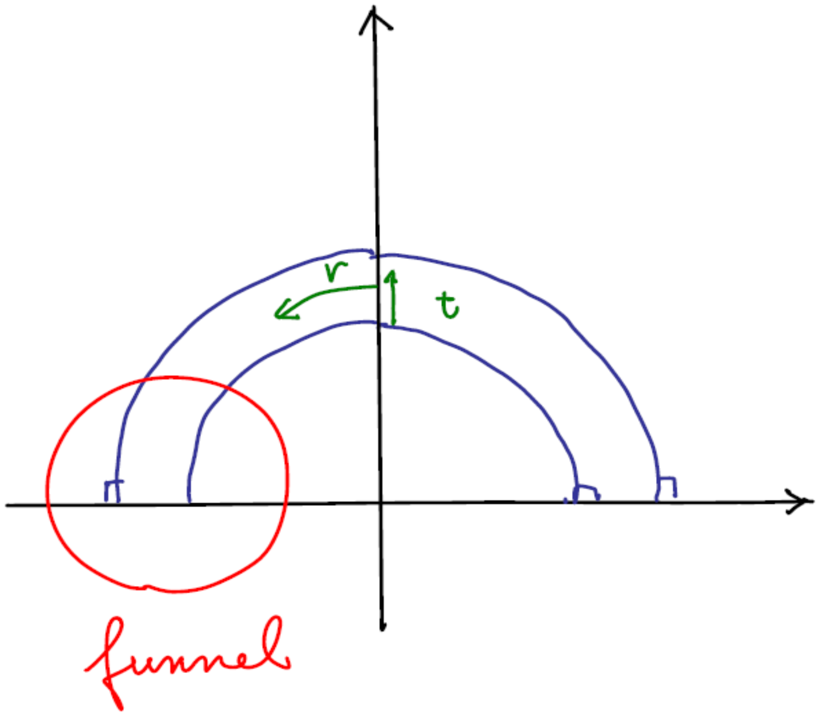
$$\mathcal{M}_\Gamma = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, \infty),$$

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$



Example 2.   Dilation

$$\Gamma : z \to \lambda z \quad (\lambda > 1)$$



$$ds^2 = (dr)^2 + \cosh^2 r (dt)^2$$

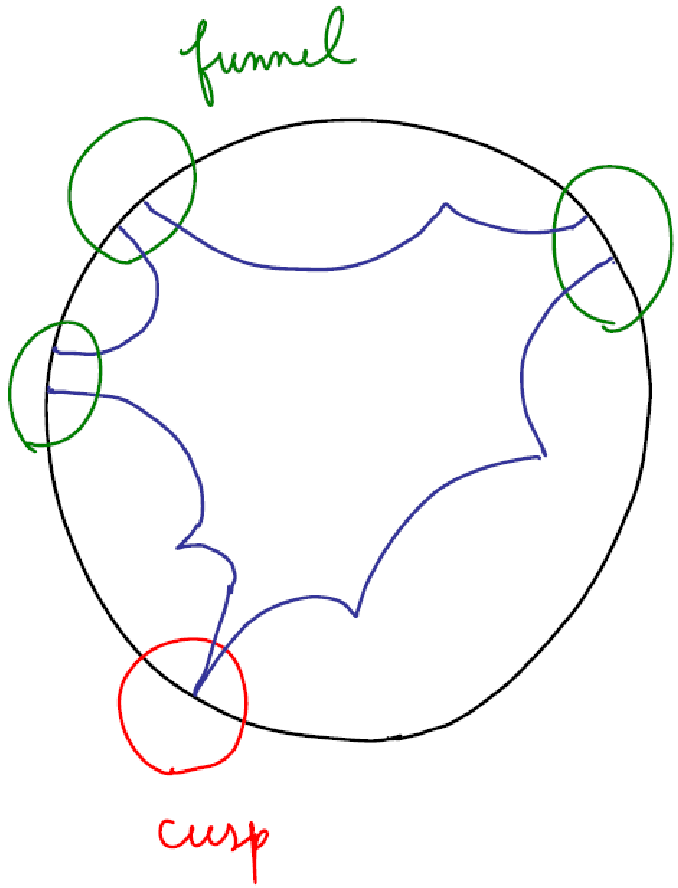
$$(y = 2e^{-r})$$

$$= \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} + \frac{y}{4}\right)^2 (dt)^2$$

Classification of 2-dim. Hyperbolic manifolds

- THEOREM. Suppose $\mathcal{M}_\Gamma$ is a 2-dim. non-elementary geometrically finite hyperbolic manifold. Then there exists a compact subset $\mathcal{K} \subset \mathcal{M}_\Gamma$ such that $\mathcal{M}_\Gamma \setminus \mathcal{K}$ is a finite disjoint union of cusps and funnels.

Most general example



Fuchsian group of the 1st kind

- THEOREM Let Γ be a Fuchsian group.

Γ is of the 1st kind

$\iff$ $\mathcal{M}_\Gamma$ has a finite area

$\implies$ $\mathcal{M}_\Gamma$ is geometrically finite

Fundamental domains for the Fuchsian group of the 1st kind

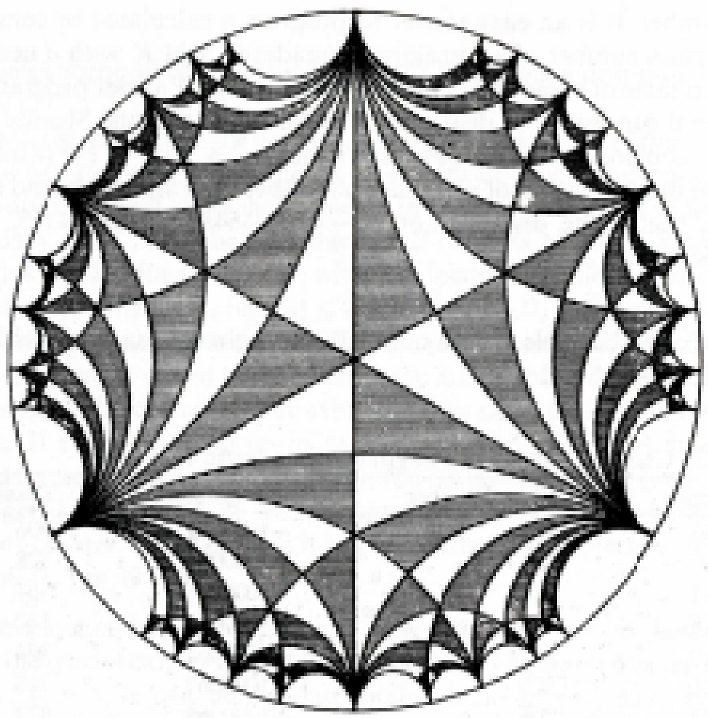


Figure 3.17. Another tessellation of the unit disc. (From Klein and Fricke [1]. Reprinted by permission of Teubner.)

Riemann surface

- If Γ is of the 1st kind, the ends of $\mathcal{M}_\Gamma$ consist only of cusps.
- Usually, one compactifies $\mathcal{M}_\Gamma$, and regards it as a Riemann surface. Then the field of meromorphic functions on $\mathcal{M}_\Gamma$ is an algebraic function field.
- There is a one to one correspondence

algebraic function fields

$\Longleftrightarrow$ compact Riemann surfaces

What does it mean?

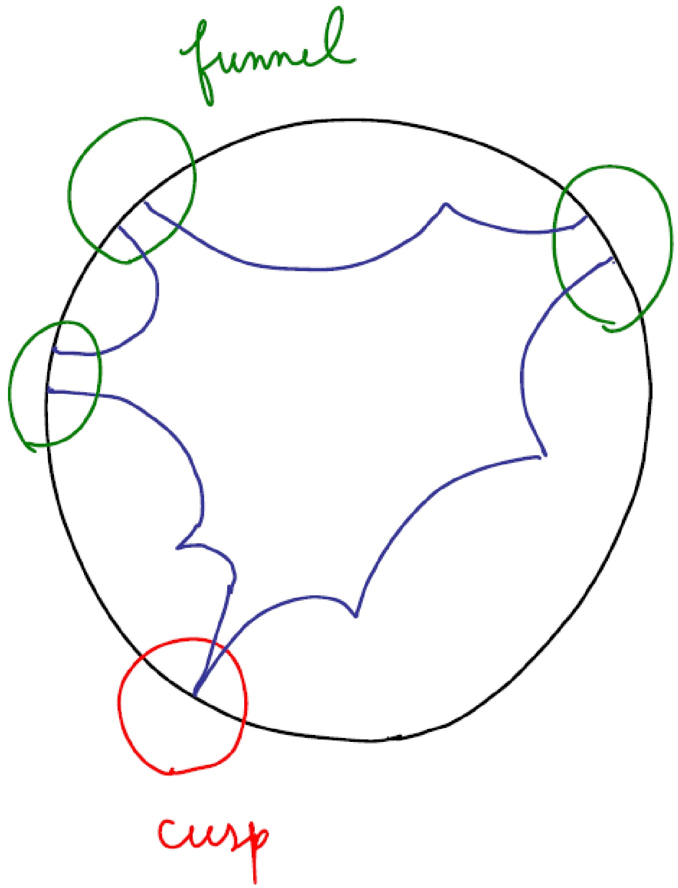
The surface is determined by a set of functions on it.

Question   How can we generalize it?

Answer

- The solution space of the **Helmholtz equation** on the manifold.
- More precisely, the **behavior of solutions** of the Helmholtz equation near infinity.
- This leads us to the **S-matrix.**

We need to be careful for singular points



Classification of the action by fixed points

$$\begin{aligned}
\text{elliptic} &\Longleftrightarrow \exists 1 \text{ fixed point} \in \mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| < 2
\end{aligned}$$

$$\begin{aligned}
\text{parabolic} &\Longleftrightarrow \exists 1 \text{ degenerate fixed point} \in \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| = 2
\end{aligned}$$

$$\begin{aligned}
\text{hyperbolic} &\Longleftrightarrow \exists 2 \text{ fixed points} \in \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| > 2
\end{aligned}$$

Isotropy group for the elliptic fixed points

- Let $\mathcal{M}_{sing}$ be the set of

ELLIPTIC FIXED POINTS

for Γ , and for $p \in \mathcal{M}_{sing}$ put

$$\mathcal{I}(p) = \{\gamma \in \Gamma\, ;\ \gamma \cdot p = p\}$$

Riemannian manifolds with singular points

- $\mathcal{M}_\Gamma$ can also be regarded as a Riemannian manifold equipped with the hyperbolic metric.
- However, at $p \in \mathcal{M}_{sing}$ this metric becomes singular.
- Around $p \in \mathcal{M}_\Gamma$, $\mathcal{M}_\Gamma$ has a special structure.

Orbifold structure

- By a suitable choice of local coordinates around $p \in \mathcal{M}_\Gamma$, the isotropy group $\mathcal{I}(p)$ turns out to be a finite rotation group.
- Then one can take a neighborhood of $p \in \mathcal{M}_{sing}$ which is like a sector with vertex at p.
- Hence $\mathcal{M}_\Gamma$ admits a local covering space around $p \in \mathcal{M}_\Gamma$, which is isometric to the hyperbolic space.

2-dim. Riemannian orbifold

- We consider a 2-dim. connected C^{∞} manifold $\mathcal{M}$, which is written as a union of open sets,

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N,$$

satisfying the following 4 assumptions:

Assumptions

(A-1) There exists $1 \leq \mu \leq N$ such that for $1 \leq i \leq \mu$, M_i is isometric to $S^1 \times (1, \infty)$ equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

(So, $M_1, \cdots, M_\mu$ have cusps at infinity.)

(A-2) For $\mu+1 \leq i \leq N$, $\mathcal{M}_i$ is diffeomorphic to $S^1 \times (0,1)$, and the metric on it has the following form :

$$ds^2 = \frac{(dy)^2 + h(x,dx) + A(x,y,dx,dy)}{y^2},$$

where $h(x,dx) = h(x)(dx)^2$ is a positive definite metric on S^1,

$$A(x, y, dx, dy)$$

$$= a(x, y)(dx)^2 + 2b(x, y)dxdy + c(x, y)(dy)^2,$$

and $a(x, y), b(x, y), c(x, y)$ satisfy

$$|\partial_x^\alpha (y\partial_y)^n d(x, y)| \le C_{\alpha n}(1 + |\log y|)^{-n-1-\epsilon_0}, \ \forall \alpha, n$$

for some $\epsilon_0 > 0$.

(We shall call $\{y = 0\}$ a regular infinity.)

(A-3) $\overline{\mathcal{K}}$ is compact.

(A-4) There exists a finite subset

$$\mathcal{M}_{sing} \subset \mathcal{K}$$

such that $\mathcal{M}$ has a C^∞ Riemannian metric g on $\mathcal{M} \setminus \mathcal{M}_{sing}$.

To each $p \in \mathcal{M}_{sing}$, there exists an open set $\widetilde{U}_p \subset \mathbf{R}^2$ such that $0 \in \widetilde{U}_p$ and $\widetilde{U}_p$ has the metric $\widetilde{g}_p$ with the following properties :

Orbifold strucure around $p \in \mathcal{M}_{sing}$

Let $U_p(\epsilon)$ and $\widetilde{B}(\epsilon)$ be ϵ -neighborhood of $p \in \mathcal{M}$ and $0 \in \mathbf{R}^2$, respectively.

We adopt the geodesic polar coordinates centered around 0 to transform the metric $\widetilde{g}_p$ on $\widetilde{B}(\epsilon) \subset \widetilde{U}(\epsilon)$ into the form

$$\widetilde{g}_p = (dr)^2 + G_p(r,\theta)(d\theta)^2,$$

$$0 < r < \epsilon,\ 0 \leq \theta < 2\pi$$

We assume that there exists an integer $n_p \geq 2$ such that by the action

$$z = (x_1, x_2) \to$$

$$\gamma \cdot z = \begin{pmatrix} \cos(2\pi/n_p) & -\sin(2\pi/n_p) \\ \sin(2\pi/n_p) & \cos(2\pi/n_p) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the metric $\widetilde{g}_p$ is invariant. Moreover, $U_p(\epsilon) \setminus \{p\}$ is isometric to the part

$$\{(r, \theta); 0 < r < \epsilon, 0 \leq \theta \leq 2\pi/n_p\}$$

where two segments $\{(r, 0); 0 < r < \epsilon\}$, $\{(r, 2\pi/n_p); 0 < r < \epsilon\}$ are identified.

Spectral properties

Let Δ_g be the Laplace-Beltrami operator of $\mathcal{M}$,

and put $$H = -\Delta_g - \frac{1}{4}$$

Then $$\sigma_{ess}(H) = [0, \infty)$$

Besov type space

- By using the diadic decomposition of the manifold (just as in the Fourier analysis on Euclidean space), one can introduce the spaces $B, \ B^*$ rigging $L^2(M)$:

$$B \subset L^2(M) \subset B^*$$

The space B^* is important. On each end, it is defined as

$$\|u\|_{\mathcal{B}^*} = \left(\sup_{R>e} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|u(y)\|_{\mathbf{H}}^2 \frac{dy}{y^2} \right)^{1/2} < \infty$$

We write $f(y) \simeq g(y)$ if

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \| f(y) - g(y) \|_{\mathbf{H}} \frac{dy}{y^2} = 0$$

Fourier transform on M

- By observing the behavior of the resolvent at infinity, one can construct the Fourier transform M.

Representation space

We put

$$h = \sum_{i=1}^{\mu} C \oplus \sum_{i=\mu+1}^{N} L^2(S^1)$$

$$\widehat{\mathcal{H}} = L^2((0,\infty); \mathbf{h}; dk)$$

$$\mathcal{F}^{(\pm)}(k) = \left(\mathcal{F}_1^{(\pm)}(k), \cdots, \mathcal{F}_N^{(\pm)}(k)\right)$$

$$\left(\mathcal{F}^{(\pm)} f\right)(k) = \mathcal{F}^{(\pm)}(k) f$$

Spectral representation

$$\mathcal{F}^{(\pm)} : \mathcal{H}_{ac}(H) \rightarrow \widehat{\mathcal{H}}$$

is unitary, and diagonalizes H :

$$\left(\mathcal{F}^{(\pm)} H f\right)(k) = k^2 \left(\mathcal{F}^{(\pm)} f\right)(k)$$

Eigenoperator

$$\mathcal{F}^{(\pm)}(k)^* : \mathbf{h} \longrightarrow \mathcal{B}^*$$

is an eigenoperator of H in the sense that

$$(H - k^2)\mathcal{F}^{(\pm)}(k)^*\phi = 0, \ \forall \phi \in \mathbf{h}$$

Characterization of the solution space for the Helmholtz equation

$$\{u \in \mathcal{B}^*; (H - k^2)u = 0\} = \mathcal{F}^{(\pm)}(k)^* \mathbf{h}$$

Theorem of Helgason

On the Poincare disc, all solutions of the Helmholtz equation

$$(-\Delta_g - E)u = 0$$

are written in terms of Poisson integrals of Sato's hyperfunction on the boundary.

Features of $\mathcal{B}^*$ space

Smallest space with respect to the decay at infinity: i.e. if

$$(-\Delta_g - E)u = 0, E > 1/4,$$

$$u \in \mathcal{B}^*, u \simeq 0,$$

then $u = 0.$

Asymptotic expansion

If $u \in \mathcal{B}^*, (H - k^2)u = 0,$

then it admits the following asymptotic expansion:

$$u \cong \omega_-(k)\sum_{j=1}^{\mu} \chi_j y^{(n-1)/2+ik}\psi_j^{(-)}$$

$$+ \ \omega_-^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j y^{(n-1)/2-ik}\psi_j^{(-)}$$

$$- \ \omega_+(k)\sum_{j=1}^{\mu} \chi_j y^{(n-1)/2+ik}\psi_j^{(+)}$$

$$- \ \omega_+^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j y^{(n-1)/2+ik}\psi_j^{(+)}$$

S-matrix

For any $\psi^{(-)} \in \mathbf{h}$, there exists a unique $\psi^{(+)} \in \mathbf{h}$ and $u \in \mathcal{B}^*$ satisfying

$$(H - k^2)u = 0$$

for which the above expansion holds. Moreover, the operator $\widehat{S}(k)$ defined by

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

is unitary on $\mathbf{h}$.

Inverse problems

- One can show that if one of the ends has a regular infinity, the corresponding component of the S-matrix (for all frequency) determines the Riemannian metric (Sa Barreto, Kurelev-I).

- So, if one of the ends is regular, the space B^* contains sufficient information to recover the manifold.

- However, it does not cover the case where all the ends have a cusp (as in the case of Fuchsian group of the 1^{st} kind).
- This is because the cusp gives only a one-dimensional contribution to the continuous spectrum.

Exponentially growing solutions

- On the end M_1, having a cusp, the Helmholtz equation takes the form

$$-y^2(\partial_y^2 + \partial_x^2)u - \frac{1}{4}u = k^2 u$$

- Expand u into a Fourier series

$$u(x,y) = \sum_{n \in \mathbf{Z}} e^{2\pi i n x} u_n(y)$$

$$y^2(-\partial_y^2 + (2\pi n)^2)u_n - \frac{1}{4}u_n = k^2 u_n$$

$$u_n(y) = \begin{cases} a_n y^{1/2} I_{-ik}(2\pi|n|y) \\ \qquad\qquad + b_n\, y^{1/2} K_{ik}(2\pi|n|y), & (n \neq 0) \\ a_0 y^{1/2-ik} + b_0 y^{1/2+ik}, & (n = 0) \end{cases}$$

Modified Bessel functions

$$\frac{dw^2}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 + \frac{\nu^2}{z^2}\right) w = 0$$

$$\begin{cases} I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}} e^z, & z \to \infty, \\ K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, & z \to \infty \end{cases}$$

The space of generalized scattering data at infinity

$$A_{\pm\infty} = \left(\bigoplus_{j=1}^{\mu} l^{2,\pm\infty}\right) \oplus \left(\bigoplus_{j=\mu+1}^{N} L^2(M_j)\right)$$

$$L^2(M_j) = L^2(S^1; h_j(x)dx)$$

$$l^{2,\infty} \ni a = (a_n)_{n\in\mathbf{Z}} \Longleftrightarrow \sum_{n\in\mathbf{Z}} |a_n|^2 \rho^{|n|} < \infty, \forall \rho > 1,$$

$$l^{2,-\infty} \ni b = (b_n)_{n\in\mathbf{Z}} \Longleftrightarrow \sum_{n\in\mathbf{Z}} |b_n|^2 \rho^{-|n|} < \infty, \exists \rho > 1,$$

Notation

- Incoming and outgoing data at infinity

$$\psi^{(-)} = (a_1, \cdots, a_\mu, \psi_{\mu+1}^{(-)}, \cdots, \psi_N^{(-)}) \in A_{-\infty},$$

$$\psi^{(+)} = (b_1, \cdots, b_\mu, \psi_{\mu+1}^{(+)}, \cdots, \psi_N^{(+)}) \in A_{+\infty}.$$

$$u_j^{(-)} = \begin{cases} \omega_c^{(-)}(k)\Big(a_{j,0}y^{1/2-ik} \\ \qquad + \sum_{n\neq 0} a_{j,n}e^{2\pi inx}y^{1/2}I_{-ik}(2\pi|n|y)\Big), & 1 \le j \le \mu \\ \omega_-(k)y^{1/2+ik}\psi_j^{(-)}(x), & \mu+1 \le j \le N, \end{cases}$$

$$u_j^{(+)} = \begin{cases} \omega_c^{(+)}(k)\Big(b_{j,0}y^{1/2+ik} \\ \qquad + \sum_{n\neq 0} b_{j,n}e^{2\pi inx}y^{1/2}K_{ik}(2\pi|n|y)\Big), & 1 \le j \le \mu \\ \omega_+(k)y^{1/2-ik}\psi_j^{(+)}(x), & \mu+1 \le j \le N, \end{cases}$$

Exponentially growing (at the cusp) solutions to the Helmholtz equation

THEOREM. Let $k > 0$ be such that $k^2 \notin \sigma_p(H)$. Then, given any incoming data $u_j^{(-)}$, $j = 1, \cdots, N$ there exists a unique solution u s. t.

$$(H - k^2)u = 0, \quad u - \sum_{j=1}^{N} \chi_j u_j^{(-)} \text{ is outgoing.}$$

For this u, there exists $\psi^{(+)} \in \mathbf{A}_\infty$ such that

(1) For $j=1,\cdots,\mu,$ there exists $y_0>0$
s.t. in M_j if $y > y_0$

$$u = u_j^{(-)} - u_j^{(+)}.$$

(2) For $j = \mu + 1, \cdots, N,$

$$u - u_j^{(-)} \cong - u_j^{(+)}.$$

This is a solution to the Helmholtz equation which is

- exponentially growing at the cusp, (i.e. non-physical at the cusp)

- polynomially bounded near the regular infinity (i.e. physical at the regular infinity).

Generalized S-matrix

- We call the operator

$$\mathbf{S}(\mathbf{k}) : \mathbf{A}_{-\infty} \ni \psi^{(-)} \longrightarrow \psi^{(+)} \in \mathbf{A}_{\infty}$$

the generalized S-matrix.

THEOREM

Suppose we are given two 2-dim. Riemannian orbifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ satisfying the above conditions.

Suppose the $(1,1)$ components of the generalized S-matrix coincide :

$$S_{11}^{(1)}(k) = S_{11}^{(2)}(k)$$

$$k > 0,\ k^2 \notin \sigma_p(H^{(1)}) \cup \sigma_p(H^{(2)}).$$

Then, there is an isometry between $M^{(1)}$ and $M^{(2)}$ in the following sense.

(1) There is a homeomorphism

$$\Phi : M^{(1)} \to M^{(2)}.$$

(2)

$$\Phi : \mathcal{M}^{(1)}_{sing} \to \mathcal{M}^{(1)}_{sing}$$

(3) $$\Phi : \mathcal{M}^{(1)} \setminus \mathcal{M}^{(1)}_{sing} \to \mathcal{M}^{(2)} \setminus \mathcal{M}^{(2)}_{sing}$$

is a Riemannian isometry.

For the case of the Fuchsian group of the 1st kind

- If the generalized S-matrices coincide, then

$\Gamma^{(1)}\backslash\mathbf{H}^2$ and $\Gamma^{(2)}\backslash\mathbf{H}^2$ are conformal, and

$\Gamma^{(1)}$ and $\Gamma^{(2)}$ are conjugate each other:

$$g\Gamma^{(1)}g^{-1} = \Gamma^{(2)}, \quad \exists g \in SL(2,\mathbf{R})$$

The basic idea of the proof

- BC-method (Boundary Control method)

  Belishev 1987

  Belishev-Kurylev 1987, 1992

Neumann-Dirichlet map

For a compact manifold M with boundary consider the Neumann problem

$$(-\Delta_g - z)u = 0, \quad in \quad M,$$
$$\partial_n u = f, \quad on \quad \partial M$$

The Neumann-Dirichlet map is defined by

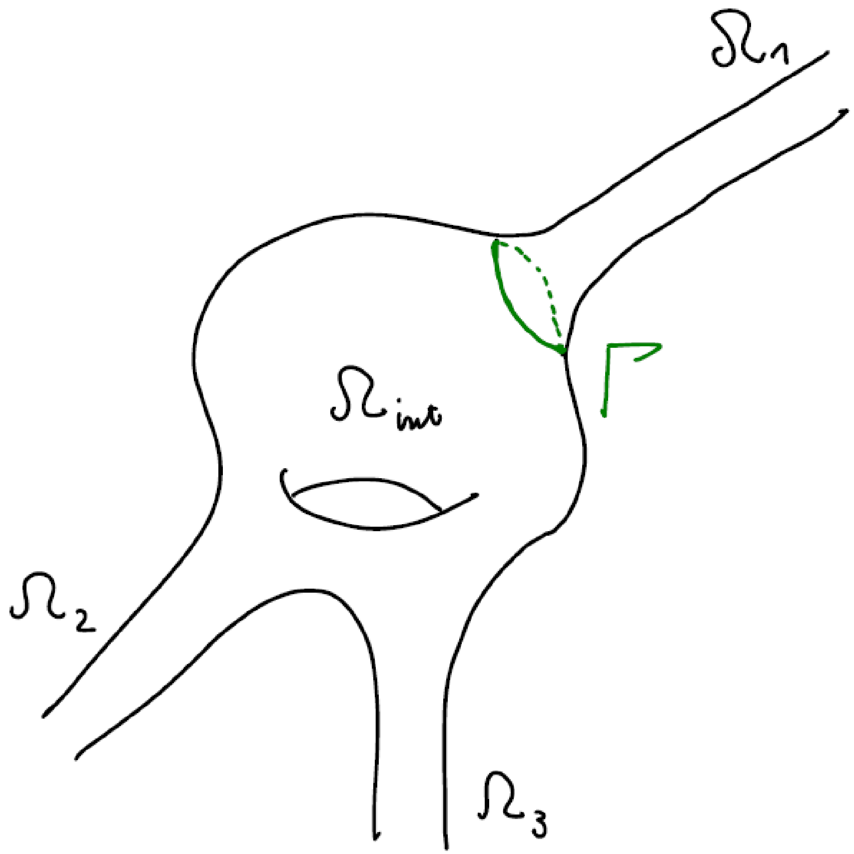
$$\Lambda(z)f = u\Big|_{\partial M}$$

Boundary control method

- From the knowledge of N-D map $\Lambda(z)$

one can recover the metric of M uniquely.

“Interior” boundary value problem



- From the generalized S-matrix, one can determine the N-D map for the interior domain with data on Γ.
- Then one can apply the BC method to recover the metric in $\mathcal{M} \setminus \mathcal{M}_{sing}$.
- Around the singular points, one needs a new issue.

Work in progress

- Extension to higher dimensions, where the manifolds at infinity are also orbifolds.