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Inverse hyperbolic problems and black holes

Let $\sum_{j,k=1}^{n} g_{jk}(x)dx_j dx_k$ be a Lorentzian metric in $\mathbf{R}^{n+1}$, $x=(x_1,...,x_n)\in \mathbf{R}^n$, $x_0 \in \mathbf{R}$ is the time variable, $g_{jk}(x)$ are independent of x_0. Consider a hyperbolic equation of the form

$$(1)\qquad \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g(x)}\, g^{jk}(x) \frac{\partial u(x_0,x)}{\partial x_k}\right) = 0$$

in $\Omega\times\mathbf{R}^n$, where Ω is a smooth bounded domain in $\mathbf{R}^n$, $g(x)=\det[g_{jk}(x)]_{j,k=0}^{n}$, $[g^{jk}(x)]_{j,k=0}^{n} = \left([g_{jk}(x)]_{j,k=0}^{n}\right)^{-1}$.

Equation (1) describes linear waves in the general relativity. Also (1) describes the wave propagation in a moving medium (for example, the light propagation in the moving dielectric or acoustic waves in a moving fluid). An interesting feature of the equation (1) is the appearance of black holes, i.e. regions $\Omega_0\times\mathbf{R}$ such that no signals (disturbances) initiated inside $\Omega_0\times\mathbf{R}$ can reach the exterior of $\Omega_0\times\mathbf{R}$. The famous example is the Schwartzchild metric

$$(1-\frac{2m}{R})dt^2 - dx^2 - dy^2 - dz^2 - \frac{4m}{R}dtdR - \frac{2m}{R}(dR)^2$$

when $R=\sqrt{x^2+y^2+z^2}$. Here $\{R=2m\}\times\mathbf{R}$ is a black hole.

Consider the initial-boundary problem for the equation (1) on $\Omega\times[0,T]$:

$$(2)\qquad u(0,x)=0,\quad x\in\Omega,\qquad u\big|_{\partial\Omega\times[0,T]} = f.$$

Let Γ be any open subset of $\partial\Omega$ and let Λ be the Dirichlet-to-Neumann operator. Consider the inverse problem of the determination of the metric by the DN operator on $\Gamma\times(0,T)$. The unique determination of the metric is possible only modulo the changes of variables

$$(3)\qquad y=\varphi(x),\quad y_0 = x_0 + a(x),$$

where $\varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\hat{\Omega}}$, $a(x)\in C^\infty(\overline{\Omega})$, $\varphi(x)=I$ on Γ, $a(x)=0$ on Γ, because such changes of variables do not change the DN operator. It is clear that in the presence of the black hole it is impossible to determine the metric inside the black hole.

An interesting problem is whether it is possible to determine the boundary of the black hole (called the event horizon). This problem is still open.

We prove the following result:

Theorem 0.1. *Let* Σ *be the ergosphere, i.e. the surface where* $g_{00}(x) = 0, g_{00}(x) > 0$ *in the exterior of* Σ. *Then the boundary measurements on* $\Gamma \times (0, +\infty)$ *uniquely determine the ergosphere (up to the change of variables (3)).*

Remarks

1) Black holes always either coincide with the ergosphere (as in the case of the Schwartzschield black hole) or are inside the ergosphere (as in the case of the Kerr black hole).

2) Our method of proof does not require advance knowledge whether the ergosphere is present or not. In the case when there is no ergosphere (and therefore there is no black hole) we recover the metric in the whole domain Ω.

3) It follows from the proof that one need an infinite time interval of boundary measurements (i.e. $\Gamma \times (0, +\infty)$) to recover the ergosphere.