INVERSE HYPERBOLIC PROBLEMS AND BLACK HOLES.

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This talk is based on the following papers:

1) Eskin, G., Optical Aharonov-Bohm effect: inverse hyperbolic problem approach, Comm. Math. Phys. 284 (2008), no.2, 317-343

2) Eskin, G., Inverse hyperbolic problems and optical black holes, Comm. Math. Phys., 291, 817-829 (2010)

3) Eskin, G., Perturbations of the Kerr black hole and the boundness of linear waves, Journal of Math. Phys., vol. 51, no.11, November, 2010

The wave equation.

Let

$$(1) \qquad \sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k$$

be a Lorentzian metric in $\mathbb{R}^{n-1}$, where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $x_0 \in \mathbb{R}$ is the time variable. We assume that $g_{jk}(x)$ are independent of x_0 (stationary metric), $g^{00}(x) > 0$, the signature of the quadratic form (1) is $(+1, -1, ..., -1)$.

Consider the wave equation of the form

$$(2) \qquad \square_g u \stackrel{def}{=} \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}}$$

$$\cdot \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g(x)} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k}\right) = 0$$

Here

$$[g^{jk}(x)]_{j,k=0}^{n} = \left([g_{jk}(x)]_{j,k=0}^{n}\right)^{-1},$$
$$g(x) = \det[g_{jk}(x)]_{j,k=0}^{n}.$$

Examples of the wave equation of the form (2).

1) General relativity.

Let metric $[g_{jk}(x)]_{j,k=0}^{n}$ be a solution of the Einstein equations in the vacuum, $n = 3$.

Consider $\mathbb{R}^3 \times \mathbb{R}$ as a pseudo-Riemannian manifold with the metric $[g_{jk}]_{j,k=0}^{n}$. It is called the space-time. Then the equation

$$\Box_g u = 0$$

describes the linear gravitational waves on the background of this metric.

Example. Schwarzschild metric.

$$\left(1 - \frac{2m}{R}\right) dt^2 - dx^2 - dy^2 - dz^2 - \frac{4m}{R} dt dR - \frac{2m}{R}(dR)^2, \tag{3}$$

where $R = \sqrt{x^2 + y^2 + z^2}$,
is the solution of the Einstein equations.

2) Propagation of light in a moving dielectric, $n = 3$.

In this case the metric $[g_{jk}]_{j,k=0}^{3}$ has the following form

(4)

$g_{jk}(x) = \eta_{jk}+(n^{-2}(x)-1)v_j v_k,\ 0 \leq j, k \leq 3,$

where $[\eta_{jk}]$ is the Lorentz (Minkowski) metric tensor,

$\eta_{00} = 1, \eta_{jj} = -1, 1 \leq j \leq 3,$

$\eta_{jk} = 0$ when $j \neq k$,

$n(x) = \sqrt{\varepsilon(x)\mu(x)}$ is the refraction index,

$v_0(x) = \left(1 - \frac{|w(x)|^2}{c^2}\right)^{-\frac{1}{2}},$

$v_j(x) = -\left(1 - \frac{|w(x)|^2}{c^2}\right)^{-\frac{1}{2}} \frac{w_j}{c},\ 1 \leq j \leq 3,$

is the four-velocity vector,

$w(x) = (w_1(x), w_2(x), w_3(x))$ is the velocity of the flow, c is the speed of light in the vacuum.

The wave equation corresponding to the metric (4) is called the Gordon equation.

3) Acoustic waves in a moving fluid.

Here the metric tensor $[g_{jk}]_{j,k=0}^{3}$ has the form

$$g_{00} = \frac{\rho}{c}(c^2 - v^2),$$

$$g_{0j} = g_{j0} = \frac{\rho}{c}v^j, \ 1 \le j \le 3,$$

$$g_{jk} = -\frac{\rho}{c}\delta_{jk}, \ 1 \le j, k \le 3, \ v^2 = \sum_{j=1}^{3}(v^j)^2,$$

$v = (v^{(1)}, v^{(2)}, v^{(3)})$ is the velocity of the fluid, ρ is the density and c is the sound speed.

Inverse problem.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$.

Consider initial-boundary value problem for the wave equation $\square_g u = 0$ in $\Omega \times \mathbb{R}$:

$$(5) \quad u(x_0, x) = 0 \quad \text{for } x_0 \ll 0,\ x \in \Omega,$$
$$u(x_0, x)\Big|_{\partial\Omega\times\mathbb{R}} = f,$$

where f has a compact support in $\partial\Omega \times \mathbb{R}$.

Let Λ be the Dirichlet-to-Neumann operator (DN operator)

$$\Lambda f = \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x)$$
$$\cdot \Big(\sum_{p,r=1}^{n} g^{pr}(x) \nu_j \nu_r \Big)^{-\frac{1}{2}} \Big|_{\partial\Omega\times\mathbb{R}}$$

Let Γ be an arbitrary open subset of $\partial\Omega$. We shall consider the inverse problem of the determination of the metric by the boundary measurements on $\Gamma \times (0, T)$, i.e. from the knowledge of Λf on $\Gamma \times (0, T)$ for any f with the support on $\overline{\Gamma} \times [0, T]$.

Let $y = \varphi(x)$ be a diffeomorphism of $\overline{\Omega}$ on $\overline{\hat{\Omega}} \subset \mathbb{R}^n$ such that $\varphi(x) = x$ on Γ_0.

Let $a(x) \in C^\infty(\overline{\Omega}),\ a(x) = 0$ on Γ.

Consider the changes of variables of the form

$$(6) \qquad y = \varphi(x), \quad y_0 = x_0 + a(x),$$

where

$$(7) \quad \varphi(x) = x \quad \text{on} \quad \Gamma, a(x) = 0 \quad \text{on} \quad \Gamma.$$

It is easy to show that the change of variables (6), (7) does not change the boundary measurements. Therefore the determination of the metric by the boundary measurements is possible only modulo the change of variables (6), (7).

Black and white holes.

An interesting feature of the equation

$$\square_g u = 0$$

is the appearance of black and white holes.

A closed domain $\Omega_0 \times \mathbb{R}$ is called a black hole if no disturbance (signal) initiated in $\Omega_0 \times \mathbb{R}$ can reach the exterior of $\Omega_0 \times \mathbb{R}$.

A domain $\Omega_0 \times \mathbb{R}$ is a white hole if no signal from the exterior of $\Omega_0 \times \mathbb{R}$ can reach the interior of $\Omega_0 \times \mathbb{R}$.

One can show that $\Omega_0 \times \mathbb{R}$ is a black or a white hole if the boundary $S_0 \times \mathbb{R}$ of $\Omega_0 \times \mathbb{R}$ is a characteristic surface, i.e.

$$(8) \qquad \sum_{j,k=1}^{n} g^{jk}(x) S_{0x_j}(x) S_{0x_k}(x) = 0$$

$$\text{when} \quad S_0(x) = 0,$$

where $S_0(x) = 0$ is the equation of $\partial\Omega_0$.

One can also show that $\Omega_0 \times \mathbb{R}$ is a black hole if

$$\sum_{j=1}^{n} g^{0j}(x) S_{0x_j}(x) < 0 \quad \text{when} \quad S_0(x) = 0,$$

and $\Omega_0 \times \mathbb{R}$ is a white hole if

$$\sum_{j=1}^{n} g^{0j}(x) S_{0x_j}(x) > 0 \quad \text{when} \quad S_0(x) = 0.$$

The boundary $S_0 \times \mathbb{R}$ of a black or a white hole is called the event horizon.

In the case of the Schwarzschield metric $\{R < 2m\} \times \mathbb{R}$ is a black hole.

The ergosphere.

The surface

$$g_{00}(x) = 0$$

is called the ergosphere. We assume that the ergosphere is a closed surface $\Delta \subset \mathbb{R}^n$ and that $g_{00}(x) > 0$ outside of Δ.

In the case of the Schwartzschield metric the ergosphere is $\{R = 2m\}$, i.e. the ergosphere coincide with the event horizon. The event horizon always either coincide with the ergosphere or it is inside the ergosphere (as in the case of the Kerr metric).

For the Gordon equation of the propagation of light in the moving medium we have that the equation of the ergosphere Δ is

$$|w|^2 - \frac{c^2}{h^2(x)} = 0.$$

When vector $w = (w_1, w_2, w_3)$ is normal to Δ for $x \in \Delta$ the ergosphere coincides with the event horizon.

Nonuniqueness of the solution of the inverse problem.

If $\overline{\Omega}_0 \subset \Omega$ and $\Omega_0 \times \mathbb{R}$ is the event horizon then we can change arbitrary the metric inside Ω_0 and this will not affect the boundary measurements on $\partial\omega_0 \times \mathbb{R}$, i.e. the solution of the inverse problem is not unique in the presence of black or white holes.

The following problem arise:

It is possible to determine the event horizon by the boundary measurements?

A partial answer is given by the following theorem.

Theorem 1. *Consider the initial-boundary value problem for the equation* $\square_g u = 0$. *Suppose* $g^{00}(x) > 0$ *in* $\overline{\Omega}$ *and* $\partial\Omega \times \mathbb{R}$ *is not characteristic. Let the ergosphere* $\Delta(x) = 0$ *be a smooth closed surface inside* Ω. *Let* Γ *be an arbitrary open subset of* $\partial\Omega$. *Then the boundary measurements on* $\Gamma \times (0, +\infty)$ *determine uniquely* $\Delta(x) = 0$ *up to a change of variables (6), (7).*

Sketch of the proof.

The proof of the Theorem is based on an extension of the Boundary Control method (BC-method).

BC-method was invented by M.Belishev and developed by Belishev, Belishev and Kurylev, Kurylev and Lassas, and others.

I proposed a new method based on the BC-method that allows to deal with new problems such as the inverse hyperbolic problems with time-dependent coefficients and the inverse problems for the wave equations of the form (2).

The proof consists of the recovery of the metric in the exterior of $\Delta = 0$ (up to a change of variables). Then by continuing we get the equation of the ergosphere. We start with the determination of the metric in a small neighborhood of Γ and gradually continue to recover metric deeper in Ω. As we proceed the time interval $(0, T)$ needed to reach a point $x \in \Omega$ increases and $T \to +\infty$ when we approach the ergosphere.

One can show that the measurements on $\Gamma \times (0, T)$ where $T < +\infty$ are not enough to recover the ergosphere.

Note that we do not need apriori knowledge of the ergosphere existence. If there is no ergosphere then we can recover the metric in the whole domain Ω in a finite time T.

References to the BC method.

1 Belishev, M., 1997, Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Problems 13, R1-R45

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4 Eskin, G., 2006, A new approach to the hyperbolic inverse problems, Inverse problems, vol. 22, No. 3, 815-831

5 Eskin, G., 2007, A new approach to the hyperbolic inverse problems II: global step, Inverse Problems 23, 2343-2356

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